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Generalized Fixed Point Algebras for Coactions of Locally Compact Quantum Groups

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Abstract

We extend the construction of generalized fixed point algebras to the setting of locally compact quantum groups following the treatment of Marc Rieffel, Ruy Exel and Ralf Meyer in the group case. We mainly follow Meyer's approach analyzing the constructions in the realm of equivariant Hilbert modules.

We generalize the notion of square-integrable Hilbert modules and prove that they are characterized by the equivariant Version of Kasparov's Stabilization Theorem. We also generalize the notion of continuous square-integrability, which is exactly what one needs in order to define generalized fixed point algebras. As in the group case, we prove that there is a correspondence between continuously square-integrable Hilbert modules over an equivariant C^*-algebra B and Hilbert modules over the reduced crossed product of B by the underlying quantum group. The generalized fixed point algebra always appears as the algebra of compact operators of the associated Hilbert module over the reduced crossed product.

As an application, we analyze the case of group coactions and show that the class of Fell bundles over a locally compact group G can be characterized by means of continuous square-integrability of coactions of G on C^*-algebras. We construct Fell bundles over G from continuously square-integrable coactions of G and vice-versa. Under certain circumstances, this correspondence provides an equivalence of categories. In this picture, the generalized fixed point algebra coincides with the unit fiber of the associated Fell bundle. Our results can be used to classify the Fell bundle structures for a given coaction.

Zusammenfassung

Wir untersuchen die Konstruktion verallgemeinerter Fixpunktalgebren für Kowirkungen lokalkompakter Quantengruppen, wobei wir den Arbeiten von Marc Rieffel, Ruy Exel und Ralf Meyer im Gruppenfall folgen. Hauptsächlich folgen wir der Arbeit von Meyer, indem wir die Konstruktion äquivarianter Hilbertmoduln analysieren.

Wir verallgemeinern den Begriff der quadrat-integriebaren Hilbertmoduln und beweisen, dass sie durch die äquivariante Version von Kasparovs Stabilisierungssatz charakterisiert sind. Wir verallgemeinern auch den Begriff der stetigen quadrat-integriebaren Hilbertmoduln. Dies ist genau, was man braucht, um verallgemeinerte Fixpunktalgebren zu definieren. Wir beweisen, dass – so wie im Gruppenfall – die stetigen quadrat-integriebaren Hilbertmoduln über einer äquivarianten C^*-algebra B zu den Hilbertmoduln über dem reduzierten veschränkten Produkt von B korrespondieren. Auf diese Weise korrespondiert die verallgemeinerte Fixpunktalgebra zur Algebra der kompakten Operatoren auf dem entsprechenden Hilbertmodul über dem reduzierten verschränkten Produkt.

Als eine Anwendung untersuchen wir den Fall von Gruppen-Kowirkungen und beweisen, dass die Klasse der Fell-Bündel über einer lokalkompakten Gruppe G durch stetige Quadrat-Integrierbarkeit von Kowirkungen von G auf C^*-Algebren charakterisiert werden kann. Wir konstruieren Fell-Bündel über G aus stetigen quadrat-integriebaren Kowirkungen von G und umgekehrt. Unter bestimmten Voraussetzungen ist diese Beziehung eine Äquivalenz von Kategorien. Die verallgemeinerte Fixpunktalgebra ist immer gegeben durch die Eins-Faser des entsprechenden Fell-Bündels. Unsere Resultate können benutzt werden, um die Fell-Bündel-Strukturen für Kowirkungen zu klassifizieren.

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Contents

Chapter 1

Introduction

1.1 The group case

Let G be a locally compact group and let X be a G-space, that is, a locally compact (Hausdorff) space with a continuous action of G. The action of G on X is called *proper* if the map $G \times X \to X \times X$, $(t, x) \mapsto (t \cdot x, x)$ is proper in the sense that inverse images of compact subsets are again compact.

Properness is a concept that enables properties of the actions of non-compact groups to resemble those of compact groups. Proper actions have many nice properties. One of the most important ones is that the orbit space $G\backslash X$ is again a locally compact (Hausdorff) space.

A program to extend this notion to the setting of noncommutative dynamical systems, that is, groups acting on C^*-algebras, was initiated by Marc Rieffel in [65]. His idea relies on one basic result, namely, the fact that for a proper G-space X, the commutative C^*-algebra associated to the orbit space (that is, the algebra $\mathcal{C}_0(G\backslash X)$ of continuous functions on $G\backslash X$ vanishing at infinity) is Morita equivalent to an ideal in the reduced crossed product $C^*_{\mathrm{r}}\big(G, \mathcal{C}_0(X)\big)$, where we let G act on $\mathcal{C}_0(X)$ in the usual way. If, in addition, the action is free, then this ideal is the whole crossed product.

The imprimitivity bimodule implementing the Morita equivalence between the algebra $\mathcal{C}_0(G\backslash X)$ and the ideal in the crossed product turns out to be a suitable completion of the space $\mathcal{C}_c(X)$ of compactly supported continuous functions on X. Based on this fact, Rieffel called a (not necessarily commutative) G-C^*-algebra, that is, a C^*-algebra A with a (strongly) continuous action of G, *proper* if there exists a dense $*$-subalgebra A_0 of A with some suitable properties (which recover more or less the properties of $\mathcal{C}_c(X)$ in the commutative case) such that from A_0 one can define a *generalized fixed point algebra* $\mathrm{Fix}(A_0)$ (which is the noncommutative analogue of $\mathcal{C}_0(G\backslash X)$) and an ideal $\mathcal{I}(A_0)$ in the reduced crossed product algebra $C^*_{\mathrm{r}}(G, A)$. Moreover, one can complete A_0 to give rise to an imprimitivity bimodule between $\mathrm{Fix}(A_0)$ and $\mathcal{I}(A_0)$. If, in addition, $\mathcal{I}(A_0)$ is the entire reduced crossed product, then the action is called *saturated*.

In the commutative case, it makes no difference to work with full or reduced crossed products because they are isomorphic if the action is proper. However, in the general case, this is not true and, as observed by Rieffel in [65], one meets some problems in trying to construct an appropriate inner product (and hence an imprimitivity bimodule) taking values in the full crossed product algebra.

The choice of $A_0 = \mathcal{C}_c(X)$ in the commutative case seems to be canonical in some sense. Also, if G is compact, any G-C^*-algebra is proper for $A_0 = A$ and the generalized fixed point algebra $\mathrm{Fix}(A)$ is the usual fixed point algebra $\mathrm{Fix}(A) = \{a \in A : \alpha_t(a) = a \text{ for all } t \in G\}$, where α denotes the action of G on A. In general, it was not clear to Rieffel how canonical the choice of A_0 is, or how the generalized fixed point algebra depends on the choice of A_0. Of course, the best case would be to have only one generalized fixed point algebra or even only one choice of A_0 by requiring some additional properties. At least, it would be desirable to have some intrinsic process which could produce some canonical choice of A_0.

Focusing on this point, Rieffel has further investigated his first definition of proper actions in a second work [66]. He came out with another definition of proper action including the first one, which we explain in some detail. A positive element $a \in A$ is called *integrable* if there exists b in the multiplier algebra $\mathcal{M}(A)$ of A such that for any positive linear functional θ on A, the function $t \mapsto \theta(\alpha_t(a))$ is integrable in the ordinary sense, and $\int_G \theta(\alpha_t(a))\,\mathrm{d}t = \theta(b)$. In this case, it is natural to write $b = \int \alpha_t(a)\,\mathrm{d}t$. However, we should point out that this integral does not converge in Bochner's sense, unless G is compact or $a = 0$, because the integrand has constant norm. The G-C^*-algebra A is called *integrable* if the space of integrable elements (that is, elements of A that can be written as a sum of positive integrable elements) is dense in A.

Integrability is closely related to the notion of properness discussed previously. Indeed, Rieffel proved in [66] that if A is proper, then it is also integrable. Furthermore, he also proved that in the commutative case $A = \mathcal{C}_0(X)$, where X is some locally compact G-space, A is integrable if and only if X is a proper G-space. Moreover, in this case $\mathcal{C}_c(X)$ consists of integrable elements and the generalized fixed point algebra is generated by the averages $\int \alpha_t(a)\,\mathrm{d}t$ with $a \in \mathcal{C}_c(X)$. Note also that if G is compact, then any G-C^*-algebra A is integrable. In fact, in this case, any element of A is integrable and the (generalized) fixed point algebra is also generated by the averages $\int \alpha_t(a)\,\mathrm{d}t$, with $a \in A$. Due to this close relation, an integrable G-C^*-algebra was also called proper by Rieffel in [66].

However, it was not clear to Rieffel in [66] whether, given an integrable G-C^*-algebra A, there is a dense subspace $A_0 \subseteq A$ yielding the properness of A (as defined in [65]) and hence the desired generalized fixed point algebra. He defined a "big generalized fixed point algebra" generated by averages that worked in the commutative case, but, in general, it was really too big to be Morita equivalent to an ideal in the reduced crossed product. As explained by Ruy Exel in [18, 19], the problem appears already in the case of Abelian groups.

Exel was more interested in another point, namely, to characterize the G-C^*-algebras appearing as dual actions on cross-sectional C^*-algebras of *Fell bundles* (also called C^*-algebraic bundles; see [23]). In order to explain this, let us assume that G is not only Abelian, but also compact. In this case we have not only the fixed point algebra, but a family of spectral subspaces of A:

$$\mathcal{A}_x := \{a \in A : \alpha_t(a) = \overline{\langle x \,|\, t\rangle} \cdot a \text{ for all } t \in G\}, \tag{1.1}$$

for any x in the Pontrjagin dual $\widehat{G}$ of G, where $\langle x \,|\, t\rangle := x(t)$. Note that $\mathcal{A}_1$ is the fixed point algebra. Since α acts by $*$-automorphisms, we have

$$\mathcal{A}_x \cdot \mathcal{A}_y \subseteq \mathcal{A}_{xy} \quad \text{and} \quad \mathcal{A}_x^* = \mathcal{A}_{x^{-1}} \quad \text{for all } x, y \in \widehat{G}. \tag{1.2}$$

Thus the family $\mathcal{A} = \{\mathcal{A}_x\}_{x\in\widehat{G}}$ forms a Fell bundle over $\widehat{G}$. There is no continuity condition because $\widehat{G}$ is discrete.

The cross-sectional C^*-algebra $C^*(\mathcal{A})$ of a Fell bundle $\mathcal{A}$ over $\widehat{G}$ always comes with a canonical action $\widehat{\alpha}$ of G, the so-called *dual action* which is characterized by $\widehat{\alpha}_t(a_x) = \overline{\langle x \,|\, t\rangle} a_x$ for all $t \in G$ and $a_x \in \mathcal{A}_x$.

The subspaces (1.1) yield a dense embedding from the algebraic direct sum $\oplus_{x\in\widehat{G}} \mathcal{A}_x$ into A which extends to a natural G-equivariant $*$-isomorphism $C^*(\mathcal{A}) \cong A$, that is, a $*$-isomorphism compatible with the G-actions.

Conversely, if we start with a Fell bundle $\mathcal{A}$ over $\widehat{G}$ and equip $C^*(\mathcal{A})$ with the dual action of G, then the spectral subspaces (1.1) recover the original bundle $\mathcal{A}$ (up to natural isomorphism). As a result, we get an equivalence between the categories of G-C^*-algebras and of Fell bundles over $\widehat{G}$.

What happens if G is Abelian but not compact? We can still consider a Fell bundle $\mathcal{B}$ over $\widehat{G}$ and the associated cross-sectional C^*-algebra $C^*(\mathcal{B})$ with the dual action of G. However, not every G-C^*-algebra A has this form. For instance, the C^*-algebra $C^*(\mathcal{B})$ is never unital, unless $\widehat{G}$ is discrete, that is, G is compact. Moreover, the subspaces (1.1) do not help for non-compact G because they are $\{0\}$ if A is the cross-sectional C^*-algebra of a Fell bundle over $\widehat{G}$. In order to find the right spectral subspaces one has to consider larger subspaces in the multiplier algebra $\mathcal{M}(A)$ of A:

$$\mathcal{M}_x(A) := \{a \in \mathcal{M}(A) : \alpha_t(a) = \overline{\langle x \,|\, t\rangle} \cdot a \text{ for all } t \in G\}. \tag{1.3}$$

These spaces contain the fibers $\mathcal{B}_x$ if A is $C^*(\mathcal{B})$, but they are too big in general (even if G is compact, $\mathcal{M}_1(A)$ is only the multiplier algebra of A_1). This is strongly related to the problem previously discussed of finding a generalized fixed point algebra.

The solution to the problem of describing the class of G-C^*-algebras appearing as cross-sectional C^*-algebras of some Fell bundle over $\widehat{G}$ was given by Exel [18, 19]. First, in [18], Exel proved that the G-C^*-algebras of the form $C^*(\mathcal{B})$ are integrable. In fact, he defined a notion of integrability for functions with values in Banach spaces, called *unconditional integrability*, generalizing the notion of integrability in Bochner's sense. When applied in the correct way to the context of groups acting on C^*-algebras, this notion recovers the integrability of the underlying action as previously discussed: a positive element a of a G-C^*-algebra A is integrable if and only if the functions $t \mapsto \alpha_t(a)b$ and $t \mapsto b\alpha_t(a)$ are unconditionally integrable which means that the nets $(\int_K \alpha_t(a)b \,\mathrm{d}t)_{K\in\mathcal{C}}$ and $(\int_K b\alpha_t(a) \,\mathrm{d}t)_{K\in\mathcal{C}}$ converge in A, where $\mathcal{C}$ is the set of all measurable relatively compact subsets of G [66, Proposition 4.4]. In this case, we also say that $t \mapsto \alpha_t(a)$ is *strictly-unconditionally integrable* and denote its strict-unconditional integral by $\int_G^{\mathrm{su}} \alpha_t(a)\,\mathrm{d}t$ which is an element of $\mathcal{M}(A)$. Moreover, whenever a is an integrable element, one can define the *Fourier coefficients*

$$E_x(a) := \int_G^{\mathrm{su}} \langle x\,|\,t\rangle \alpha_t(a)\,\mathrm{d}t \quad \text{for all } x \in \widehat{G},$$

and a short computation shows that $E_x(a)$ belongs to the spectral subspace $\mathcal{M}_x(A)$.

The main result of [18] says that if A is the cross-sectional C^*-algebra $C^*(\mathcal{B})$ of a Fell bundle $\mathcal{B} = \{\mathcal{B}_x\}_{x\in\widehat{G}}$ over $\widehat{G}$, and if we equip it with the dual action of G, then any element a in the linear span $\mathcal{W}_{\mathcal{B}}$ of $\mathcal{C}_c(\mathcal{B}) * \mathcal{C}_c(\mathcal{B})$ (where $\mathcal{C}_c(\mathcal{B})$ is the space of compactly supported continuous sections of $\mathcal{B}$ and $*$ denotes the convolution product) is integrable and $E_x(a) = a(x)$ for all $x \in \widehat{G}$. This implies in particular that the fibers $\mathcal{B}_x$ of $\mathcal{B}$ can be recovered from the Fourier coefficients:

$$\mathcal{B}_x = \overline{\{E_x(a) : a \in \mathcal{W}_{\mathcal{B}}\}}, \tag{1.4}$$

where the overline above denotes the norm closure in $\mathcal{M}(A)$.

In the converse direction, Exel proved the following result [18]:

Theorem 1.1.1. *Let G be an Abelian locally compact group and let A be a G-C^*-algebra. Then A is isomorphic to the cross-sectional C^*-algebra of some Fell bundle $\mathcal{B}$ over $\widehat{G}$ if and only if there is a dense subspace $\mathcal{W} \subseteq A$ with $\mathcal{W}^* = \mathcal{W}$ consisting of integrable elements and such that the following property holds*:

Relative continuity: *for all $a, b \in \mathcal{W}$, we have*

$$\lim_{y\to 1} \|E_{xy}(a)E_z(b) - E_x(a)E_{yz}(b)\| = 0 \quad \textit{uniformly in } x, z \in \widehat{G}.$$

If A is of the form $C^*(\mathcal{B})$ for some Fell bundle $\mathcal{B}$ over $\widehat{G}$, then the subspace $\mathcal{W}_{\mathcal{B}}$ satisfies the hypothesis above, that is, it is relatively continuous. And in this case, the Fell bundle constructed by Exel recovers the original Fell bundle $\mathcal{B}$ which essentially follows from the equation $E_x(a) = a(x)$, $a \in \mathcal{W}_{\mathcal{B}}$ mentioned above.

At a first glance, it is not transparent what relative continuity really means. But, as proved by Exel, it is equivalent to the requirement that some natural operators belong to the crossed product algebra [19, Theorem 7.5]. Due to this fact, if relative continuity is present, then it is possible to construct a generalized fixed point algebra which is Morita equivalent to an ideal in the crossed product [19, Section 9]. For instance, if A is the G-C^*-algebra $C^*(\mathcal{B})$, and if we choose the relatively continuous subspace $\mathcal{W}_{\mathcal{B}}$, then the corresponding generalized fixed point algebra is the unit fiber $\mathcal{B}_1$. Thus relative continuity is closely related to the notion of proper action defined by Rieffel in [65] and, in particular, this is a sufficient condition to find the generalized fixed point algebra that Rieffel was looking for in [66].

However, some things were not clear in [19] (see Questions 9.4, 9.5 and 11.16) and essentially these were the same doubts that Rieffel had in [66]:

Question 1.1.2. **(1)** Suppose that A is an integrable G-C^*-algebra. Is there a dense, relatively continuous subspace of A?

(2) Are the generalized fixed point algebras associated to two different maximal relatively continuous subspaces always the same?

The answers to these questions were given by Ralf Meyer in [48] where he also generalized the notion of relative continuity to non-Abelian groups.

First, let us recall a previous work of Meyer [47] where he generalizes the notion of integrability to the setting of group actions on Hilbert modules. Let G be a (not necessarily Abelian) locally compact group, let B be a G-C^*-algebra and suppose that $\mathcal{E}$ is a Hilbert B,G-module, that is, a Hilbert B-modules with a continuous action γ of G compatible with the action β of G on B. Given an element $\xi \in \mathcal{E}$, we can define the following maps:

$$\begin{aligned} \langle\!\langle \xi | \colon \mathcal{E} \to \mathcal{C}_b(G,B), &\qquad (\langle\!\langle \xi | \eta)(t) := \langle \gamma_t(\xi) \,|\, \eta \rangle, \\ |\xi\rangle\!\rangle \colon \mathcal{C}_c(G,B) \to \mathcal{E}, &\qquad |\xi\rangle\!\rangle f := \int_G \gamma_t(\xi) \cdot f(t) \,\mathrm{d}t. \end{aligned}$$

We call $\xi \in \mathcal{E}$ *square-integrable* if $\langle\!\langle \xi | \eta \in L^2(G,B)$ for all $\eta \in \mathcal{E}$. In this case, $\langle\!\langle \xi |$ becomes an adjointable operator $\mathcal{E} \to L^2(G,B)$, whose adjoint extends $|\xi\rangle\!\rangle$ to an adjointable operator $L^2(G,B) \to \mathcal{E}$; we denote these extensions by $\langle\!\langle \xi |$ and $|\xi\rangle\!\rangle$ as well. Conversely, if $|\xi\rangle\!\rangle$ extends to an *adjointable* operator $L^2(G,B) \to \mathcal{E}$, then ξ is square-integrable. We say that $\mathcal{E}$ is square-integrable if the space $\mathcal{E}_{\mathrm{si}}$ of square-integrable elements is dense in $\mathcal{E}$.

The basic example of a square-integrable Hilbert B,G-module is $L^2(G,B)$ endowed with the diagonal action $\beta \otimes \lambda$, where we identify $L^2(G,B) \cong B \otimes L^2(G)$ and write λ for the left regular representation of G. Moreover, one can prove that direct sums or G-invariant direct summands of square-integrable Hilbert B,G-modules are again square-integrable. In particular, $\mathcal{H}_B := \bigoplus_{n \in \mathbb{N}} L^2(G,B)$ is square-integrable, and this turns out to be the universal example in the sense that it contains all the other countably generated square-integrable Hilbert B,G-modules. In fact, concerning square-integrability, the main result in [47] is the following G-equivariant version of the Kasparov Stabilization Theorem:

Theorem 1.1.3. *Let $\mathcal{E}$ be a countably generated Hilbert B,G-modules. Then the following assertions are equivalent:*

(i) *$\mathcal{E}$ is square-integrable,*

(ii) *$\mathcal{K}(\mathcal{E})$ is integrable (or proper in Rieffel's sense [66]),*

(iii) *$\mathcal{E} \oplus \mathcal{H}_B \cong \mathcal{H}_B$ as Hilbert B,G-modules,*

(iv) *$\mathcal{E}$ is a G-invariant direct summand of $\mathcal{H}_B$.*

Now we turn our attention to the second work of Meyer [48]. Given square-integrable elements $\xi, \eta \in \mathcal{E}$, we write $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle := \langle\!\langle \xi | \circ | \eta \rangle\!\rangle$ and $|\xi\rangle\!\rangle\langle\!\langle \eta| := |\xi\rangle\!\rangle \circ \langle\!\langle \eta |$. A short computation shows that the operators $\langle\!\langle \xi | : \mathcal{E} \to L^2(G,B)$ and $|\eta\rangle\!\rangle : L^2(G,B) \to \mathcal{E}$ are G-equivariant. In particular, so are the operators $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in \mathcal{L}\big(L^2(G,B)\big)$ and $|\xi\rangle\!\rangle\langle\!\langle \eta| \in \mathcal{L}(\mathcal{E})$, where for any two Hilbert B-modules $\mathcal{E}_1$ and $\mathcal{E}_2$, we denote by $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ the space of all adjointable operators $\mathcal{E}_1 \to \mathcal{E}_2$. We also write $\mathcal{L}^G(\mathcal{E}_1, \mathcal{E}_2)$ for the subspace of G-equivariant operators. Note that the space of G-equivariant operators $\mathcal{L}^G(\mathcal{E})$ is (canonically isomorphic to) the big fixed point algebra $\mathcal{M}_1\big(\mathcal{K}(\mathcal{E})\big)$ and should contain a generalized fixed point algebra. This indicates that the operators $|\xi\rangle\!\rangle\langle\!\langle\eta|$ may generate a candidate for the generalized fixed point algebra. On the other hand, the reduced crossed product algebra $C^*_{\mathrm{r}}(G, B)$ has a canonical realization as a C^*-subalgebra of $\mathcal{L}^G\big(L^2(G, B)\big)$. Our basic principle is that a generalized fixed point algebra should be Morita equivalent to some ideal in the reduced crossed product. This naturally leads us to the following definition ([48, Definition 6.1]):

Definition 1.1.4. A subset $\mathcal{R} \subseteq \mathcal{E}$ consisting of square-integrable elements is called *relatively continuous* if $\langle\!\langle\mathcal{R}\,|\,\mathcal{R}\rangle\!\rangle := \{\langle\!\langle\xi\,|\,\eta\rangle\!\rangle : \xi, \eta \in \mathcal{R}\} \subseteq C^*_{\mathrm{r}}(G, B)$.

Given a relatively continuous subset $\mathcal{R} \subseteq \mathcal{E}$, we define

$$\mathcal{F}(\mathcal{E}, \mathcal{R}) := \overline{\operatorname{span}}\big(|\mathcal{R}\rangle\!\rangle \circ C^*_{\mathrm{r}}(G, B)\big) \subseteq \mathcal{L}^G\big(L^2(G, B), \mathcal{E}\big).$$

By definition of relative continuity, $\mathcal{F}(\mathcal{E}, \mathcal{R})$ is a *concrete* Hilbert $C^*_{\mathrm{r}}(G, B)$-module in the sense that it is a closed subspace of $\mathcal{L}^G\big(L^2(G, B), \mathcal{E}\big)$ satisfying

$$\mathcal{F}(\mathcal{E}, \mathcal{R}) \circ C^*_{\mathrm{r}}(G, B) \subseteq \mathcal{F}(\mathcal{E}, \mathcal{R}) \quad \text{and} \quad \mathcal{F}(\mathcal{E}, \mathcal{R})^* \circ \mathcal{F}(\mathcal{E}, \mathcal{R}) \subseteq C^*_{\mathrm{r}}(G, B).$$

A concrete Hilbert $C^*_{\mathrm{r}}(G, B)$-module can be regarded as an abstract Hilbert $C^*_{\mathrm{r}}(G, B)$-module in the obvious way. Conversely, any abstract Hilbert $C^*_{\mathrm{r}}(G, B)$-module $\mathcal{F}$ can be represented in an essentially unique way in $\mathcal{L}^G(L^2(G, B), \mathcal{E}_{\mathcal{F}})$, where $\mathcal{E}_{\mathcal{F}}$ is the balanced tensor product $\mathcal{F} \otimes_{C^*_{\mathrm{r}}(G,B)} L^2(G, B)$ ([48, Theorem 5.3]).

The algebra of compact operators on $\mathcal{F}(\mathcal{E}, \mathcal{R})$ is canonically isomorphic to the closed linear span of $\mathcal{F}(\mathcal{E}, \mathcal{R}) \circ \mathcal{F}(\mathcal{E}, \mathcal{R})^* \subseteq \mathcal{L}^G(\mathcal{E})$ which we denote by $\mathrm{Fix}(\mathcal{E}, \mathcal{R})$ and call the generalized fixed point algebra associated to the pair $(\mathcal{E}, \mathcal{R})$. It is therefore Morita equivalent to the ideal $\mathcal{I}(\mathcal{E}, \mathcal{R}) := \overline{\operatorname{span}}\big(\mathcal{F}(\mathcal{E}, \mathcal{R})^* \circ \mathcal{F}(\mathcal{E}, \mathcal{R})\big) \subseteq C^*_{\mathrm{r}}(G, B)$ (and $\mathcal{F}(\mathcal{E}, \mathcal{R})$ can be viewed as an imprimitivity Hilbert bimodule implementing this Morita equivalence).

In general, there are many relatively continuous subspaces $\mathcal{R} \subseteq \mathcal{E}$ yielding the same Hilbert $C^*_{\mathrm{r}}(G, B)$-module $\mathcal{F} = \mathcal{F}(\mathcal{E}, \mathcal{R})$. However, we can control this by imposing some more natural conditions on $\mathcal{R}$. We say that $\mathcal{R}$ is *complete* if it is a G-invariant B-submodule of $\mathcal{E}$ (that is, $\gamma_t(\mathcal{R}) \subseteq \mathcal{R}$ and $\mathcal{R} \cdot B \subseteq \mathcal{R}$) which is closed with respect the si-norm: $\|\xi\|_{\mathrm{si}} := \|\xi\| + \||\xi\rangle\!\rangle\|$. The *completion* of $\mathcal{R}$ is the smallest complete subspace $\mathcal{R}_{\mathrm{c}}$ containing $\mathcal{R}$. If $\mathcal{R}$ is complete, then the Hilbert module $\mathcal{F}(\mathcal{E}, \mathcal{R})$ is just the closure of $|\mathcal{R}\rangle\!\rangle$ and, as a consequence, the generalized fixed point algebra $\mathrm{Fix}(\mathcal{E}, \mathcal{R})$ and the ideal $\mathcal{I}(\mathcal{E}, \mathcal{R})$ are just the closed linear spans of $|\mathcal{R}\rangle\!\rangle\langle\!\langle\mathcal{R}|$ and $\langle\!\langle\mathcal{R}\,|\,\mathcal{R}\rangle\!\rangle$, respectively. Moreover, we always have $\mathcal{F}(\mathcal{E}, \mathcal{R}) = \mathcal{F}(\mathcal{E}, \mathcal{R}_{\mathrm{c}})$ for any relatively continuous subset $\mathcal{R}$ and hence we can replace $\mathcal{R}$ by its completion to get the same results.

If we restrict to complete subspaces, then $\mathcal{R}$ is uniquely determined by the Hilbert module $\mathcal{F}(\mathcal{E}, \mathcal{R})$. In fact, Theorem 6.1 in [48] says the following:

Theorem 1.1.5. *Let $\mathcal{E}$ be a Hilbert B, G-module. Then the map $\mathcal{R} \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R})$ is a bijection between complete, relatively continuous subspaces $\mathcal{R} \subseteq \mathcal{E}$ and concrete Hilbert $C^*_{\mathrm{r}}(G, B)$-modules $\mathcal{F} \subseteq \mathcal{L}^G\big(L^2(G, B), \mathcal{E}\big)$. The inverse map is given by the assignment $\mathcal{F} \mapsto \mathcal{R}_{\mathcal{F}} := \{\xi \in \mathcal{E}_{\mathrm{si}} : |\xi\rangle\!\rangle \in \mathcal{F}\}$. Moreover, $\mathcal{R}$ is dense in $\mathcal{E}$ if and only if $\mathcal{F}(\mathcal{E}, \mathcal{R})$ is* essential*, that is, the linear span of $\mathcal{F}(\mathcal{E}, \mathcal{R})(L^2(G, B))$ is dense in $\mathcal{E}$.*

A *continuously square-integrable Hilbert B, G-module* is a pair $(\mathcal{E}, \mathcal{R})$ consisting of a Hilbert B, G-module $\mathcal{E}$ and a dense, complete, relatively continuous subspace $\mathcal{R} \subseteq \mathcal{E}$. This class forms a category if we take *$\mathcal{R}$-continuous* G-equivariant operators as morphisms, that is, G-equivariant operators that are compatible with the relatively continuous subspaces in the obvious way ([48]).

The construction $(\mathcal{E}, \mathcal{R}) \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R})$ is a functor from the category of continuously square-integrable Hilbert B, G-modules to the category of Hilbert $C^*_{\mathrm{r}}(G, B)$-modules with morphisms as usual. Theorem 1.1.5 and the fact that any abstract Hilbert module can be realized as a concrete one imply that $(\mathcal{E}, \mathcal{R}) \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R})$ induces a bijection between the isomorphism classes. Moreover, this construction is natural and yields an equivalence between the respective categories ([48, Theorem 6.2]).

Using this correspondence, Meyer could give a negative answer to the above questions. In fact, considering the case where G is an Abelian second countable locally compact group and $B = \mathbb{C}$, we get that separable continuously square-integrable G-Hilbert spaces correspond to Hilbert $C^*_{\mathrm{r}}(G) \cong \mathcal{C}_0(\widehat{G})$-modules, that is, continuous fields of separable Hilbert spaces over $\widehat{G}$. On the other hand, the G-equivariant version of Kasparov's Stabilization Theorem implies that separable square-integrable G-Hilbert spaces correspond to measurable fields of separable Hilbert spaces over $\widehat{G}$ ([48, Section 8]). Analyzing the subtle difference between these two classes one arrives at the conclusion that not every square-integrable Hilbert space has a dense relatively continuous subspace. In fact, the examples considered in [48] show that there are square-integrable Hilbert spaces where $\{0\}$ is the unique relatively continuous subspace. And it is also shown in [48] that maximal relatively continuous subspaces do not yield isomorphic generalized fixed point algebras in general. Moreover, there is a canonical correspondence between relatively continuous subspaces of a Hilbert module and of its algebra of compact operators ([48, Theorem 7.2]).[1] Thus these counterexamples also yield counterexamples in the realm of G-C^*-algebras.

Rieffel introduced integrability as a non-commutative generalization of proper actions on spaces but the results above indicate that integrability is closer to *stability* than to properness and that some special properties of proper actions are not captured by integrability. There are, however, some special situations where this in fact happens. In [48], a G-C^*-algebra B is called *spectrally proper* if the induced action of G on the primitive ideal space is proper ([48, Definition 9.2]). This generalizes the notion of proper actions in the sense of Kasparov [35]. If B is spectrally proper, then every Hilbert B, G-module $\mathcal{E}$ is square-integrable and there is a unique dense, complete, relatively continuous subspace of $\mathcal{E}$ ([48, Theorem 9.1]). As a consequence, the functor

$$\mathcal{F} \mapsto \mathcal{F} \otimes_{C^*_{\mathrm{r}}(G,B)} L^2(G, B)$$

is an equivalence between the categories of Hilbert $C^*_{\mathrm{r}}(G, B)$-modules and Hilbert B, G-modules ([48, Corollary 9.1]).

[1] This correspondence is bijective if G is exact. In particular, it is bijective if G is Abelian which is the case here.

1.2 The quantum case: the main results of this thesis

The main goal of this thesis is to generalize the concepts and results above to the setting of locally compact quantum groups in the sense of Kustermans and Vaes [41].

This work is divided into five parts. The first part (Chapter 2) is a preliminary background containing notions and results necessary throughout the rest of the work.

In the second part (Chapter 3) we define the notion of integrable coactions of a locally compact quantum group $\mathcal{G}$ on C^*-algebras generalizing the notion of integrable (or proper) actions of groups mentioned above. The basic ingredient here is the existence of a Haar weight on $\mathcal{G}$ which naturally leads us to the setting of locally compact quantum groups.

As an immediate consequence of the definition, we get that every coaction of a compact quantum group is integrable. On the other hand, if the locally compact quantum group $\mathcal{G}$ is not compact, then we always have non-integrable coactions. For instance, trivial coactions or coactions on unital C^*-algebras are not integrable, unless $\mathcal{G}$ is compact. However, the class of integrable coactions is always very huge. A natural coaction to consider is the comultiplication of $\mathcal{G}$ itself. We prove that it is always integrable, for any locally compact quantum group. Moreover, given coactions γ_A and γ_B of $\mathcal{G}$ on C^*-algebras A and B, respectively, and given a nondegenerate $\mathcal{G}$-equivariant $*$-homomorphism $\pi : A \to \mathcal{M}(B)$, if γ_A is integrable, then so is γ_B. As a consequence, we get that any dual coaction is integrable. In particular, if $\mathcal{G}$ is regular, the dual coaction of $\mathcal{G}$ on the algebra of compact operators $\mathcal{K} := \mathcal{K}\big(L^2(\mathcal{G})\big)$ is integrable, where $L^2(\mathcal{G})$ denotes the L^2-Hilbert space associated to the Haar weight of $\mathcal{G}$. Moreover, even if $\mathcal{G}$ is not regular, $\mathcal{K}$ always has a canonical coaction of $\mathcal{G}$, and it is always integrable. More generally, we can always furnish the tensor product $A \otimes \mathcal{K}$ with a coaction of $\mathcal{G}$ (whenever A has a coaction of $\mathcal{G}$) and this coaction is also always integrable. In particular, any coaction is Morita equivalent to an integrable coaction.

In the third part of this work (Chapter 4) we generalize the notion of square-integrable actions of groups to the setting of coactions of locally compact quantum group on Hilbert modules. After defining the notion of a square-integrable element ξ in a $\mathcal{G}$-equivariant Hilbert module, the main point is to define the bra-ket operators $\langle\!\langle\xi|$ and $|\xi\rangle\!\rangle$. This uses the notion of KSGNS-constructions for the Haar weight of $\mathcal{G}$. Once we have the bra-ket operators, an important step is to establish their equivariance. This is straightforward in the group case, but it requires some work in quantum setting.

Let B be a $\mathcal{G}$-C^*-algebra, that is, a C^*-algebra with a continuous coaction of $\mathcal{G}$. As in the group case, the basic example of a square-integrable Hilbert $B,\mathcal{G}$-module is $B \otimes L^2(\mathcal{G})$ endowed with a canonical coaction of $\mathcal{G}$. In fact, concerning square-integrability, our main result is the quantum version of the equivariant Kasparov Stabilization Theorem.[2] After establishing some basic properties of the bra-ket operators, the proof of this theorem is almost the same as in the group case. The basic difference comes from the fact that the L^1-algebra of a locally compact quantum group $\mathcal{G}$ does not have a bounded approximate unit in general. This happens if and only if $\mathcal{G}$ is co-amenable. This brings about some technical problems because we need to use the Banach L^1-action induced by the underlying

[2] See Theorem 4.5.6

coaction.

The main part of this work (Chapter 5) contains the definition of relative continuity and generalized fixed point algebras in the setting of coactions of locally compact quantum groups on Hilbert modules. Once we have the bra-ket operators, the definitions are exactly the same as in the group case. Given a relatively continuous subset $\mathcal{R}$ in a Hilbert $B,\mathcal{G}$-module $\mathcal{E}$, we define, as in the group case, a concrete Hilbert module $\mathcal{F}(\mathcal{E},\mathcal{R})$ over the reduced crossed product $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ (the reason for this notation will be clear later). Again, the algebra of compact operators on $\mathcal{F}(\mathcal{E},\mathcal{R})$ is (canonically isomorphic to) the generalized fixed point algebra $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ and therefore, it is Morita equivalent to the ideal $\mathcal{I}(\mathcal{E},\mathcal{R}) := \overline{\mathrm{span}}\big(\mathcal{F}(\mathcal{E},\mathcal{R})^* \circ \mathcal{F}(\mathcal{E},\mathcal{R})\big)$ in $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. If $\mathcal{I}(\mathcal{E},\mathcal{R})$ is equal to $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, then we say that $\mathcal{R}$ is *saturated*.

One of the first examples that we analyze is the coaction of $\mathcal{G}$ on itself via the comultiplication. We already mentioned that this coaction is always integrable, but here is where the first difference appears: there is a non-zero relatively continuous subset of $\mathcal{G}$ if and only if $\mathcal{G}$ is semi-regular. Moreover, there is a saturated relatively continuous subset of $\mathcal{G}$ if and only if $\mathcal{G}$ is regular.

If $\mathcal{G}$ is compact, then any subset $\mathcal{R} \subseteq \mathcal{E}$ is relatively continuous and the generalized fixed point algebra $\mathrm{Fix}(\mathcal{E}) = \mathrm{Fix}(\mathcal{E},\mathcal{E})$ is the usual fixed point algebra which is therefore Morita equivalent to an ideal in the reduced crossed product.

The most important example is the Hilbert $B,\mathcal{G}$-module $B \otimes L^2(\mathcal{G})$. We prove that we always can find a dense, relatively continuous subspace $\mathcal{R}_0 \subseteq B \otimes L^2(\mathcal{G})$ such that $\mathcal{F}\big(B \otimes L^2(\mathcal{G}),\mathcal{R}_0\big) = B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. In particular, this shows that reduced crossed products appear as generalized fixed point algebras.

Next, we analyze some completeness conditions of relatively continuous subsets. Again, the possible non-co-amenability of $\mathcal{G}$ brings about some technical problems. As in the group case, we can define complete subspaces, but it turns out that completeness alone is not enough in general and we need an extra condition that we call *s-completeness*. This is a sort of "slice map property" and this is where the script "s" comes from. If $\mathcal{G}$ is co-amenable, then this condition reduces to completeness. Another natural condition on a complete subspace $\mathcal{R}$ is *essentialness* which, roughly speaking, means that the L^1-action on $\mathcal{R}$ is nondegenerate. In this case, we also say that $\mathcal{R}$ is e-complete. Again, if $\mathcal{G}$ is co-amenable, then essentialness is automatic. Having these completeness conditions we can then define a *continuously square-integrable Hilbert $B,\mathcal{G}$-module* to be a pair $(\mathcal{E},\mathcal{R})$, where $\mathcal{E}$ is Hilbert $B,\mathcal{G}$-modules, and $\mathcal{R}$ is a dense, complete, relatively continuous subspace. If, in addition, $\mathcal{R}$ is s-complete (resp. e-complete) then we say that $(\mathcal{E},\mathcal{R})$ is an s-continuously (resp. e-continuously) square-integrable Hilbert $B,\mathcal{G}$-module.

One of our main results is a quantum version of Meyer's Theorem 1.1.5 above. If we replace completeness by s-completeness, then the result remains almost unchanged in the quantum setting.[3] As in the group case, this implies that the construction $(\mathcal{E},\mathcal{R}) \mapsto \mathcal{F}(\mathcal{E},\mathcal{R})$ is an equivalence between the categories of s-continuously square-integrable Hilbert $B,\mathcal{G}$-modules and Hilbert modules over the reduced crossed product $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. The inverse construction is given by the assignment $\mathcal{F} \mapsto (\mathcal{E}_{\mathcal{F}},\mathcal{R}_{\mathcal{F}})$, where $\mathcal{E}_{\mathcal{F}} := \mathcal{F} \otimes_{B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}} \big(B \otimes L^2(\mathcal{G})\big)$

[3]See Theorem 5.4.4

and $\mathcal{R}_{\mathcal{F}}$ is the s-completion of the algebraic tensor product $\mathcal{F} \odot_{B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}} \mathcal{R}_0$.

It is an important question whether there is a canonical choice for a dense, s-complete, relatively continuous subspace in a given Hilbert $B,\mathcal{G}$-module $\mathcal{E}$. In particular, it is also important to know when such a choice is unique. We say that $\mathcal{E}$ is *$\mathcal{R}$-proper*, if there is a unique dense, s-complete, relatively continuous subspace of $\mathcal{E}$. If $\mathcal{G}$ is compact, then any Hilbert $B,\mathcal{G}$-module $\mathcal{E}$ is $\mathcal{R}$-proper and $\mathcal{R} = \mathcal{E}$ is the unique dense, s-complete, relatively continuous subspace of $\mathcal{E}$. This implies that the functor $\mathcal{F} \mapsto \mathcal{F} \otimes_{B \rtimes \widehat{\mathcal{G}}^{\mathrm{c}}} \big(B \otimes L^2(\mathcal{G})\big)$ is an equivalence between the categories of Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-modules and Hilbert $B,\mathcal{G}$-modules. As already mentioned above, in the case of groups, if B is a spectrally proper G-C^*-algebra, then every Hilbert B,G-module is $\mathcal{R}$-proper. For non-compact quantum groups, it is not clear how to find non-trivial examples satisfying this strong form of properness. At least, we prove that $\mathcal{G}$ itself is an $\mathcal{R}$-proper $\mathcal{G}$-C^*-algebra if (and only if) $\mathcal{G}$ is semi-regular.

In the final part of this work we analyze group coactions, that is, coactions of the locally compact quantum group $C^*_{\mathrm{r}}(G)$, where G is some locally compact group. It turns out that there is a strong connection between continuously square-integrable $\widehat{G}$-C^*-algebras (that is, $C^*_{\mathrm{r}}(G)$-C^*-algebras) and Fell bundles over G.

Let $\mathcal{B}$ be a Fell bundle over G. Then we can still consider its cross-sectional C^*-algebra $C^*(\mathcal{B})$, and it comes with a *dual coaction* of G (which corresponds to the dual action of $\widehat{G}$ if G is Abelian). Moreover, we also have a reduced cross-sectional C^*-algebra $C^*_{\mathrm{r}}(\mathcal{B})$, which also carries a dual coaction of G. Thus, given a Fell bundle over G, we have two $\widehat{G}$-C^*-algebras $C^*(\mathcal{B})$ and $C^*_{\mathrm{r}}(\mathcal{B})$. If G is amenable, these two $\widehat{G}$-C^*-algebras are isomorphic.

We can characterize the $\widehat{G}$-C^*-algebras $C^*(\mathcal{B})$ and $C^*_{\mathrm{r}}(\mathcal{B})$ by means of continuous square-integrability. In fact, first we prove that $\mathcal{C}_c(\mathcal{B})$ is a relatively continuous subspace of $C^*(\mathcal{B})$, and the same is true for the copy of $\mathcal{C}_c(\mathcal{B})$ in $C^*_{\mathrm{r}}(\mathcal{B})$. This gives rise to two continuously square-integrable $\widehat{G}$-C^*-algebras $\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ and $\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$, where $A_{(\mathrm{r})}(\mathcal{B}) := C^*_{(\mathrm{r})}(\mathcal{B})$ and $\mathcal{R}_{(\mathrm{r})}(\mathcal{B})$ is the completion of $\mathcal{C}_c(\mathcal{B})$ in $C^*_{(\mathrm{r})}(\mathcal{B})$.

Conversely, given any continuously square-integrable $\widehat{G}$-C^*-algebra $(A,\mathcal{R})$, we construct a Fell bundle $\mathcal{B} = \mathcal{B}(A,\mathcal{R})$ over G together with two canonical equivariant surjections $\kappa : C^*(\mathcal{B}) \to A$ and $\nu : A \to C^*_{\mathrm{r}}(\mathcal{B})$. Moreover, if A is a maximal $\widehat{G}$-C^*-algebra, then κ is an isomorphism, and if A is a reduced $\widehat{G}$-C^*-algebra, then ν is an isomorphism. In general, $C^*_{\mathrm{r}}(\mathcal{B})$ is a reduction of A, and if $C^*(\mathcal{B})$ is maximal, then it is a maximalization of A. It is an open problem whether $C^*(\mathcal{B})$ is a maximal $\widehat{G}$-C^*-algebra for every Fell bundle $\mathcal{B}$ over G. This is the case if G is discrete or amenable. In general, we say that G has the *maximality property* if this happens.

The relatively continuous subspaces $\mathcal{R}(\mathcal{B})$ and $\mathcal{R}_{\mathrm{r}}(\mathcal{B})$ are essential. Thus the pairs $\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ and $\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$ are, in fact, e-continuously square-integrable $\widehat{G}$-C^*-algebras. Conversely, if we start with an e-continuously square-integrable $\widehat{G}$-C^*-algebra $(A,\mathcal{R})$ and define $\mathcal{B} := \mathcal{B}(A,\mathcal{R})$, then the canonical surjections κ and ν yield natural isomorphisms $(A,\mathcal{R}) \cong \big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ if A is maximal and $(A,\mathcal{R}) \cong \big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$ if A is reduced. Moreover, if $(A,\mathcal{R})$ is of the form $\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ or $\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$ for some Fell bundle $\mathcal{B}$, then the Fell bundle $\mathcal{B}(A,\mathcal{R})$ is naturally isomorphic to $\mathcal{B}$. As a result, the constructions

$$\mathcal{B} \mapsto \big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big) \quad \text{and} \quad (A, \mathcal{R}) \mapsto \mathcal{B}(A, \mathcal{R})$$

provide an equivalence between the categories of Fell bundles over G and e-continuously square-integrable reduced $\widehat{G}$-C^*-algebras. If G has the maximality property, then the constructions

$$\mathcal{B} \mapsto \big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big) \quad \text{and} \quad (A, \mathcal{R}) \mapsto \mathcal{B}(A, \mathcal{R})$$

provide an equivalence between the categories of Fell bundles over G and e-continuously square-integrable maximal $\widehat{G}$-C^*-algebras. In this picture, the generalized fixed point algebra fixed point algebra $\mathrm{Fix}(A, \mathcal{R})$ coincides with the unit fiber of $\mathcal{B}(A, \mathcal{R})$.

These last two results were the initial motivation for this work. We started trying to generalize the theory of continuous square-integrability to coactions of groups. But it turned out that the difficulties and techniques used in this case (weight theory) are basically the same as for general locally compact quantum groups.

If G is discrete, our results specialize to the well-known fact that the categories of Fell bundles over G and maximal (or reduced) $\widehat{G}$-C^*-algebras are equivalent. Given a $\widehat{G}$-C^*-algebra A, the corresponding Fell bundle over G is canonical one:

$$\mathcal{B}_t = \{a \in A : \gamma_A(a) = a \otimes \lambda_t\} \quad \text{for all } t \in G.$$

Since G is discrete, there is no continuity condition.

If G is amenable, $\big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big) \cong \big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$, and any $\widehat{G}$-C^*-algebra is both reduced and maximal. Thus, if G is amenable, our results specialize to the fact that the categories of Fell bundles over G and continuously square-integrable $\widehat{G}$-C^*-algebras are equivalent.

Our results not only generalize Exel's Theorem 1.1.1 above, but they also allow us to say exactly how many Fell bundle structures a given $\widehat{G}$-C^*-algebra A has. In fact, if the coaction on A is maximal (resp. reduced), then isomorphism classes of *full* (*resp. reduced*) *Fell bundle structures* for A, that is, isomorphism classes of pairs $(\mathcal{B}, \pi)$, where $\mathcal{B}$ is a Fell bundle over G and π is an equivariant isomorphism $\pi : C^*(\mathcal{B}) \to A$ (resp. $\pi : C^*_{\mathrm{r}}(\mathcal{B}) \to A$), correspond bijectively to dense, e-complete, relatively continuous subspaces $\mathcal{R} \subseteq A$.

This implies, in particular, that A is a maximal (resp. reduced) $\mathcal{R}$-proper $\widehat{G}$-C^*-algebra if and only if there is, up to isomorphism, a unique full (resp. reduced) Fell bundle structure for A. In general, there are integrable $\widehat{G}$-C^*-algebras without or with several non-isomorphic Fell bundle structures.

Chapter 2

Preliminary background

This preparatory chapter contains a review of some important concepts that are necessary in the subsequent chapters of this work. We start by recalling the notion of Hilbert (bi)modules and some related concepts like (bi)module homomorphisms, internal and external tensor products and linking algebras. Next, we review some basic notions in weight theory, which is one of the most important technical tools of this work. Of essential importance are the concepts of slice maps with weights and their KSGNS-constructions. For C^*-algebras, these concepts have been developed by Kustermans and Vaes in connection with their work on locally compact quantum groups. However, we also need an analogous construction for Hilbert modules and we use linking algebra techniques in order to generalize Kustermans and Vaes constructions to this setting. In the three final sections, we review the notions of locally compact quantum groups and their coactions and crossed products.

We will assume familiarity with the rudiments of the theory of Banach algebras and C^*-algebras, such as can be found in [10, 11, 50, 58, 68, 69].

2.1 Hilbert modules and their morphisms

This section contains some basic facts on Hilbert modules. We mainly follow [15].

Definition 2.1.1. Let B be a C^*-algebra. A (*right*) *Hilbert B-module* is a (complex) vector space $\mathcal{E}$ which is a right B-module equipped with a *B-inner product*, that is, a sesquilinear map

$$\mathcal{E} \times \mathcal{E} \to B, \quad (\xi, \eta) \mapsto \langle \xi | \eta \rangle_B$$

satisfying

$$\langle \xi | \eta \cdot b \rangle_B = \langle \xi | \eta \rangle_B b, \quad \langle \xi | \eta \rangle_B^* = \langle \eta | \xi \rangle_B, \quad \langle \xi | \xi \rangle_B \geq 0, \quad \text{and} \quad \langle \xi | \xi \rangle_B = 0 \Rightarrow \xi = 0$$

for all $\xi, \eta \in \mathcal{E}$ and $b \in B$, and which is complete with respect to the induced norm $\|\xi\| := \|\langle \xi | \xi \rangle_B\|^{\frac{1}{2}}$. We say that $\mathcal{E}$ is *full* if $\overline{\operatorname{span}} \langle \mathcal{E} | \mathcal{E} \rangle_B = B$.

Analogously, one defines *left* Hilbert B-modules.

Definition 2.1.2. Let A and B be C^*-algebras. A *right Hilbert A,B-bimodule* is a right Hilbert B-module which is also a nondegenerate left A-module (that is, $\overline{\operatorname{span}}\, A\cdot\mathcal{E}=\mathcal{E}$), and satisfies

(i) $a\cdot(\xi\cdot b)=(a\cdot\xi)\cdot b$ and

(ii) $\langle a\cdot\xi|\eta\rangle_B=\langle\xi|a^*\cdot\eta\rangle_B$,

for all $\xi,\eta\in\mathcal{E}$, $a\in A$ and $b\in B$.

We write ${}_A\mathcal{E}_B$ to indicate all the data. If, in addition, $\mathcal{E}$ is also a left Hilbert A-module such that

$$ {}_A\langle\xi|\eta\rangle\cdot\zeta=\xi\langle\eta|\zeta\rangle_B $$

for all $\xi,\eta,\zeta\in\mathcal{E}$, then we say that $\mathcal{E}$ is a *Hilbert A,B-bimodule* or also a *partial imprimitivity A,B-bimodule* (see [15, Definition 1.5]). If both inner products ${}_A\langle\cdot|\cdot\rangle$ and $\langle\cdot|\cdot\rangle_B$ are full, then $\mathcal{E}$ is called an *imprimitivity Hilbert A,B-module* or also an *A,B-Morita equivalence.* In this case, A and B are called *Morita equivalent.*

Given Hilbert B-modules $\mathcal{E},\mathcal{F}$, we denote by $\mathcal{L}(\mathcal{E},\mathcal{F})$ the set of *adjointable* maps, that is, maps $T:\mathcal{E}\to\mathcal{F}$ for which there is a (necessarily unique) *adjoint* map $T^*:\mathcal{F}\to\mathcal{E}$ satisfying

$$ \langle T\xi|\eta\rangle_B=\langle\xi|T^*\eta\rangle_B,\qquad\text{for all }\xi\in\mathcal{E},\ \eta\in\mathcal{F}. $$

Adjointable maps are automatically B-linear and bounded, and $\mathcal{L}(\mathcal{E},\mathcal{F})$ is a Banach space with the operator norm $\|T\|:=\sup\{\|T\xi\|:\|\xi\|\le 1\}$. Moreover, $\mathcal{L}(\mathcal{E}):=\mathcal{L}(\mathcal{E},\mathcal{E})$ is a C^*-algebra. We denote by $\mathcal{K}(\mathcal{E},\mathcal{F})$ the set compact operators $\mathcal{E}\to\mathcal{F}$ which is, by definition, the closed linear span in $\mathcal{L}(\mathcal{E},\mathcal{F})$ of the rank-one operators $|\xi\rangle\langle\eta|$ with $\xi,\eta\in\mathcal{E}$, which are defined by:

$$ |\xi\rangle\langle\eta|\zeta:=\xi\langle\eta|\zeta\rangle_B\qquad\text{for all }\zeta\in\mathcal{E}. $$

Then $\mathcal{K}(\mathcal{E},\mathcal{F})$ is also a Banach space with the operator norm, and $\mathcal{K}(\mathcal{E})$ is a closed $*$-ideal of $\mathcal{L}(\mathcal{E})$. Moreover, we have $\mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)\cong\mathcal{L}(\mathcal{E})$.

We recall that $\mathcal{L}(\mathcal{E},\mathcal{F})$ has a natural structure of Hilbert $\mathcal{L}(\mathcal{F}),\mathcal{L}(\mathcal{E})$-bimodule given by (see [15, Proposition 1.10]):

$$ P\cdot T:=P\circ T,\quad T\cdot Q:=T\circ Q,\quad {}_{\mathcal{L}(\mathcal{F})}\langle T|S\rangle:=T\circ S^*,\quad \langle T|S\rangle_{\mathcal{L}(\mathcal{E})}:=T^*\circ S $$

for all $P\in\mathcal{L}(\mathcal{F})$, $T,S\in\mathcal{L}(\mathcal{E},\mathcal{F})$ and $Q\in\mathcal{L}(\mathcal{E})$. Analogously, $\mathcal{K}(\mathcal{E},\mathcal{F})$ has a natural structure of Hilbert $\mathcal{K}(\mathcal{F}),\mathcal{K}(\mathcal{E})$-bimodule. Moreover, $\mathcal{K}(\mathcal{E},\mathcal{F})$ can also be considered as a Hilbert $\mathcal{L}(\mathcal{F}),\mathcal{L}(\mathcal{E})$-bimodule, and in this way, it is a Hilbert $\mathcal{L}(\mathcal{F}),\mathcal{L}(\mathcal{E})$-submodule of $\mathcal{L}(\mathcal{E},\mathcal{F})$ (that is, a closed subspace which is invariant under the left and right actions).

Definition 2.1.3. Let $\mathcal{E}$ be a Hilbert A,B-bimodule and define $\tilde{\mathcal{E}}=\{\tilde{\xi}:\xi\in\mathcal{E}\}$, where $\mathcal{E}\ni\xi\mapsto\tilde{\xi}\in\tilde{\mathcal{E}}$ is, by definition, an anti-linear map and satisfies:

$$ b\cdot\tilde{\xi}:=\widetilde{\xi\cdot b^*}\quad \tilde{\xi}\cdot a:=\widetilde{a^*\cdot\xi},\quad {}_B\langle\tilde{\xi}|\tilde{\eta}\rangle:=\langle\xi|\eta\rangle_B\quad\text{and}\quad\langle\tilde{\xi}|\tilde{\eta}\rangle_A:={}_A\langle\xi|\eta\rangle. $$

Then $\tilde{\mathcal{E}}$ is a Hilbert B,A-bimodule, called the *dual* of $\mathcal{E}$. If no confusion appears, we shall also denote the dual of $\mathcal{E}$ by $\mathcal{E}^*$.

If $\mathcal{E}$ is just a Hilbert B-module, and we consider it as a Hilbert $\mathcal{K}(\mathcal{E}), B$-bimodule, then the map $\mathcal{E} \ni \xi \mapsto |\xi\rangle \in \mathcal{K}(B, \mathcal{E})$, where $|\xi\rangle b := \xi \cdot b$ for all $b \in B$, is an isomorphism of Hilbert $\mathcal{K}(\mathcal{E}), B$-bimodules. The adjoint of $|\xi\rangle$, denoted by $\langle\xi|$, is given by $\langle\xi|\eta = \langle\xi|\eta\rangle_B$. Moreover, the map $\tilde{\mathcal{E}} \ni \tilde{\xi} \mapsto \langle\xi| \in \mathcal{K}(\mathcal{E}, B)$ is an isomorphism of Hilbert $B, \mathcal{K}(\mathcal{E})$-bimodules. Thus $\mathcal{K}(\mathcal{E}, B)$ is, up to isomorphism, the dual of $\mathcal{E}$. Sometimes, we shall also identify $\mathcal{E} \cong \mathcal{K}(B, \mathcal{E})$, and in this way identify each $\xi \in \mathcal{E}$ with the operator $|\xi\rangle \in \mathcal{K}(B, \mathcal{E})$. In this case, we shall also use the notation $\xi^* = \langle\xi| \in \mathcal{K}(\mathcal{E}, B) \cong \mathcal{E}^*$.

More generally, assume that $\mathcal{E}, \mathcal{F}$ are Hilbert B-modules and consider the Hilbert $\mathcal{K}(\mathcal{F}), \mathcal{K}(\mathcal{E})$-bimodule $\mathcal{K}(\mathcal{E}, \mathcal{F})$. Then it is easy to see that the map $\widetilde{\mathcal{K}(\mathcal{E}, \mathcal{F})} \ni \tilde{x} \mapsto x^* \in \mathcal{K}(\mathcal{F}, \mathcal{E})$ is an isomorphism $\widetilde{\mathcal{K}(\mathcal{E}, \mathcal{F})} \cong \mathcal{K}(\mathcal{F}, \mathcal{E})$ of Hilbert $\mathcal{K}(\mathcal{E}), \mathcal{K}(\mathcal{F})$-bimodules. Analogously, considering the Hilbert $\mathcal{L}(\mathcal{F}), \mathcal{L}(\mathcal{E})$-bimodule $\mathcal{L}(\mathcal{E}, \mathcal{F})$ the map $\widetilde{\mathcal{L}(\mathcal{E}, \mathcal{F})} \ni \tilde{x} \mapsto x^* \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ is an isomorphism $\widetilde{\mathcal{L}(\mathcal{E}, \mathcal{F})} \cong \mathcal{L}(\mathcal{F}, \mathcal{E})$ of Hilbert $\mathcal{L}(\mathcal{E}), \mathcal{L}(\mathcal{F})$-bimodules.

Definition 2.1.4. Let $\mathcal{E}$ be a right Hilbert A, B-bimodule. The *multiplier bimodule* of $\mathcal{E}$, denoted by $\mathcal{M}(\mathcal{E})$, is by definition, $\mathcal{L}(B, \mathcal{E})$ considered as a right Hilbert $\mathcal{M}(A), \mathcal{M}(B)$-bimodule, where the actions and the $\mathcal{M}(B)$-inner product are defined by:

$$(a \cdot T)b := a \cdot (Tb), \quad (T \cdot c)b := T(cb), \text{ and } \langle T|S\rangle_{\mathcal{M}(B)} := T^*S$$

for all $a \in \mathcal{M}(A)$, $T, S \in \mathcal{L}(B, \mathcal{E})$, $c \in \mathcal{M}(B)$ and $b \in B$.

In other words, if we identify $\mathcal{M}(B) \cong \mathcal{L}(B)$, then $\mathcal{M}(\mathcal{E})$ is the right Hilbert $\mathcal{L}(B)$-module $\mathcal{L}(B, \mathcal{E})$ and we forget the left Hilbert $\mathcal{L}(\mathcal{E})$-module structure and replace it by the left action of $\mathcal{M}(A)$ defined above. In general, even if $\mathcal{E}$ is a Hilbert A, B-bimodule, $\mathcal{M}(\mathcal{E})$ is not a Hilbert $\mathcal{M}(A), \mathcal{M}(B)$-bimodule. This happens if the A-inner product is full. In fact, in this case one can identify $\mathcal{M}(A) \cong \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big) \cong \mathcal{L}(\mathcal{E})$, and in this way $\mathcal{M}(\mathcal{E})$ is isomorphic to the Hilbert $\mathcal{L}(\mathcal{E}), \mathcal{L}(B)$-bimodule $\mathcal{L}(B, \mathcal{E})$.

Note that, identifying $\mathcal{E} \cong \mathcal{K}(B, \mathcal{E})$ (as Hilbert $\mathcal{K}(\mathcal{E}), B$-bimodules), we get a canonical embedding $\mathcal{E} \hookrightarrow \mathcal{M}(\mathcal{E})$ which is given by the map $\xi \mapsto |\xi\rangle$. This embedding is also compatible with the left A- and $\mathcal{M}(A)$-actions on $\mathcal{E}$ and $\mathcal{M}(\mathcal{E})$, respectively, in the sense that $|a \cdot \xi\rangle = a \cdot |\xi\rangle$ for all $a \in A$ and $\xi \in \mathcal{E}$.

Definition 2.1.5. Let $\mathcal{E}$ be a right-Hilbert A, B-bimodule. The *strict topology* on $\mathcal{M}(\mathcal{E})$ is the locally convex topology generated by the seminorms $m \mapsto \|Tm\|$ and $m \mapsto \|mb\|$ for all $T \in \mathcal{K}(\mathcal{E})$ and $b \in B$. If a net (m_i) converges strictly to m, then we write $m = \operatorname{s-lim} m_i$.

Note that the strict topology on $\mathcal{M}(\mathcal{E})$ has nothing to do with the left A-action, and it depends only on the Hilbert $\mathcal{K}(\mathcal{E}), B$-bimodule $\mathcal{K}(B, \mathcal{E}) \cong_{\mathcal{K}(\mathcal{E})} \mathcal{E}_B$. The natural embedding $\mathcal{E} \hookrightarrow \mathcal{M}(\mathcal{E})$ has dense image with respect to the strict topology, and we have $\mathcal{K}(\mathcal{E}) \cdot \mathcal{M}(\mathcal{E}) \subseteq \mathcal{E}$ and $\mathcal{M}(\mathcal{E}) \cdot B \subseteq \mathcal{E}$ (see [15, Proposition 1.27]). Moreover, $\mathcal{M}(\mathcal{E})$ is maximal with respect to these properties (see [15, Proposition 1.28]).

Definition 2.1.6. Let ${}_A\mathcal{E}_B$ and ${}_C\mathcal{F}_D$ be right-Hilbert bimodules, and suppose that $\phi : A \to \mathcal{M}(C)$ and $\psi : B \to \mathcal{M}(D)$ are $*$-homomorphisms. A linear map $\Phi : \mathcal{E} \to \mathcal{M}(\mathcal{F})$ is called a *ϕ, ψ-compatible right-Hilbert bimodule homomorphism* if

(i) $\Phi(a \cdot \xi) = \phi(a) \cdot \Phi(\xi)$,

(ii) $\Phi(\xi \cdot b) = \Phi(\xi) \cdot b$, and

(iii) $\langle \Phi(\xi) | \Phi(\eta) \rangle_{\mathcal{M}(D)} = \psi(\langle \xi | \eta \rangle_B)$ for all $a \in A$, $\xi, \eta \in \mathcal{E}$ and $b \in B$.

We call ϕ and ψ the *coefficient maps* and write ${}_\phi\Phi_\psi : {}_A\mathcal{E}_B \to \mathcal{M}({}_C\mathcal{F}_\mathcal{D})$ to indicate all the data.

We say that Φ is *nondegenerate* if both ϕ and ψ are nondegenerate and Φ satisfies $\overline{\mathrm{span}}(\Phi(\mathcal{E}) \cdot D) = \mathcal{F}$.

We say that Φ is a *right-Hilbert bimodule isomorphism* of $\mathcal{E}$ onto $\mathcal{F}$ if ϕ and ψ are isomorphisms of A and B onto C and D, respectively, and Φ is a bijection of $\mathcal{E}$ onto $\mathcal{F}$.

Any right-Hilbert bimodule homomorphism is automatically norm-decreasing, and if ψ is isometric, then so is Φ. In particular, any right-Hilbert bimodule isomorphism is automatically isometric.

In the situation above, if $\mathcal{E}$ and $\mathcal{F}$ are Hilbert bimodules and ${}_\phi\Phi_\psi$ is a nondegenerate right-Hilbert bimodule homomorphism, then the extra structure is automatically preserved (as well as possible; see [15, Lemma 1.18]). In this case, we also say that Φ is a Hilbert bimodule homomorphism.

Any nondegenerate right-Hilbert bimodule homomorphism ${}_\phi\Phi_\psi : {}_A\mathcal{E}_B \to \mathcal{M}({}_C\mathcal{F}_D)$ has a unique strictly continuous extension, which we also denote by

$$ {}_\phi\Phi_\psi : {}_{\mathcal{M}(A)}\, \mathcal{M}(\mathcal{E})_{\mathcal{M}(B)} \to \mathcal{M}({}_C\mathcal{F}_D). $$

Remark 2.1.7. Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules and consider the Hilbert $\mathcal{L}(\mathcal{F}), \mathcal{L}(\mathcal{E})$-bimodule $\mathcal{X} := \mathcal{L}(\mathcal{E}, \mathcal{F})$ and also the Hilbert $\mathcal{K}(\mathcal{F}), \mathcal{K}(\mathcal{E})$-bimodule $\mathcal{Y} := \mathcal{K}(\mathcal{E}, \mathcal{F})$. Thus, by definition, $\mathcal{M}(\mathcal{Y})$ is a right-Hilbert $\mathcal{M}\big(\mathcal{K}(\mathcal{F})\big), \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$-bimodule. Moreover, by the maximal property of $\mathcal{M}(\mathcal{Y})$ (see [15, Proposition 1.28]), it follows that there is an embedding

$$ \Phi : \mathcal{L}(\mathcal{E}, \mathcal{F}) \to \mathcal{M}\big(\mathcal{K}(\mathcal{E}, \mathcal{F})\big) $$

of right-Hilbert $\mathcal{M}\big(\mathcal{K}(\mathcal{E})\big), \mathcal{M}\big(\mathcal{K}(\mathcal{F})\big)$-bimodules (where we use the canonical identifications $\mathcal{L}(\mathcal{E}) \cong \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$ and $\mathcal{L}(\mathcal{F}) \cong \mathcal{M}\big(\mathcal{K}(\mathcal{F})\big)$ and forget the left-Hilbert structure of $\mathcal{L}(\mathcal{E}, \mathcal{F})$). In fact, this map is given by $\Phi(x)(S) := x \circ S$, for $x \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, and $S \in \mathcal{K}(\mathcal{E})$. Note that $\Phi(x)$ is, in fact, an adjointable operator $\mathcal{K}(\mathcal{E}) \to \mathcal{K}(\mathcal{E}, \mathcal{F})$ with $\Phi(x)^*(T) = x^* \circ T$ for all $T \in \mathcal{K}(\mathcal{E}, \mathcal{F})$. It is not true in general that Φ is surjective. In fact, we are now going to describe the image of Φ. For this, we need a preparation.

Proposition 2.1.8. *Let $\mathcal{E}, \mathcal{F}$ be Hilbert B-modules, and on the Hilbert $\mathcal{L}(\mathcal{F}), \mathcal{L}(\mathcal{E})$-bimodule $\mathcal{X} := \mathcal{L}(\mathcal{E}, \mathcal{F})$ define the following topology $\tau_{\mathcal{E},\mathcal{F}}$:*

$$ x_i \to x \text{ with respect to } \tau_{\mathcal{E},\mathcal{F}} \text{ if and only if } x_i S \to xS, \text{ and } Tx_i \to Tx \text{ (in norm)} $$

for all $S \in \mathcal{K}(\mathcal{E})$ and $T \in \mathcal{K}(\mathcal{F})$. In other words, $\tau_{\mathcal{E},\mathcal{F}}$ is the locally convex topology generated by the seminorms $x \mapsto \|xS\|$ and $x \mapsto \|Tx\|$ for all $S \in \mathcal{K}(\mathcal{E})$ and $T \in \mathcal{K}(\mathcal{F})$. Then, with respect to $\tau_{\mathcal{E},\mathcal{F}}$,

(i) *$\mathcal{K}(\mathcal{E},\mathcal{F})$ is dense in $\mathcal{L}(\mathcal{E},\mathcal{F})$, and*

(ii) *$\mathcal{L}(\mathcal{E},\mathcal{F})$ is complete.*

In other words, $\mathcal{L}(\mathcal{E},\mathcal{F})$ is the completion of $\mathcal{K}(\mathcal{E},\mathcal{F})$ with respect to $\tau_{\mathcal{E},\mathcal{F}}$.

Proof. Let $L := \mathcal{M}\big(\mathcal{K}(\mathcal{F}\oplus\mathcal{E})\big) \cong \mathcal{L}(\mathcal{F}\oplus\mathcal{E})$. Note that there are canonical identifications $\mathcal{K}(\mathcal{F}\oplus\mathcal{E}) \cong \begin{pmatrix} \mathcal{K}(\mathcal{F}) & \mathcal{K}(\mathcal{E},\mathcal{F}) \\ \mathcal{K}(\mathcal{F},\mathcal{E}) & \mathcal{K}(\mathcal{F}) \end{pmatrix}$ and $\mathcal{L}(\mathcal{F}\oplus\mathcal{E}) \cong \begin{pmatrix} \mathcal{L}(\mathcal{F}) & \mathcal{L}(\mathcal{E},\mathcal{F}) \\ \mathcal{L}(\mathcal{F},\mathcal{E}) & \mathcal{L}(\mathcal{F}) \end{pmatrix}$. Under this identifications, a net $(x_i) \subseteq \mathcal{L}(\mathcal{E},\mathcal{F})$ converges to $x \in \mathcal{L}(\mathcal{E},\mathcal{F})$ with respect to $\tau_{\mathcal{E},\mathcal{F}}$, if and only if $\begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ strictly in L. In fact, given $\begin{pmatrix} T & \xi \\ \eta & S \end{pmatrix} \in \mathcal{K}(\mathcal{F}\oplus\mathcal{E})$, we have $\eta \in \mathcal{K}(\mathcal{F},\mathcal{E})$, which is a Hilbert $\mathcal{K}(\mathcal{E}),\mathcal{K}(\mathcal{F})$-bimodule. Cohen's Factorization Theorem implies that $\eta = R\zeta$, where $R \in \mathcal{K}(\mathcal{E})$ and $\zeta \in \mathcal{K}(\mathcal{F},\mathcal{E})$. Thus, if $x_i \to x$ with respect to $\tau_{\mathcal{E},\mathcal{F}}$, then

$$\begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & \xi \\ \eta & S \end{pmatrix} = \begin{pmatrix} x_i\eta & x_iS \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} x\eta & xS \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & \xi \\ \eta & S \end{pmatrix}.$$

Analogously, we have

$$\begin{pmatrix} T & \xi \\ \eta & S \end{pmatrix}\begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} T & \xi \\ \eta & S \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}.$$

This means that $\begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ strictly in L. Conversely, assume that $\begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ strictly in L. Thus

$$\begin{pmatrix} x_i\eta & x_iS \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} x\eta & xS \\ 0 & 0 \end{pmatrix},$$

in the norm topology and hence

$$\begin{aligned}\begin{pmatrix} 0 & x_iS \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_i\eta & x_iS \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x\eta & xS \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & xS \\ 0 & 0 \end{pmatrix},\end{aligned}$$

in the norm topology. It follows that $x_iS \to xS$ for all $S \in \mathcal{K}(\mathcal{E})$. Similarly, one proves that $Tx_i \to Tx$ for all $T \in \mathcal{K}(\mathcal{F})$. Therefore $x_i \to x$ with respect to $\tau_{\mathcal{E},\mathcal{F}}$.

In the same way one proves that a net (x_i) in $\mathcal{L}(\mathcal{E},\mathcal{F})$ is Cauchy with respect to $\tau_{\mathcal{E},\mathcal{F}}$ if and only if the net $\begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix}$ is Cauchy with respect to the strict topology on L. The assertions now follow from the fact that $\mathcal{K}(\mathcal{F}\oplus\mathcal{E})$ is strictly dense in $\mathcal{M}\big(\mathcal{K}(\mathcal{F}\oplus\mathcal{E})\big)$ and $\mathcal{M}\big(\mathcal{K}(\mathcal{F}\oplus\mathcal{E})\big)$ is strictly complete. □

Definition 2.1.9. The topology $\tau_{\mathcal{E},\mathcal{F}}$ defined in Proposition 2.1.8 will be called the *bi-strict* topology. If (x_i) converges bi-strictly to x, then we write $x = \text{ss-lim}\, x_i$.

Remark 2.1.10. We have seen in the proof of Proposition 2.1.8 that, under the canonical identification

$$\mathcal{M}\big(\mathcal{K}(\mathcal{F}\oplus\mathcal{E})\big) \cong \mathcal{L}(\mathcal{F}\oplus\mathcal{E}) \cong \begin{pmatrix} \mathcal{L}(\mathcal{F}) & \mathcal{L}(\mathcal{E},\mathcal{F}) \\ \mathcal{L}(\mathcal{F},\mathcal{E}) & \mathcal{L}(\mathcal{F}) \end{pmatrix},$$

a net $\begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix}$ converges strictly to $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ in $\mathcal{M}\big(\mathcal{K}(\mathcal{F}\oplus\mathcal{E})\big)$ if and only if x_i converges bi-strictly to x in $\mathcal{L}(\mathcal{E},\mathcal{F})$. More generally, one can also show, using a similar idea, that a net $\begin{pmatrix} z_i & x_i \\ y_i & w_i \end{pmatrix}$ converges strictly to $\begin{pmatrix} z & x \\ y & w \end{pmatrix}$ in $\mathcal{M}\big(\mathcal{K}(\mathcal{F}\oplus\mathcal{E})\big)$ if and only if x_i and y_i converge bi-strictly to x and y in $\mathcal{L}(\mathcal{E},\mathcal{F})$ and $\mathcal{L}(\mathcal{F},\mathcal{E})$, respectively, and z_i and w_i converge strictly to z and w in $\mathcal{M}\big(\mathcal{K}(\mathcal{F})\big)$ and $\mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$, respectively.

Note that, if $\mathcal{E} = B$, then $\mathcal{K}(\mathcal{F}\oplus B) \cong L(\mathcal{F})$, the linking algebra of $\mathcal{F}$, and (hence) $\mathcal{M}\big(\mathcal{K}(\mathcal{F}\oplus B)\big) \cong \mathcal{M}\big(L(\mathcal{F})\big)$. In particular, we get as a consequence, that the strict topology on $\mathcal{M}\big(L(\mathcal{F})\big)$ coincides with the strict topologies on the corners $\mathcal{M}\big(\mathcal{K}(\mathcal{F})\big)$, $\mathcal{M}(\mathcal{F})$ and $\mathcal{M}(B)$, and with the bi-strict topology on the corner $\mathcal{L}(\mathcal{F},B)$.

Proposition 2.1.11. *Let $\mathcal{E},\mathcal{F}$ be Hilbert B-modules and consider the embedding Φ : $\mathcal{L}(\mathcal{E},\mathcal{F}) \to \mathcal{M}\big(\mathcal{K}(\mathcal{E},\mathcal{F})\big)$ described in Remark* 2.1.7. *Then*

$$\operatorname{Ran}\Phi = \big\{y \in \mathcal{M}\big(\mathcal{K}(\mathcal{E},\mathcal{F})\big) : \mathcal{K}(\mathcal{F})\cdot y \subseteq \mathcal{K}(\mathcal{E},\mathcal{F})\big\},$$

where $\cdot$ denotes the left action of $\mathcal{M}\big(\mathcal{K}(\mathcal{F})\big)$ on $\mathcal{M}\big(\mathcal{K}(\mathcal{E},\mathcal{F})\big)$, and we have identified $\mathcal{K}(\mathcal{E},\mathcal{F}) \subseteq \mathcal{M}\big(\mathcal{K}(\mathcal{E},\mathcal{F})\big)$. In particular, if $\mathcal{K}\big(\mathcal{K}(\mathcal{E},\mathcal{F})\big) = \mathcal{K}(\mathcal{F})$, then Φ is surjective, and therefore it is an isomorphism of Hilbert bimodules.

Proof. Since $\mathcal{K}(\mathcal{F})\mathcal{L}(\mathcal{E},\mathcal{F}) \subseteq \mathcal{K}(\mathcal{E},\mathcal{F})$ and since Φ preserves the left module structures and is the identity on $\mathcal{K}(\mathcal{E},\mathcal{F})$, we have

$$\operatorname{Ran}\Phi \subseteq \big\{y \in \mathcal{M}\big(\mathcal{K}(\mathcal{E},\mathcal{F})\big) : \mathcal{K}(\mathcal{F})\cdot y \subseteq \mathcal{K}(\mathcal{E},\mathcal{F})\big\}.$$

For the converse inclusion take $y \in \mathcal{M}\big(\mathcal{K}(\mathcal{E},\mathcal{F})\big)$ and suppose that $T\cdot y \in \mathcal{K}(\mathcal{E},\mathcal{F})$ for all $T \in \mathcal{K}(\mathcal{F})$. Let (e_i) be an approximate unit for $\mathcal{K}(\mathcal{F})$, and define the net $y_i := e_i \cdot y \in \mathcal{K}(\mathcal{E},\mathcal{F})$. We have

$$Ty_i = Te_i \cdot y \to T\cdot y, \quad T \in \mathcal{K}(\mathcal{F}),$$

and

$$y_iS = (e_i\cdot y)S = e_i(yS) \to yS, \quad S \in \mathcal{K}(\mathcal{E}),$$

because $yS \in \mathcal{K}(\mathcal{E},\mathcal{F})$, which is a Hilbert $\mathcal{K}(\mathcal{F}),\mathcal{K}(\mathcal{E})$-bimodule. It follows that (y_i) is Cauchy with respect to the bi-strict topology in $\mathcal{L}(\mathcal{E},\mathcal{F})$. By Proposition 2.1.8, there is $x \in \mathcal{L}(\mathcal{E},\mathcal{F})$ such that $y_i \to x$ bi-strictly. Finally, note that for all $S \in \mathcal{K}(\mathcal{E})$

$$\Phi(x)S = x\circ S = \lim_i y_iS = \lim_i (e_i\cdot y)S = \lim_i e_i(yS) = yS.$$

Therefore $y = \Phi(x) \in \operatorname{Ran}\Phi$. $\square$

Remark 2.1.12. We keep the notations of Remark 2.1.7.

(1) If $\mathcal{K}(\mathcal{Y}) \neq \mathcal{K}(\mathcal{F})$, then Φ may not be surjective. In fact, take $\mathcal{F} = B$, with B unital. Then $\mathcal{L}(\mathcal{E}, B) = \mathcal{K}(\mathcal{E}, B) = \mathcal{E}^*$. So we have $\mathcal{X} = \mathcal{Y} = \mathcal{K}(\mathcal{E}, B)$ and hence $\mathcal{X}$ embeds in $\mathcal{M}(\mathcal{Y})$, but this embedding is not surjective if we take, for example, $\mathcal{E} = I$ to be an ideal of B without unit (considered as a Hilbert B-module), because in this case we have $\mathcal{X} \cong I$ (considered as Hilbert $\mathcal{M}(B), \mathcal{M}(I)$-bimodules), and $\mathcal{M}(\mathcal{Y}) \cong \mathcal{M}(I)$ (considered as right-Hilbert $\mathcal{M}(B), \mathcal{M}(I)$-bimodules), and Φ is (identified with) the inclusion of I into $\mathcal{M}(I)$ which is not surjective if I is not unital.

(2) One sufficient condition for $\mathcal{K}(\mathcal{Y}) = \mathcal{K}(\mathcal{F})$ is when $\mathcal{E}$ is full. In fact, this is exactly [43, Proposition 7.1]. But note that this condition is not necessary, because if $\mathcal{E} = \mathcal{F}$, then we always have $\mathcal{K}(\mathcal{Y}) = \mathcal{K}(\mathcal{F})$.

(3) Note also that the condition $\mathcal{K}(\mathcal{Y}) = \mathcal{K}(\mathcal{F})$ is not necessary for the surjectivity of Φ. In fact, one can take the trivial example $\mathcal{E} = \{0\}$ and $\mathcal{F} \neq \{0\}$. In this case we have $\mathcal{X} = \mathcal{Y} = \mathcal{M}(\mathcal{Y}) = \{0\}$ and hence Φ is surjective, but $\mathcal{K}(\mathcal{X}) = \{0\} \neq \mathcal{K}(\mathcal{F})$.

Later, we shall also need the following topology.

Definition 2.1.13. Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules. The *$\mathcal{K}$-strong topology* on $\mathcal{L}(\mathcal{E}, \mathcal{F})$ is the locally convex topology generated by the seminorms $x \mapsto \|xS\|$ for all $S \in \mathcal{K}(\mathcal{E})$.

Since $\mathcal{E} = \mathcal{K}(\mathcal{E}) \cdot \mathcal{E}$, if a net (x_i) converges $\mathcal{K}$-strongly to x in $\mathcal{L}(\mathcal{E}, \mathcal{F})$, then it also converges strongly, that is, $x_i\xi \to x\xi$ for all $\xi \in \mathcal{E}$. The converse also holds if the net (x_i) is bounded.

Note also that a net (x_i) converges bi-strictly to x in $\mathcal{L}(\mathcal{E}, \mathcal{F})$ if and only if (x_i) converges $\mathcal{K}$-strongly to x in $\mathcal{L}(\mathcal{E}, \mathcal{F})$ and (x_i^*) converges $\mathcal{K}$-strongly to x^* in $\mathcal{L}(\mathcal{F}, \mathcal{E})$.

2.2 Tensor products

Definition 2.2.1. Let ${}_A\mathcal{E}_B$ and ${}_B\mathcal{F}_C$ be right-Hilbert bimodules. The *balanced tensor product* $\mathcal{E} \otimes_B \mathcal{F}$ is the right-Hilbert A, C-bimodule defined as the completion of the algebraic tensor product $\mathcal{E} \odot \mathcal{F}$ with respect to the C-inner product defined on elementary tensors by

$$\langle \xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2 \rangle_C := \langle \eta_1 | \langle \xi_1 | \xi_2 \rangle_B \cdot \eta_2 \rangle_C$$

for all $\xi_1, \xi_2 \in \mathcal{E}$ and $\eta_1, \eta_2 \in \mathcal{F}$. The module actions are defined by

$$a \cdot (\xi \otimes \eta) := a \cdot \xi \otimes \eta, \quad \text{and} \quad (\xi \otimes \eta) \cdot c := \xi \otimes \eta \cdot c$$

for all $a \in A$, $\xi \in \mathcal{E}$, $\eta \in \mathcal{F}$ and $c \in C$. The balanced tensor product $\mathcal{E} \otimes_B \mathcal{F}$ is also sometimes called the *internal tensor product*. If B acts on $\mathcal{F}$ via a homomorphism $\pi : B \to \mathcal{L}(\mathcal{F})$, then $\mathcal{E} \otimes_B \mathcal{F}$ is also denoted by $\mathcal{E} \otimes_\pi \mathcal{F}$. If $\xi \in \mathcal{E}$ and $\eta \in \mathcal{F}$, then we also write $\xi \otimes_B \eta$ (or $\xi \otimes_\pi \eta$) to indicate the respective element in $\mathcal{E} \otimes_B \mathcal{F}$ (or $\mathcal{E} \otimes_\pi \mathcal{F}$).

Given right-Hilbert bimodules ${}_D\mathcal{X}_E$ and ${}_E\mathcal{Y}_F$, there is a natural embedding

$$\iota : \mathcal{M}(\mathcal{X}) \otimes_{\mathcal{M}(E)} \mathcal{M}(\mathcal{Y}) \hookrightarrow \mathcal{M}(\mathcal{X} \otimes_E \mathcal{Y})$$

given by $\iota(m)f = m \cdot f$ and $\iota(m)^*x = \langle m|x\rangle_{\mathcal{M}(F)}$ for all $m \in \mathcal{M}(\mathcal{X}) \otimes_{\mathcal{M}(E)} \mathcal{M}(\mathcal{Y})$, $f \in F$ and $x \in \mathcal{X} \otimes_E \mathcal{Y}$ (see [15, Lemma 1.32]). We shall view $\mathcal{M}(\mathcal{X}) \otimes_{\mathcal{M}(E)} \mathcal{M}(\mathcal{Y})$ as a subspace of $\mathcal{M}(\mathcal{X} \otimes_E \mathcal{Y})$ via the map ι. The following result is Proposition 1.34 in [15].

Proposition 2.2.2. *If* ${}_\phi\Phi_\psi \colon {}_A\mathcal{E}_B \to \mathcal{M}({}_D\mathcal{X}_E)$ *and* ${}_\psi\Psi_\gamma \colon {}_B\mathcal{F}_C \to \mathcal{M}({}_E\mathcal{Y}_F)$ *are right-Hilbert bimodule homomorphisms, then there is a* ϕ, γ*-compatible right-Hilbert bimodule homomorphism*

$$\Phi \otimes_B \Psi : \, {}_A(\mathcal{E} \otimes_B \mathcal{F})_C \to \mathcal{M}\big({}_D(\mathcal{X} \otimes_E \mathcal{Y})_F\big)$$

such that

$$(\Phi \otimes_B \Psi)(\xi \otimes_B \eta) = \Phi(\xi) \otimes_E \Psi(\eta), \quad \xi \in \mathcal{E}, \eta \in \mathcal{F}.$$

If Φ *and* Ψ *are nondegenerate, then so is* $\Phi \otimes_B \Psi$.

For C^*-algebras, the symbol $\otimes$ will always denote the *minimal* tensor product.

Definition 2.2.3. Let ${}_A\mathcal{E}_B$ and ${}_C\mathcal{F}_D$ be right-Hilbert bimodules. The *external tensor product* $\mathcal{E} \otimes \mathcal{F}$ is the right-Hilbert $A \otimes C, B \otimes D$-bimodule defined as the completion of the algebraic tensor product $\mathcal{E} \odot \mathcal{F}$ with respect to the inner product defined on elementary tensors by

$$\langle \xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2 \rangle_{B \otimes D} := \langle \xi_1 | \xi_2 \rangle_B \otimes \langle \eta_1 | \eta_2 \rangle_D$$

for all $\xi_1, \xi_2 \in \mathcal{E}$ and $\eta_1, \eta_2 \in \mathcal{F}$. The module actions are defined by

$$(a \otimes c) \cdot (\xi \otimes \eta) := a \cdot \xi \otimes c \cdot \eta, \quad \text{and} \quad (\xi \otimes \eta)(b \otimes d) := \xi \cdot b \otimes \eta \cdot d$$

for all $a \in A$, $c \in C$, $\xi \in \mathcal{E}$, $\eta \in \mathcal{F}$, $b \in B$ and $d \in D$.

Given right-Hilbert bimodules ${}_E\mathcal{X}_F$ and ${}_G\mathcal{Y}_H$, there is a natural embedding

$$\iota : \mathcal{M}(\mathcal{X}) \otimes \mathcal{M}(\mathcal{Y}) \hookrightarrow \mathcal{M}(\mathcal{X} \otimes \mathcal{Y})$$

which is given by $\iota(m)b = m \cdot b$ and $\iota(m)^*x = \langle m|x\rangle_{\mathcal{M}(F)\otimes\mathcal{M}(H)}$ for all $m \in \mathcal{M}(\mathcal{X}) \otimes \mathcal{M}(\mathcal{Y})$, $b \in F \otimes H$ and $x \in \mathcal{X} \otimes \mathcal{Y}$. As for balanced tensor products, we shall view $\mathcal{M}(\mathcal{X}) \otimes \mathcal{M}(\mathcal{Y})$ as a subspace of $\mathcal{M}(\mathcal{X} \otimes \mathcal{Y})$ via the map ι. The following result is Proposition 1.38 in [15].

Proposition 2.2.4. *If* ${}_\phi\Phi_\psi \colon {}_A\mathcal{E}_B \to \mathcal{M}({}_E\mathcal{X}_F)$ *and* ${}_\beta\Phi_\gamma \colon {}_C\mathcal{F}_D \to \mathcal{M}({}_G\mathcal{Y}_H)$ *are right-Hilbert bimodule homomorphisms, then there is a* $\phi \otimes \beta, \psi \otimes \gamma$*-compatible right-Hilbert bimodule homomorphism*

$$\Phi \otimes \Psi : {}_{A \otimes C}(\mathcal{E} \otimes \mathcal{F})_{B \otimes D} \to \mathcal{M}\big({}_{E \otimes G}(\mathcal{X} \otimes \mathcal{Y})_{F \otimes H}\big)$$

such that

$$(\Phi \otimes \Psi)(\xi \otimes \eta) = \Phi(\xi) \otimes \Psi(\eta), \quad \xi \in \mathcal{E}, \eta \in \mathcal{F}.$$

If Φ *and* Ψ *are nondegenerate, then so is* $\Phi \otimes \Psi$.

The proof of the following proposition is straightforward and we omit it.

Proposition 2.2.5. *Let $\mathcal{E}_1, \mathcal{F}_1$ be Hilbert B-modules, and let $\mathcal{E}_2, \mathcal{F}_2$ be C-modules. Then the map*

$$\Phi : \mathcal{L}(\mathcal{E}_1, \mathcal{F}_1) \otimes \mathcal{L}(\mathcal{E}_2, \mathcal{F}_2) \to \mathcal{L}(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$$

given by $\Phi(x \otimes y)(\xi_1 \otimes \xi_2) = x\xi_1 \otimes y\xi_2$ for all $x \in \mathcal{L}(\mathcal{E}_1, \mathcal{F}_1)$, $y \in \mathcal{L}(\mathcal{E}_2, \mathcal{F}_2)$, $\xi_1 \in \mathcal{E}_1$ and $\xi_2 \in \mathcal{E}_2$ is an isometric ϕ, ψ-compatible Hilbert bimodule homomorphism (that is, an embedding of Hilbert bimodules), where

$$\phi : \mathcal{L}(\mathcal{F}_1) \otimes \mathcal{L}(\mathcal{F}_2) \to \mathcal{L}(\mathcal{F}_1 \otimes \mathcal{F}_2) \quad \text{and} \quad \psi : \mathcal{L}(\mathcal{E}_1 \otimes \mathcal{E}_2) \to \mathcal{L}(\mathcal{E}_1 \otimes \mathcal{E}_2)$$

are the natural (isometric) homomorphisms given by $\phi(S \otimes T)(\eta_1 \otimes \eta_2) = S\xi_1 \otimes T\xi_2$ for all $S \in \mathcal{L}(\mathcal{F}_1)$, $T \in \mathcal{L}(\mathcal{F}_2)$, $\eta_1 \in \mathcal{F}_1$ and $\eta_2 \in \mathcal{F}_2$. And analogously for ψ.

Moreover, when all the maps are restricted to the compact operators, we get an isomorphism

$$\mathcal{K}(\mathcal{E}_1, \mathcal{F}_1) \otimes \mathcal{K}(\mathcal{E}_2, \mathcal{F}_2) \cong \mathcal{K}(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$$

of Hilbert $\mathcal{K}(\mathcal{F}_1) \otimes \mathcal{K}(\mathcal{F}_2) \cong \mathcal{K}(\mathcal{F}_1 \otimes \mathcal{F}_2), \mathcal{K}(\mathcal{E}_1) \otimes \mathcal{K}(\mathcal{E}_2) \cong \mathcal{K}(\mathcal{E}_2 \otimes \mathcal{E}_2)$-bimodules.

Definition 2.2.6. Let $\mathcal{E}$ be a right-Hilbert A, B-bimodule and let $\mathcal{G}$ be a C^*-algebra. The *$\mathcal{G}$-multiplier bimodule* of $\mathcal{E} \otimes \mathcal{G}$ is defined by

$$\tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G}) := \big\{ m \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G}) : m(1 \otimes \mathcal{G}),\, (1 \otimes \mathcal{G})m \subseteq \mathcal{E} \otimes \mathcal{G} \big\}.$$

The *$\mathcal{G}$-strict topology* on $\tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G})$ is the locally convex topology generated by the seminorms $m \mapsto \|m(1 \otimes x)\|$ and $m \mapsto \|(1 \otimes x)m\|$ for all $x \in \mathcal{G}$.

In particular, we have the $\mathcal{G}$-multipliers $\tilde{\mathcal{M}}(A \otimes \mathcal{G})$ and $\tilde{\mathcal{M}}(B \otimes \mathcal{G})$ which are C^*-subalgebras of $\mathcal{M}(A \otimes \mathcal{G})$ and $\mathcal{M}(B \otimes \mathcal{G})$, respectively. The $\mathcal{G}$-multiplier bimodule $\tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G})$ defined above is, in fact, a right-Hilbert $\tilde{\mathcal{M}}(A \otimes \mathcal{G}), \tilde{\mathcal{M}}(B \otimes \mathcal{G})$-bimodule with respect to the bimodule structure on $\mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ restricted to $\tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G})$. Moreover, it is the $\mathcal{G}$-strict completion of $\mathcal{E} \otimes \mathcal{G}$ (see [15, Lemma 1.40]).

Proposition 2.2.7 (Proposition 1.42 in [15])**.** *Suppose that ${}_A\mathcal{E}_B$ and ${}_C\mathcal{X}_D$ are right-Hilbert bimodules and suppose that ${}_\psi\Phi_\phi : {}_A\mathcal{E}_B \to \mathcal{M}({}_C\mathcal{X}_D)$ is a (possibly degenerate) right-Hilbert bimodule homomorphism. If $\mathcal{G}$ and $\mathcal{H}$ are C^*-algebras and $\Psi : \mathcal{G} \to \mathcal{M}(\mathcal{H})$ is a nondegenerate $*$-homomorphism, then there is a unique bimodule homomorphism*

$${}_{\overline{\psi \otimes \Psi}}\overline{\Phi \otimes \Psi}_{\overline{\phi \otimes \Psi}} : {}_{\tilde{\mathcal{M}}(A \otimes \mathcal{G})}\, \tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G})_{\tilde{\mathcal{M}}(B \otimes \mathcal{G})} \longrightarrow {}_{\mathcal{M}(C \otimes \mathcal{H})}\, \mathcal{M}(\mathcal{X} \otimes \mathcal{H})_{\mathcal{M}(D \otimes \mathcal{H})}$$

extending $\Phi \otimes \Psi : \mathcal{E} \otimes \mathcal{G} \to \mathcal{M}(\mathcal{X} \otimes \mathcal{H})$. The map $\overline{\Phi \otimes \Psi}$ is $\mathcal{G}$-strict to strict continuous. If $\Phi(\mathcal{E}) \subseteq \mathcal{X}$, then $\overline{\Phi \otimes \Psi}\big(\tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G})\big) \subseteq \tilde{\mathcal{M}}(\mathcal{X} \otimes \mathcal{H})$ and $\overline{\Phi \otimes \Psi}$ is $\mathcal{G}$-strict to $\mathcal{H}$-strict continuous.

We shall denote the unique extension $\overline{\Phi \otimes \Psi}$ above also by $\Phi \otimes \Psi$. Of course, we do the same for the coefficient maps. If Φ and Ψ are isometric, then so is its extension $\Phi \otimes \Psi : \tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G}) \to \mathcal{M}(\mathcal{X} \otimes \mathcal{H})$ ([15, Proposition 1.45]). An important case is when Ψ is the identity map. In this case we have:

Proposition 2.2.8 (Lemma 1.46 in [15]). *Suppose that $\Phi : {}_A\mathcal{E}_B \to \mathcal{M}({}_C\mathcal{X}_D)$ is an isometric right-Hilbert bimodule homomorphism with $\Phi(\mathcal{E}) \subseteq \mathcal{X}$. Then the isometry $\Phi \otimes \mathrm{id}_{\mathcal{G}} : \tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G}) \to \tilde{\mathcal{M}}(\mathcal{X} \otimes \mathcal{G})$ has image*

$$M = \big\{ m \in \tilde{\mathcal{M}}(\mathcal{X} \otimes \mathcal{G}) : m(1 \otimes \mathcal{G}),\, (1 \otimes \mathcal{G})m \subseteq (\Phi \otimes \mathrm{id}_{\mathcal{G}})(\mathcal{X} \otimes \mathcal{G}) \big\}.$$

2.3 Linking algebras

Definition 2.3.1. Let $\mathcal{E}$ be a Hilbert A, B-bimodule. The *linking algebra* of $\mathcal{E}$ is the $*$-algebra

$$L(\mathcal{E}) := \begin{pmatrix} A & \mathcal{E} \\ \tilde{\mathcal{E}} & B \end{pmatrix} = \left\{ \begin{pmatrix} a & \xi \\ \tilde{\eta} & b \end{pmatrix} : a \in A,\, \xi, \eta \in \mathcal{E},\, b \in B \right\},$$

with operations

$$\begin{pmatrix} a_1 & \xi_1 \\ \tilde{\eta}_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & \xi_2 \\ \tilde{\eta}_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + {}_A\langle \xi_1 | \eta_2 \rangle & a_1 \cdot \xi_2 + \xi_1 \cdot b_2 \\ \tilde{\eta}_1 \cdot a_2 + b_1 \cdot \tilde{\eta}_2 & \langle \eta_1 | \xi_2 \rangle_B + b_1 b_2 \end{pmatrix}$$

and

$$\begin{pmatrix} a & \xi \\ \tilde{\eta} & b \end{pmatrix}^* = \begin{pmatrix} a^* & \eta \\ \tilde{\xi} & b^* \end{pmatrix}.$$

There is a natural left action of $L(\mathcal{E})$ on the right-Hilbert B-module $\mathcal{E} \oplus B$, that is, a homomorphism $L(\mathcal{E}) \to \mathcal{L}(\mathcal{E} \oplus B)$, which is injective on $\mathcal{E}$, $\tilde{\mathcal{E}}$ and B, and a natural right action of $L(\mathcal{E})$ on the left Hilbert A-module $A \oplus \mathcal{E}$, that is, a homomorphism $L(\mathcal{E}) \to \mathcal{L}(A \oplus \mathcal{E})$, which is injective on A, $\mathcal{E}$ and $\tilde{\mathcal{E}}$. The norm on $L(\mathcal{E})$ is, by definition, the maximum of the respective operator norms in $\mathcal{L}(\mathcal{E} \oplus B)$ and $\mathcal{L}(A \oplus \mathcal{E})$. With this norm, $L(\mathcal{E})$ is a C^*-algebra (see [15, Section 1.5]).

The linking algebra $L(\mathcal{E})$ contains copies of A, $\mathcal{E}$, $\tilde{\mathcal{E}}$ and B that can be recovered by the projections $p, q \in \mathcal{M}\big(L(\mathcal{E})\big)$ defined by

$$p := \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad q := \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}.$$

For example, we have

$$pL(\mathcal{E})p = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \cong A.$$

If $\mathcal{E}$ is just a (right) Hilbert B-module, then we consider $\mathcal{E}$ as a Hilbert $\mathcal{K}(\mathcal{E}), B$-bimodule, and in this way the linking algebra of $\mathcal{E}$ is given by

$$L(\mathcal{E}) = \begin{pmatrix} \mathcal{K}(\mathcal{E}) & \mathcal{E} \\ \tilde{\mathcal{E}} & B \end{pmatrix}.$$

In this case, we have $\tilde{\mathcal{E}} \cong \mathcal{K}(\mathcal{E}, B)$ and the linking algebra is isomorphic to $\mathcal{K}(\mathcal{E} \oplus B)$. Moreover,

$$\mathcal{M}\big(L(\mathcal{E})\big) \cong \mathcal{M}\big(\mathcal{K}(\mathcal{E} \oplus B)\big) \cong \mathcal{L}(\mathcal{E} \oplus B) \cong \begin{pmatrix} \mathcal{L}(\mathcal{E}) & \mathcal{L}(B, \mathcal{E}) \\ \mathcal{L}(\mathcal{E}, B) & \mathcal{L}(B) \end{pmatrix}$$
$$\cong \begin{pmatrix} \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big) & \mathcal{M}(\mathcal{E}) \\ \widetilde{\mathcal{M}(\mathcal{E})} & \mathcal{M}(B) \end{pmatrix} = L\big(\mathcal{M}(\mathcal{E})\big) \quad \text{(see [15, Proposition 1.51])}.$$

Let $\mathcal{E}$ be a Hilbert B-module and let H be a Hilbert space. Later, we shall need a matrix representation of the Hilbert $L(\mathcal{E})$-module $L(\mathcal{E}) \otimes H$ which we describe in what follows. Define the following linear space

$$L(\mathcal{E}; H) := \begin{pmatrix} \mathcal{K}(\mathcal{E}) \otimes H & \mathcal{E} \otimes H \\ \mathcal{E}^* \otimes H & B \otimes H \end{pmatrix} :=$$
$$\left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} : x_1 \in \mathcal{K}(\mathcal{E}) \otimes H, x_2 \in \mathcal{E} \otimes H, x_3 \in \mathcal{E}^* \otimes H \text{ and } x_4 \in B \otimes H \right\}$$

Then $L(\mathcal{E}; H)$ is a Hilbert $L(\mathcal{E})$-module, where all the structure is given by matrix multiplication. Thus the $L(\mathcal{E})$-inner product is defined by

$$\left\langle \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \middle| \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \right\rangle := \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^* \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$
$$= \begin{pmatrix} x_1^* & x_3^* \\ x_2^* & x_4^* \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} x_1^* y_1 + x_3^* y_3 & x_1^* y_2 + x_3^* y_4 \\ x_2^* y_1 + x_4^* y_3 & x_2^* y_2 + x_4^* y_4 \end{pmatrix}.$$

All the products above make sense if we identify $\mathcal{K}(\mathcal{E}) \otimes H \cong \mathcal{K}(\mathcal{E}, \mathcal{E} \otimes H)$, $\mathcal{E} \otimes H \cong \mathcal{K}(B, \mathcal{E} \otimes H)$, $\mathcal{E}^* \otimes H \cong \mathcal{K}(\mathcal{E}, B \otimes H)$ and $B \otimes H \cong \mathcal{K}(B, B \otimes H)$. The right $L(\mathcal{E})$-module action is defined in a similar way. Note that if $\begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix}$ is a Cauchy net in $L(\mathcal{E}; H)$, then it follows from the definition of the inner product that $(x_i - x_j)^*(x_i - x_j) + (z_i - z_j)^*(z_i - z_j) \to 0$ and $(y_i - y_j)^*(y_i - y_j) + (w_i - w_j)^*(w_i - w_j) \to 0$ in $\mathcal{K}(\mathcal{E})$ and B, respectively, and therefore (x_i), (y_i), (z_i) and (w_i) are Cauchy nets in $K(\mathcal{E}) \otimes H$, $\mathcal{E} \otimes H$, $\mathcal{E}^* \otimes H$ and $B \otimes H$, respectively. Thus $L(\mathcal{E}; H)$ is complete and therefore a Hilbert $L(\mathcal{E})$-module. Note that a similar argument shows that a net

$$\begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix} \to \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in L(\mathcal{E}; H)$$

if and only if $x_i \to x$, $y_i \to y$, $z_i \to z$ and $w_i \to w$. In particular,

$$\begin{pmatrix} \mathcal{K}(\mathcal{E}) \odot H & \mathcal{E} \odot H \\ \mathcal{E}^* \odot H & B \odot H \end{pmatrix}$$

is dense in $L(\mathcal{E}; H)$.

Now note that there is a canonical map

$$\Phi : L(\mathcal{E}) \odot H \to \begin{pmatrix} \mathcal{K}(\mathcal{E}) \odot H & \mathcal{E} \odot H \\ \mathcal{E}^* \odot H & B \odot H \end{pmatrix} \subseteq L(\mathcal{E}; H)$$

such that

$$\Phi\left(\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \otimes v\right) = \begin{pmatrix} k \otimes v & \xi \otimes v \\ \eta^* \otimes v & b \otimes v \end{pmatrix}$$

for all $k \in \mathcal{K}(\mathcal{E})$, $\xi, \eta \in \mathcal{E}$, $b \in B$ and $v \in H$. Moreover, it is easy to see that Φ preserves all the Hilbert $L(\mathcal{E})$-module structure, and therefore it extends to an isomorphism of Hilbert $L(\mathcal{E})$-modules $L(\mathcal{E}) \otimes H \cong L(\mathcal{E}; H)$. Under this identification we can describe $\mathcal{M}\big(L(\mathcal{E}) \otimes H\big) \cong \mathcal{M}\big(L(\mathcal{E}; H)\big)$. For this we define

$$\bar{L}(\mathcal{E}; H) := \begin{pmatrix} \mathcal{L}(\mathcal{E}, \mathcal{E} \otimes H) & \mathcal{L}(B, \mathcal{E} \otimes H) \\ \mathcal{L}(\mathcal{E}, B \otimes H) & \mathcal{L}(B, B \otimes H) \end{pmatrix}.$$

As for $L(\mathcal{E}; H)$, one can prove that $\bar{L}(\mathcal{E}; H)$ has a natural structure of Hilbert $L\big(\mathcal{M}(\mathcal{E})\big) \cong \mathcal{M}\big(L(\mathcal{E})\big)$-module, which is given by matrix multiplication, analogous to $L(\mathcal{E}; H)$. Moreover, identifying $\mathcal{K}(\mathcal{E}) \otimes H \cong \mathcal{K}(\mathcal{E}, \mathcal{E} \otimes H)$, $\mathcal{E} \otimes H \cong \mathcal{K}(B, \mathcal{E} \otimes H)$, $\mathcal{E}^* \otimes H \cong \mathcal{K}(\mathcal{E}, B \otimes H)$ and $B \otimes H \cong \mathcal{K}(B, B \otimes H)$, we get a canonical inclusion of $L(\mathcal{E}; H)$ into $\bar{L}(\mathcal{E}; H)$, and via this inclusion, $L(\mathcal{E}; H)$ is a Hilbert $L\big(\mathcal{M}(\mathcal{E})\big)$-submodule of $\bar{L}(\mathcal{E}; H)$ (where $L(\mathcal{E}; H)$ is considered as a Hilbert $L\big(\mathcal{M}(\mathcal{E})\big)$-module in the usual way). And it is easy to see that $\bar{L}(\mathcal{E}; H) \cdot L(\mathcal{E}) \subseteq L(\mathcal{E}; H)$. It follows (from [15, Proposition 1.28]) that $\bar{L}(\mathcal{E}; H)$ embeds as a Hilbert $L\big(\mathcal{M}(\mathcal{E})\big) \cong \mathcal{M}\big(L(\mathcal{E})\big)$-submodule of $\mathcal{M}\big(L(\mathcal{E}; H)\big) = \mathcal{L}\big(L(\mathcal{E}), L(\mathcal{E}; H)\big)$. In fact, this embedding is given by the map

$$\Psi : \bar{L}(\mathcal{E}; H) \to \mathcal{M}\big(L(\mathcal{E}; H)\big),$$

$$x \mapsto \Psi(x) := [a \mapsto x \cdot a].$$

Proposition 2.3.2. *The map Ψ above is surjective, and therefore is an isomorphism $\bar{L}(\mathcal{E}; H) \cong \mathcal{M}\big(L(\mathcal{E}; H)\big) \cong \mathcal{M}\big(L(\mathcal{E}) \otimes H\big)$ of Hilbert $L\big(\mathcal{M}(\mathcal{E})\big) \cong \mathcal{M}\big(L(\mathcal{E})\big)$-modules.*

Proof. Take any $T \in \mathcal{M}\big(L(\mathcal{E}; H)\big) = \mathcal{L}\big(L(\mathcal{E}), L(\mathcal{E}; H)\big)$ and define the following maps

$$T_1 : \mathcal{E} \to \mathcal{E} \otimes H, \quad T_1\xi := T\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}_{12},$$

$$T_2 : B \to \mathcal{E} \otimes H, \quad T_2 b := T\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}_{12},$$

$$T_3 : \mathcal{E} \to B \otimes H, \quad T_3\xi := T\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}_{22},$$

and

$$T_4 : \mathcal{E} \to B \otimes H, \quad T_4 b := T\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}_{22},$$

where we are using the notation m_{ij} for the (i,j)-element of a matrix m. The maps T_k, $k = 1,2,3,4$ are, in fact, adjointable operators, and the adjoints are given by the following formulas:

$$T_1^* x := T^* \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}_{12}, \quad T_2^* x := T^* \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}_{22},$$

$$T_3^* y := T^* \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}_{12}, \quad T_4^* y := T^* \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}_{22}$$

for all $x \in \mathcal{E} \otimes H$ and $y \in B \otimes H$. For example, to check the formula for the adjoint of T_1, we use the relations

$$\langle Z_{12}|\xi\rangle = \left\langle Z \middle| \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \right\rangle_{22}, \quad \langle x|W_{12}\rangle = \left\langle \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \middle| W \right\rangle_{22}$$

for all $Z \in L(\mathcal{E})$, $\xi \in \mathcal{E}$, $x \in \mathcal{E} \otimes H$, and $W \in L(\mathcal{E}; H)$ to get

$$\begin{aligned} \langle T_1^* x|\xi\rangle &= \left\langle T^* \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}_{12} \middle| \xi \right\rangle = \left\langle T^* \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \right\rangle_{22} \\ &= \left\langle \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| T \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \right\rangle_{22} = \left\langle \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| T \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}_{12} \right\rangle = \langle x|T_1\xi\rangle. \end{aligned}$$

Analogously, one can check the formulas for the adjoints of T_2, T_3, T_4. Thus we get the element $x := \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \bar{L}(\mathcal{E}; H)$, and now we show that $\Psi(x) = T$. In fact, it is easy to check the following relations

$$W_{11}\zeta = \left(W \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix} \right)_{12}, \quad W_{21}\zeta = \left(W \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix} \right)_{22},$$

$$W_{12}b = \left(W \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right)_{12}, \quad W_{22}b = \left(W \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right)_{22}$$

for all $W \in L(\mathcal{E}; H)$, $\zeta \in \mathcal{E}$ and $b \in B$. So, for example, we have

$$\begin{aligned} \left(T \begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \right)_{11} \zeta &= \left(T \begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix} \right)_{12} \\ &= \left(T \begin{pmatrix} 0 & k\zeta \\ 0 & \langle \eta|\zeta\rangle \end{pmatrix} \right)_{12} \\ &= \left(T \begin{pmatrix} 0 & k\zeta \\ 0 & 0 \end{pmatrix} \right)_{12} + \left(T \begin{pmatrix} 0 & 0 \\ 0 & \langle \eta|\zeta\rangle \end{pmatrix} \right)_{12} \\ &= T_1 k\zeta + T_2\langle \eta|\zeta\rangle = (T_1 k + T_2\eta^*)\zeta. \end{aligned}$$

And on the other hand

$$\Psi(x) \begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} =$$

$$\begin{pmatrix} T_1k + T_2\eta^* & T_1\xi + T_2b \\ T_3kT_4\eta^* & T_3\xi + T_4b \end{pmatrix}.$$

Thus

$$\left(T\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix}\right)_{11} = \left(\Psi(x)\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix}\right)_{11}.$$

Analogously, one proves that

$$\left(T\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix}\right)_{ij} = \left(\Psi(x)\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix}\right)_{ij},$$

for the other indexes $i, j \in \{1, 2\}$. Therefore $\Psi(x) = T$ as desired. □

Thus we have canonical isomorphisms

$$L(\mathcal{E}) \otimes H \cong \begin{pmatrix} \mathcal{K}(\mathcal{E}) \otimes H & \mathcal{E} \otimes H \\ \mathcal{E}^* \otimes H & B \otimes H \end{pmatrix} \cong \begin{pmatrix} \mathcal{K}(\mathcal{E}, \mathcal{E} \otimes H) & \mathcal{K}(B, \mathcal{E} \otimes H) \\ \mathcal{K}(\mathcal{E}, B \otimes H) & \mathcal{K}(B, B \otimes H) \end{pmatrix},$$

as Hilbert $L(\mathcal{E})$-modules, and

$$\mathcal{M}\big(L(\mathcal{E}) \otimes H\big) \cong \begin{pmatrix} \mathcal{L}(\mathcal{E}, \mathcal{E} \otimes H) & \mathcal{L}(B, \mathcal{E} \otimes H) \\ \mathcal{L}(\mathcal{E}, B \otimes H) & \mathcal{L}(B, B \otimes H) \end{pmatrix},$$

as Hilbert $\mathcal{M}\big(L(\mathcal{E})\big) \cong L\big(\mathcal{M}(\mathcal{E})\big)$-modules.

Note that, by definition, $\mathcal{L}(B, \mathcal{E} \otimes H) = \mathcal{M}(\mathcal{E} \otimes H)$ and $\mathcal{L}(B, B \otimes H) = \mathcal{M}(B \otimes H)$, and we also have $\mathcal{L}(\mathcal{E}, \mathcal{E} \otimes H) \cong \mathcal{M}\big(\mathcal{K}(\mathcal{E}) \otimes H\big)$ (this follows from Proposition 2.1.11). Moreover, $\mathcal{L}(\mathcal{E}, B \otimes H)$ can be embedded as a Hilbert $\mathcal{L}(\mathcal{E}) \cong \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$-submodule of $\mathcal{M}(\mathcal{E}^* \otimes H)$ (see Remark 2.1.7), but this embedding is not surjective in general (it is if $\mathcal{E}$ is full; see Remark 2.1.12(2)).

Remark 2.3.3. Let A, B be C^*-algebras and let $\mathcal{E}$ be a Hilbert B-module. Then there is a canonical isomorphism $L(\mathcal{E}) \otimes A \cong L(\mathcal{E} \otimes A)$ (see [15, Remark 1.50]). More generally, if $\mathcal{F}$ is a Hilbert A-module, then one can identify

$$L(\mathcal{E}) \otimes \mathcal{F} \cong \begin{pmatrix} \mathcal{K}(\mathcal{E}) \otimes \mathcal{F} & \mathcal{E} \otimes \mathcal{F} \\ \mathcal{E}^* \otimes \mathcal{F} & B \otimes \mathcal{F} \end{pmatrix} \cong \begin{pmatrix} \mathcal{K}(\mathcal{E} \otimes A, \mathcal{E} \otimes \mathcal{F}) & \mathcal{K}(B \otimes A, \mathcal{E} \otimes \mathcal{F}) \\ \mathcal{K}(\mathcal{E} \otimes A, B \otimes \mathcal{F}) & \mathcal{K}(B \otimes A, B \otimes \mathcal{F}) \end{pmatrix},$$

as Hilbert $L(\mathcal{E}) \otimes A \cong L(\mathcal{E} \otimes A)$-modules, and also

$$\mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{F}\big) \cong \begin{pmatrix} \mathcal{L}(\mathcal{E} \otimes A, \mathcal{E} \otimes \mathcal{F}) & \mathcal{L}(B \otimes A, \mathcal{E} \otimes \mathcal{F}) \\ \mathcal{L}(\mathcal{E} \otimes A, B \otimes \mathcal{F}) & \mathcal{L}(B \otimes A, B \otimes \mathcal{F}) \end{pmatrix},$$

as Hilbert $\mathcal{M}\big(L(\mathcal{E}) \otimes A\big) \cong L\big(\mathcal{M}(\mathcal{E} \otimes A)\big)$-modules, where all the structure is given by matrix multiplication. In fact, a short proof for this assertions (and hence also for the case of a Hilbert space $\mathcal{F} = H$) is the following. We can identify $L(\mathcal{E}) \cong \mathcal{K}(\mathcal{E} \oplus B)$ and $\mathcal{F} \cong \mathcal{K}(A, \mathcal{F})$. Proposition 2.2.5 yields

$$\begin{aligned} L(\mathcal{E}) \otimes \mathcal{F} &\cong \mathcal{K}(\mathcal{E} \oplus B) \otimes \mathcal{K}(A, \mathcal{F}) \\ &\cong \mathcal{K}\big((\mathcal{E} \oplus B) \otimes A, (\mathcal{E} \oplus B) \otimes \mathcal{F}\big) \\ &\cong \mathcal{K}\big((\mathcal{E} \otimes A) \oplus (B \otimes A), (\mathcal{E} \otimes \mathcal{F}) \oplus (B \otimes \mathcal{F})\big) \\ &\cong \begin{pmatrix} \mathcal{K}(\mathcal{E} \otimes A, \mathcal{E} \otimes \mathcal{F}) & \mathcal{K}(B \otimes A, \mathcal{E} \otimes \mathcal{F}) \\ \mathcal{K}(\mathcal{E} \otimes A, B \otimes \mathcal{F}) & \mathcal{K}(B \otimes A, B \otimes \mathcal{F}) \end{pmatrix}. \end{aligned}$$

Since $(\mathcal{E} \otimes A) \oplus (B \otimes A)$ is a full Hilbert $B \otimes A$-module, it follows from Remark 2.1.12(2) and Proposition 2.1.11 that

$$\begin{aligned} \mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{F}\big) &\cong \mathcal{M}\Big(\mathcal{K}\big((\mathcal{E} \otimes A) \oplus (B \otimes A), (\mathcal{E} \otimes \mathcal{F}) \oplus (B \otimes \mathcal{F})\big)\Big) \\ &\cong \mathcal{L}\big((\mathcal{E} \otimes A) \oplus (B \otimes A), (\mathcal{E} \otimes \mathcal{F}) \oplus (B \otimes \mathcal{F})\big) \\ &\cong \begin{pmatrix} \mathcal{L}(\mathcal{E} \otimes A, \mathcal{E} \otimes \mathcal{F}) & \mathcal{L}(B \otimes A, \mathcal{E} \otimes \mathcal{F}) \\ \mathcal{L}(\mathcal{E} \otimes A, B \otimes \mathcal{F}) & \mathcal{L}(B \otimes A, B \otimes \mathcal{F}) \end{pmatrix}. \end{aligned}$$

2.4 Weight theory

It will turn out that one of the main tools in this work is weight theory. In this section, we recall some basic notions. We refer to [42] for details.

Definition 2.4.1. Let $\mathcal{G}$ be a C^*-algebra. A *weight* on $\mathcal{G}$ is a map $\varphi : \mathcal{G}^+ \to [0, \infty]$ such that

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathcal{G}^+$, and

(ii) $\varphi(rx) = r\varphi(x)$ for all $x \in \mathcal{G}^+$, $r \in \mathbb{R}^+$ (here we use the convention $0 \cdot \infty = 0$).

The weight φ is called *faithful* if for every $x \in \mathcal{G}$, $\varphi(x^*x) = 0$ implies $x = 0$. For a weight φ we use the following standard notations

$$\mathcal{M}_\varphi^+ := \{x \in \mathcal{G}^+ : \varphi(x) < \infty\}, \quad \mathcal{M}_\varphi := \operatorname{span} \mathcal{M}_\varphi^+, \quad \mathcal{N}_\varphi := \{x \in \mathcal{G} : x^*x \in \mathcal{M}_\varphi^+\}.$$

Then $\mathcal{M}_\varphi$ is a $*$-subalgebra of $\mathcal{G}$, $\mathcal{N}_\varphi$ is a left ideal of $\mathcal{G}$ and $\mathcal{M}_\varphi$ is the linear span of $\mathcal{N}_\varphi^*\mathcal{N}_\varphi$. Moreover, $\mathcal{M}_\varphi^+$ is a *hereditary cone* [1] in $\mathcal{G}^+$ and $\mathcal{M}_\varphi^+ = \mathcal{G}^+ \cap \mathcal{M}_\varphi$. We have

$$\mathcal{M}_\varphi^+ \text{ is dense in } \mathcal{G}^+ \iff \mathcal{M}_\varphi \text{ is dense in } \mathcal{G} \iff \mathcal{N}_\varphi \text{ is dense in } \mathcal{G}.$$

If the equivalent conditions above are satisfied, then we say that φ is *densely defined*. We also denote by φ the unique linear extension of φ to $\mathcal{M}_\varphi$. We say that φ is *lower semi-continuous* if $\{x \in \mathcal{G}^+ : \varphi(x) \leq c\}$ is closed for all $c \in \mathbb{R}^+$ or, equivalently, for every net (x_i) in A^+ and $x \in A^+$, $x_i \to x$ implies $\varphi(x) \leq \liminf\big(\varphi(x_i)\big)$.

[1] This means that if $x, y \in \mathcal{M}_\varphi^+$ and $r \in \mathbb{R}^+$, then $x + ry \in \mathcal{M}_\varphi^+$ (that is, $\mathcal{M}_\varphi^+$ is a *cone*) and if $x \in \mathcal{G}^+$, $y \in \mathcal{M}_\varphi^+$ and if $x \leq y$, then $x \in \mathcal{M}_\varphi^+$ (that is, $\mathcal{M}_\varphi^+$ is *hereditary*).

A weight φ is called *proper* if it is non-zero, densely defined and lower semi-continuous. We only consider proper weights in this work.

Define the sets

$$\mathcal{F}_\varphi := \{\omega \in \mathcal{G}_+^* : \omega(x) \leq \varphi(x) \text{ for all } x \in \mathcal{G}^+\}$$

and

$$\mathcal{G}_\varphi := \{\lambda\omega : \omega \in \mathcal{F}_\varphi, \lambda \in (0,1)\} \subseteq \mathcal{F}_\varphi.$$

If we endow $\mathcal{F}_\varphi$ with the natural order of $\mathcal{G}_+^*$ then $\mathcal{G}_\varphi$ is a directed subset of $\mathcal{F}_\varphi$, so that $\mathcal{G}_\varphi$ can be used as the index set of a net. If φ is lower semi-continuous, we have ([42, Theorem 1.6])

$$\varphi(x) = \sup\{\omega(x) : \omega \in \mathcal{F}_\varphi\} \quad \text{for all } x \in \mathcal{G}^+. \tag{2.1}$$

It follows that

$$\varphi(x) = \lim_{\omega \in \mathcal{G}_\varphi} \omega(x) \quad \text{for all } x \in \mathcal{M}_\varphi. \tag{2.2}$$

The weight φ can be naturally extended to the multiplier algebra $\mathcal{M}(\mathcal{G})$ by setting

$$\bar{\varphi}(x) := \sup\{\omega(x) : \omega \in \mathcal{F}_\varphi\} \quad \text{for all } x \in \mathcal{M}(\mathcal{G})^+, \tag{2.3}$$

where each $\omega \in \mathcal{G}^*$ is extended to $\mathcal{M}(\mathcal{G})$ as usual. Then $\bar{\varphi}$ is the unique strictly lower semi-continuous weight on $\mathcal{M}(\mathcal{G})$ extending φ. We shall also denote the extension $\bar{\varphi}$ by φ and use the notations $\bar{\mathcal{M}}_\varphi^+ = \mathcal{M}_{\bar{\varphi}}^+$, $\bar{\mathcal{M}}_\varphi = \mathcal{M}_{\bar{\varphi}}$ and $\bar{\mathcal{N}}_\varphi = \mathcal{N}_{\bar{\varphi}}$. Equation (2.2) can be generalized:

$$\varphi(x) = \lim_{\omega \in \mathcal{G}_\varphi} \omega(x) \quad \text{for all } x \in \bar{\mathcal{M}}_\varphi. \tag{2.4}$$

2.4.1 Slice maps

Let A and $\mathcal{G}$ be arbitrary C^*-algebras. Given $\omega \in \mathcal{G}^*$ there is a unique bounded linear *slice map* $\mathrm{id}_A \otimes \omega : A \otimes \mathcal{G} \to A$ satisfying $(\mathrm{id}_A \otimes \omega)(a \otimes b) = a\omega(b)$ for all $a \in A$ and $b \in B$. It can be extended to a strictly continuous linear map $\mathcal{M}(A \otimes \mathcal{G}) \to \mathcal{M}(A)$ which is also denoted by $\mathrm{id}_A \otimes \omega$ (see, for example, [15] for further details). This can be generalized to weights as follows. Let φ be a proper weight on $\mathcal{G}$. We know that $\mathcal{G}_\varphi$ is a directed set and hence, given $x \in \mathcal{M}(A \otimes \mathcal{G})$, we can consider the net $\big((\mathrm{id}_A \otimes \omega)(x)\big)_{\omega \in \mathcal{G}_\varphi}$ in $\mathcal{M}(A)$. We define the following set

$$\bar{\mathcal{M}}_{\mathrm{id}_A \otimes \varphi}^+ := \{x \in \mathcal{M}(A \otimes \mathcal{G})^+ : \big((\mathrm{id}_A \otimes \omega)(x)\big)_{\omega \in \mathcal{G}_\varphi} \text{ converges strictly in } \mathcal{M}(A)\}.$$

For $x \in \bar{\mathcal{M}}_{\mathrm{id}_A \otimes \varphi}^+$ we define $(\mathrm{id}_A \otimes \varphi)(x) := \operatorname*{s\text{-}lim}_{\omega \in \mathcal{G}_\varphi} (\mathrm{id}_A \otimes \omega)(x)$, where the script "s" stands for strict limit. We list some properties of the slice map $\mathrm{id}_A \otimes \varphi$ (see [42, Result 3.6]):

1. for $x, y \in \bar{\mathcal{M}}_{\mathrm{id}_A \otimes \varphi}^+$ and $c \in \mathbb{R}^+$, we have $x + cy \in \bar{\mathcal{M}}_{\mathrm{id}_A \otimes \varphi}^+$ and

$$(\mathrm{id}_A \otimes \varphi)(x + cy) = (\mathrm{id}_A \otimes \varphi)(x) + c(\mathrm{id}_A \otimes \varphi)(y),$$

2. if $y \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$ and $x \in \mathcal{M}(A\otimes\mathcal{G})^+$ and if $x \leq y$, then $x \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$ and
$$(\mathrm{id}_A\otimes\varphi)(x) \leq (\mathrm{id}_A\otimes\varphi)(y),$$
3. for $a \in \mathcal{M}(A)^+$ and $b \in \bar{\mathcal{M}}^+_\varphi$, we have $a\otimes b \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$ and $(\mathrm{id}_A\otimes\varphi)(a\otimes b) = a\varphi(b)$.

Finally, we define $\bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi} := \operatorname{span}\bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$ and also denote by $\mathrm{id}_A\otimes\varphi$ the unique linear extension of $\mathrm{id}_A\otimes\varphi$ to $\bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$. We also define
$$\bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi} := \{x \in \mathcal{M}(A\otimes\mathcal{G}) : x^*x \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}\}.$$
So like before, $\bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi} = \bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}\cap\mathcal{M}(A\otimes\mathcal{G})^+$ is a hereditary cone in $\mathcal{M}(A\otimes\mathcal{G})^+$, $\bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$ is a hereditary $*$-subalgebra of $\mathcal{M}(A\otimes\mathcal{G})$, $\bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi}$ is a left ideal in $\mathcal{M}(A\otimes\mathcal{G})$ and $\bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi} = \bar{\mathcal{N}}^*_{\mathrm{id}_A\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi}$.

Notice that, given $x \in \mathcal{M}(A\otimes\mathcal{G})^+$, the net $\big((\mathrm{id}_A\otimes\omega)(x)\big)_{\omega\in\mathcal{G}_\varphi}$ is increasing in $\mathcal{M}(A)^+$. The following two lemmas characterize strict convergence of such nets.

Lemma 2.4.2 (Result 3.4 in [42])**.** *Let A be a C^*-algebra, and suppose that (a_i) is an increasing net in $\mathcal{M}(A)^+$. Then (a_i) converges strictly if and only if the net (b^*a_ib) converges in norm for all $b \in A$.*

Lemma 2.4.3 (Lemma 3.12 in [42])**.** *Let (a_i) be an increasing net in $\mathcal{M}(A)^+$ and suppose that $a \in \mathcal{M}(A)^+$. Then (a_i) converges strictly to a if and only if $\theta(a_i)$ converges to $\theta(a)$ for all $\theta \in A^*_+$.*

Later we shall also need the following slight modification of Lemma 2.4.2:

Lemma 2.4.4. *Let $\mathcal{E}$ be a Hilbert B-module. Denote $A = \mathcal{K}(\mathcal{E})$ and identify $\mathcal{M}(A) \cong \mathcal{L}(\mathcal{E})$. Let (T_i) be an increasing net in $\mathcal{M}(A)^+$. Then (T_i) converges strictly in $\mathcal{M}(A)$ if and only if $\langle\eta|T_i\eta\rangle$ converges in B (in norm) for every $\eta \in \mathcal{E}$.*

Proof. The proof is very similar to the one in [42, Result 3.4] (that is, Lemma 2.4.2) which is exactly this lemma in the case $\mathcal{E}$ is itself a C^*-algebra. If (T_i) converges strictly in $\mathcal{M}(A)$, then $\langle\eta|T_i\eta\rangle$ converges in B for every $\eta \in \mathcal{E}$ because $\mathcal{E} = \mathcal{K}(\mathcal{E})\cdot\mathcal{E}$. Suppose conversely that $\langle\eta|T_i\eta\rangle$ converges in A for every $\eta \in \mathcal{E}$. Since (T_i) is an increasing net, the same argument of [42, 3.4] using the uniform boundedness principle shows that (T_i) is bounded. Let $M > 0$ such that $\|T_i\| \leq M$. If $\eta,\zeta \in \mathcal{E}$ and $S := |\eta\rangle\langle\zeta|$ then $S^*T_iS = |\zeta\langle\eta|T_i\eta\rangle\rangle\langle\zeta|$, so that (S^*T_iS) converges for every $S \in \operatorname{span}|\mathcal{E}\rangle\langle\mathcal{E}|$. From [42, 3.2] we have
$$\|T_iS - T_jS\|^2 \leq \|T_i - T_j\|\|S^*(T_i - T_j)S\| \leq 2M\|S^*(T_i - T_j)S\|.$$
It follows that (T_iS) is Cauchy, and therefore convergent, for all $S \in \operatorname{span}|\mathcal{E}\rangle\langle\mathcal{E}|$. Since (T_i) is bounded we get that (T_iS) converges for all $S \in A$. Define $T : A \to A$, $T(S) = \lim T_iS$. The same argument of [42, 3.4] shows that $T \in \mathcal{M}(A)$ and T_i converges strictly to T. $\square$

Using the lemmas above one can prove the following useful result (Propositions 3.9 and 3.14 in [42]).

Proposition 2.4.5. *Let $x \in \mathcal{M}(A\otimes\mathcal{G})^+$. Then $x \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$ if and only if there is $a \in \mathcal{M}(A)^+$ such that $(\theta\otimes\mathrm{id}_\mathcal{G})(x) \in \bar{\mathcal{M}}^+_\varphi$ and $\varphi\big((\theta\otimes\mathrm{id}_\mathcal{G})(x)\big) = \theta(a)$ for all $\theta \in A^*_+$. In this case,* $(\mathrm{id}_A\otimes\varphi)(x) = a$.

2.4.2 KSGNS-Constructions

Given a weight φ on a C^*-algebra $\mathcal{G}$, we can associate to it a GNS-construction (see [42, Definition 1.2]). It is by definition a triple (H, π, Λ) where H is a Hilbert space, $\Lambda : \mathcal{N}_\varphi \to H$ is a linear map with dense image such that

$$\langle \Lambda(a) | \Lambda(b) \rangle = \varphi(b^* a) \text{ for all } a, b \in \mathcal{N}_\varphi,$$

and π is a $*$-representation of $\mathcal{G}$ on H such that $\pi(a)\Lambda(b) = \Lambda(ab)$ for all $a \in \mathcal{G}$ and $b \in \mathcal{N}_\varphi$. A GNS-construction is unique up to unitary transformation.

If the weight φ is lower semi-continuous and (H, π, Λ) is a GNS-construction for φ, then Λ is a closed map and π is a nondegenerate $*$-representation of $\mathcal{G}$ on H and we have $\pi(a)\Lambda(b) = \Lambda(ab)$ for all $a \in \mathcal{M}(\mathcal{G})$ and $b \in \mathcal{N}_\varphi$ (see [42, Result 2.3]). Moreover, the map $\Lambda : \mathcal{N}_\varphi \to H$ is strictly closable and if we denote by $\bar{\Lambda} : \mathcal{D}(\bar{\Lambda}) \to H$ its strict closure, then $\mathcal{D}(\bar{\Lambda}) = \bar{\mathcal{N}}_\varphi$ and $(H, \pi, \bar{\Lambda})$ is a GNS-construction for the extension of φ to the multiplier algebra $\mathcal{M}(\mathcal{G})$ (here $\pi : \mathcal{M}(\mathcal{G}) \to \mathcal{L}(H)$ denotes the strict extension of π) (see [38, Proposition 2.6]). We put $\Lambda(a) := \bar{\Lambda}(a)$ for all $a \in \bar{\mathcal{N}}_\varphi$, that is, we use the same symbol for the strict extension of Λ.

Given a C^*-algebra A, there is also a sort of KSGNS-construction[2] for the "C^*-valued weight" $\mathrm{id}_A \otimes \varphi$, which we describe in the next proposition (see [42, 3.18, 3.23, 3.27] for the proof). This result will be one of the main tools in this work.

Proposition 2.4.6. *Let A and $\mathcal{G}$ be C^*-algebras, let φ be a proper weight on $\mathcal{G}$ and let (H, π, H) be a GNS-construction for φ. Then there is a unique linear map*

$$\mathrm{id}_A \otimes \Lambda : \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi} \to \mathcal{M}(A \otimes H) = \mathcal{L}(A, A \otimes H)$$

such that $(\mathrm{id}_A \otimes \Lambda)(x)^* \big(b \otimes \Lambda(s)\big) = (\mathrm{id}_A \otimes \varphi)\big(x^*(b \otimes s)\big)$ for all $x \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$, $b \in A$ and $s \in \mathcal{N}_\varphi$. One has the following properties:

(i) $(\mathrm{id}_A \otimes \Lambda)(y)^* (\mathrm{id}_A \otimes \Lambda)(x) = (\mathrm{id}_A \otimes \varphi)(y^* x)$ *for all* $x, y \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$*;*

(ii) $(\mathrm{id}_A \otimes \Lambda)(b \otimes s) = b \otimes \Lambda(s)$ *for all* $b \in \mathcal{M}(A)$ *and* $s \in \bar{\mathcal{N}}_\varphi$*;*

(iii) $(\mathrm{id}_A \otimes \Lambda)(xy) = (\mathrm{id}_A \otimes \pi)(x)(\mathrm{id}_A \otimes \Lambda)(y)$ *for all* $x \in \mathcal{M}(A \otimes \mathcal{G})$ *and* $y \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$*;*

(iv) $x(b \otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$ *for all* $x \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$ *and* $b \in \mathcal{M}(A)$*, and*

$$(\mathrm{id}_A \otimes \Lambda)\big(x(b \otimes 1)\big) = (\mathrm{id}_A \otimes \Lambda)(x) b; \text{ and}$$

(v) $\mathrm{id}_A \otimes \Lambda$ *is closed for the strict topology of* $\mathcal{M}(A \otimes \mathcal{G})$ *and the strong topology of* $\mathcal{M}(A \otimes H) = \mathcal{L}(A, A \otimes H)$*.*

Remark 2.4.7. The converse of (iv) is also true: if $x \in \mathcal{M}(A \otimes \mathcal{G})$ and $x(b \otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$ for all $b \in A$, then $x \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$. In fact,

$$x(b \otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi} \iff (b^* \otimes 1) x^* x (b \otimes 1) \in \bar{\mathcal{M}}^+_{\mathrm{id}_A \otimes \varphi} \iff$$

[2] KSGNS stands for Kasparov, Stinespring, Gelfand, Naimark and Segal.

$$\exists\,\text{s-}\lim_{\omega\in\mathcal{G}_\varphi}(\mathrm{id}_A\otimes\omega)\big((b^*\otimes 1)x^*x(b\otimes 1)\big)=\lim_{\omega\in\mathcal{G}_\varphi}b^*(\mathrm{id}_A\otimes\omega)(x^*x)b.$$

Since $b\in A$ is arbitrary, the result follows from Lemma 2.4.4.

We shall need the following result of [42, Proposition 3.38].

Lemma 2.4.8. *Let A and B be C^*-algebras and suppose that $\rho : A\to\mathcal{M}(B)$ is a strict completely positive mapping* (see [43])*. Let φ be a proper weight on $\mathcal{G}$ with GNS-construction (H,π,Λ). Then the following holds*:

(i) *For all $x\in\bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$, the element $(\rho\otimes\mathrm{id}_{\mathcal{G}})(x)$ belongs to $\bar{\mathcal{M}}_{\mathrm{id}_B\otimes\varphi}$ and*

$$(\mathrm{id}_B\otimes\varphi)(\rho\otimes\mathrm{id}_{\mathcal{G}})(x)=\rho(\mathrm{id}_A\otimes\varphi)(x).$$

(ii) *For all $x\in\bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi}$, the element $(\rho\otimes\mathrm{id}_{\mathcal{G}})(x)$ belongs to $\bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ and*

$$(1_B\otimes v^*)(\mathrm{id}_B\otimes\Lambda)\big((\rho\otimes\mathrm{id}_{\mathcal{G}})(x)\big)=\rho\big((1_A\otimes v^*)(\mathrm{id}_A\otimes\Lambda)(x)\big)\quad\textit{for all } v\in H.^3$$

Lemma 2.4.8 can be applied to nondegenerate $*$-homomorphisms and to bounded functionals (because these are linear combinations of states). In fact, these are the only cases we are going to use.

2.4.3 KMS-Weights

The idea behind KMS-weights is to control the non-commutativity of the underlying algebra. We discuss some basic properties of KMS-weights.[4] We refer to [36] for full details.

Definition 2.4.9. Let $\mathcal{G}$ be a C^*-algebra and let φ be a proper weight on $\mathcal{G}$. Suppose that there is a (continuous) one-parameter group of automorphisms $\sigma=(\sigma_t)_{t\in\mathbb{R}}$, that is, an action of $\mathbb{R}$ on $\mathcal{G}$, such that

(i) $\varphi\big(\sigma_t(x)\big)=\varphi(x)$ for all $x\in\mathcal{G}^+$, that is, φ is invariant under σ, and

(ii) for all $x\in\mathcal{D}(\sigma_{\frac{\mathrm{i}}{2}})$ we have $\varphi(x^*x)=\varphi\big(\sigma_{\frac{\mathrm{i}}{2}}(x)\sigma_{\frac{\mathrm{i}}{2}}(x)^*\big)$.

Then φ is called a *KMS-weight* and σ is called a *modular group* for φ.

If φ is faithful, then the modular group is uniquely determined. We have used above the analytic extension of σ. By definition, for each $z\in\mathbb{C}$, σ_z is the closed densely defined operator in $\mathcal{G}$ whose domain $\mathcal{D}(\sigma_z)$ is the set of elements $a\in\mathcal{G}$ such that there is a function $f: S(z)\to\mathcal{G}$, where $S(z)$ denotes the horizontal strip $\{y\in\mathbb{C}: 0\le\operatorname{Im}y\le\operatorname{Im}z\}$, satisfying

(i) f is continuous on $S(z)$,

[3] Here we identify $H\cong\mathcal{L}(\mathbb{C},H)$, so that any $v\in H$ is seen as an operator $v\in\mathcal{L}(\mathbb{C},H)$, $v(z)=v\cdot z\in H$ for all $z\in\mathbb{C}$. Its adjoint $v^*: H\to\mathbb{C}$ is given by $v^*(w)=\langle v,w\rangle$, where we suppose that the inner product $\langle\cdot,\cdot\rangle$ is linear in the second variable.

[4] The expression KMS refers to the mathematicians Kubo, Martin and Schwinger.

(ii) f is analytic on $S(z)^0$, the interior of $S(z)$ and

(iii) $f(x) = \sigma_x(a)$ for all $x \in \mathbb{R}$.

Then, by definition, we have $\sigma_z(a) = f(z)$. An element $a \in \mathcal{G}$ is called *analytic* with respect to σ if $a \in \mathcal{D}(\sigma_z)$ for all $z \in \mathbb{C}$. If a is analytic then the function $z \mapsto \sigma_z(a)$ is an analytic function $\mathbb{C} \to \mathcal{G}$.

The space of all analytic elements with respect to σ is dense in $\mathcal{G}$. Moreover, given any $a \in \mathcal{G}$ the element

$$a_n := \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2 t^2) \sigma_t(a)\, \mathrm{d}t$$

is analytic for all $n \in \mathbb{N}$ and the sequence (a_n) converges to a. Using similar ideas one can also prove the following result (see [36] for details).

Lemma 2.4.10. *Suppose that φ is a KMS-weight on $\mathcal{G}$ with a modular group $\sigma = \{\sigma_t\}_{t\in\mathbb{R}}$. There exists a bounded net $\{e_j\}$ in $\mathcal{N}_\varphi$ of analytic elements with respect to σ, such that $\sigma_z(e_j) \to 1$ strictly for all $z \in \mathbb{C}$.*

In connection with KMS-weights we introduce the *Tomita $*$-algebra* as follows:

$$\begin{aligned} \mathcal{T}_\varphi := \{x \in \mathcal{G} : x \text{ is analytic with respect to } \sigma \\ \text{and } \sigma_z(x) \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*, \text{ for all } z \in \mathbb{C}\}, \end{aligned} \tag{2.5}$$

where φ a KMS-weight and $\{\sigma_t\}_{t\in\mathbb{R}}$ is a modular group of φ. It can be proved that $\mathcal{T}_\varphi$ is, in fact, a dense $*$-subalgebra of $\mathcal{G}$ ([38]). The Tomita $*$-algebra is very useful in technical situations.

Recall that a *core* of a closed (unbounded) linear map T is a subspace C of the domain $\mathcal{D}(T)$ of T such that the closure of the restriction $T|_C$ is equal to T.

Lemma 2.4.11 (Lemma 5.13 in [38])**.** *The Tomita $*$-algebra $\mathcal{T}_\varphi$ is a core for Λ.*

The next result (Proposition 1.12 in [42]) gives some basic properties of KMS-weights.

Proposition 2.4.12. *Let φ be a KMS-weight on a C^*-algebra $\mathcal{G}$ with a modular group σ. Let (H, π, Λ) be a GNS-construction for φ. Then the following properties hold:*

(i) *There is a unique anti-unitary operator J on H satisfying $J\Lambda(x) = \Lambda\big(\sigma_{\frac{\mathrm{i}}{2}}(x)^*\big)$ for all $x \in \mathcal{N}_\varphi \cap \mathcal{D}(\sigma_{\frac{\mathrm{i}}{2}})$.*

(ii) *If $x \in \mathcal{N}_\varphi$ and $a \in \mathcal{D}(\sigma_{\frac{\mathrm{i}}{2}})$, then $xa \in \mathcal{N}_\varphi$ and $\Lambda(xa) = J\pi\big(\sigma_{\frac{\mathrm{i}}{2}}(a)\big)^* J\Lambda(x)$.*

(iii) *If $x \in \mathcal{M}_\varphi$ and $a \in \mathcal{D}(\sigma_{-\mathrm{i}})$, then ax and $x\sigma_{-\mathrm{i}}(a)$ belong to $\mathcal{M}_\varphi$ and*

$$\varphi(ax) = \varphi\big(x\sigma_{-\mathrm{i}}(a)\big).$$

The anti-unitary operator J above is called the *modular conjugation* of φ in the GNS-construction (H, π, Λ). There is a strictly positive operator ∇ on H satisfying $\nabla^{it}\big(\Lambda(a)\big) = \Lambda\big(\sigma_t(a)\big)$ for all $a \in \mathcal{N}_\varphi$ and $t \in \mathbb{R}$. The operator ∇ is called the *modular operator* of φ in the GNS-construction (H, π, Λ). The modular conjugation and the modular operator do not depend on σ, but only on φ. In fact, there is a densely defined operator T on H such that $\Lambda(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*)$ is a core for T and $T(\Lambda(a) = \Lambda(a^*)$ for all $a \in \mathcal{N}_\varphi \cap \mathcal{N}_f^*$. We have $\nabla = T^*T$ and $T = J\nabla^{\frac{1}{2}} = \nabla^{-\frac{1}{2}}J$.

The next result shows that the properties above can be extended to slice maps with φ (see [42, Proposition 3.28]).

Proposition 2.4.13. *Let A be any C^*-algebra. Let J be the modular conjugation of φ in the GNS-construction (H, π, Λ).*

(i) *If $x \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$ and $a \in \mathcal{D}(\sigma_{\frac{\mathrm{i}}{2}})$, then $x(1 \otimes a) \in \bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$ and*

$$(\mathrm{id}_A \otimes \Lambda)\big(x(1 \otimes a)\big) = \big(1 \otimes J\pi\big(\sigma_{\frac{\mathrm{i}}{2}}(a)\big)^* J\big)(\mathrm{id}_A \otimes \Lambda)(x).$$

(ii) *If $x \in \bar{\mathcal{M}}_{\mathrm{id}_A \otimes \varphi}$ and $a \in \mathcal{D}(\sigma_{-\mathrm{i}})$, then $(1 \otimes a)x$ and $x\big(1 \otimes \sigma_{-\mathrm{i}}(a)\big)$ belong to $\bar{\mathcal{M}}_{\mathrm{id}_A \otimes \varphi}$ and*

$$(\mathrm{id}_A \otimes \varphi)\big((1 \otimes a)x\big) = (\mathrm{id}_A \otimes \varphi)\big(x\big(1 \otimes \sigma_{-\mathrm{i}}(a)\big)\big).$$

2.4.4 Generalized KSGNS-constructions

Later we shall need a generalization of the KSGNS-construction in Proposition 2.4.6 to the context of Hilbert modules. We develop this generalization in this section using linking algebra techniques.

Given a Hilbert B-module $\mathcal{E}$ and a C^*-algebra $\mathcal{G}$, we are going to see that there exists a slice map $\mathrm{id}_\mathcal{E} \otimes \omega : \mathcal{E} \otimes \mathcal{G} \to \mathcal{E}$ (which is a bounded linear map satisfying $(\mathrm{id}_\mathcal{E} \otimes \omega)(\xi \otimes s) = \xi\omega(s)$ for all $\xi \in \mathcal{E}$ and $s \in \mathcal{G}$) for any $\omega \in \mathcal{G}^*$. As for C^*-algebras one can also extend $\mathrm{id}_\mathcal{E} \otimes \omega$ continuously with respect to the strict topology to get a map $\mathrm{id}_\mathcal{E} \otimes \omega : \mathcal{M}(\mathcal{E} \otimes \mathcal{G}) \to \mathcal{M}(\mathcal{E})$. So we can try to imitate the case of C^*-algebras to define the slice map $\mathrm{id}_\mathcal{E} \otimes \varphi$, for a given proper weight φ on $\mathcal{G}$. We have to take care here because it does not make sense to speak of positive elements in a Hilbert module, so that we cannot define a set like $\bar{\mathcal{M}}^+_{\mathrm{id}_\mathcal{E} \otimes \varphi}$. However, we are going to see that we can define a set like $\bar{\mathcal{M}}_{\mathrm{id}_\mathcal{E} \otimes \varphi}$ using the linking algebra of $\mathcal{E}$.

Recall that the linking algebra of $\mathcal{E}$, denoted by $L(\mathcal{E})$, is the C^*-algebra of all 2×2 matrices of the form

$$\begin{pmatrix} T & \eta \\ \zeta^* & c \end{pmatrix}, \qquad \text{where } T \in \mathcal{K}(\mathcal{E}),\ \eta, \zeta \in \mathcal{E},\ c \in B,$$

with natural operations and norm.

By Remark 2.3.3, we have canonical identifications

$$L(\mathcal{E}) \otimes \mathcal{G} \cong \begin{pmatrix} \mathcal{K}(\mathcal{E}) \otimes \mathcal{G} & \mathcal{E} \otimes \mathcal{G} \\ \mathcal{E}^* \otimes \mathcal{G} & B \otimes \mathcal{G} \end{pmatrix}.$$

and

$$\mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{G}\big) \cong \begin{pmatrix} \mathcal{L}(\mathcal{E} \otimes \mathcal{G}) & \mathcal{L}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G}) \\ \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G}) & \mathcal{L}(B \otimes \mathcal{G}) \end{pmatrix},$$

in such a way that $\begin{pmatrix} T & \eta \\ \zeta^* & c \end{pmatrix} \otimes r$ is identified with $\begin{pmatrix} T \otimes r & \eta \otimes r \\ \zeta^* \otimes r & c \otimes r \end{pmatrix}$ for all $T \in \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$, $\eta, \zeta \in \mathcal{M}(\mathcal{E})$, $c \in \mathcal{M}(B)$, and $r \in \mathcal{M}(\mathcal{G})$. Using these identifications we define for each $\omega \in \mathcal{G}^*$ a slice map $\mathrm{id}_{\mathcal{E}} \otimes \omega : \mathcal{M}(\mathcal{E} \otimes \mathcal{G}) \to \mathcal{M}(\mathcal{E})$ by the equation

$$(\mathrm{id}_{L(\mathcal{E})} \otimes \omega) \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\mathrm{id}_{\mathcal{E}} \otimes \omega)(x) \\ 0 & 0 \end{pmatrix}.$$

Note that, if $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are the corner projections in $\mathcal{M}\big(L(\mathcal{E})\big)$, then we have for all $x \in \mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{G}\big)$

$$p(\mathrm{id}_{L(\mathcal{E})} \otimes \omega)(x)q = (\mathrm{id}_{L(\mathcal{E})} \otimes \omega)\big((p \otimes 1)x(q \otimes 1)\big).$$

In particular,

$$p(\mathrm{id}_{L(\mathcal{E})} \otimes \omega) \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} q = (\mathrm{id}_{L(\mathcal{E})} \otimes \omega) \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}.$$

Thus $\mathrm{id}_{\mathcal{E}} \otimes \omega$ is well-defined. Analogously, we can define a slice map

$$\mathrm{id}_{\mathcal{E}^*} \otimes \omega : \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G}) \to \mathcal{L}(\mathcal{E}, B)$$

by the equation

$$(\mathrm{id}_{L(\mathcal{E})} \otimes \omega) \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{E}^*} \otimes \omega)(y) & 0 \end{pmatrix}.$$

Again using that $q(\mathrm{id}_{L(\mathcal{E})} \otimes \omega)(x)p = (\mathrm{id}_{L(\mathcal{E})} \otimes \omega)\big((q \otimes 1)x(p \otimes 1)\big)$ for all $x \in \mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{G}\big)$, we see that $\mathrm{id}_{\mathcal{E}^*} \otimes \omega$ is well-defined. Moreover, using also the relations

$$p(\mathrm{id}_{L(\mathcal{E})} \otimes \omega)(x)p = (\mathrm{id}_{L(\mathcal{E})} \otimes \omega)\big((p \otimes 1)x(p \otimes 1)\big),$$
$$q(\mathrm{id}_{L(\mathcal{E})} \otimes \omega)(x)q = (\mathrm{id}_{L(\mathcal{E})} \otimes \omega)\big((q \otimes 1)x(q \otimes 1)\big)$$

for all $x \in \mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{G}\big)$, we get

$$(\mathrm{id}_{L(\mathcal{E})} \otimes \omega) \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \omega)(x_1) & (\mathrm{id}_{\mathcal{E}} \otimes \omega)(x_2) \\ (\mathrm{id}_{\mathcal{E}^*} \otimes \omega)(x_3) & (\mathrm{id}_{B} \otimes \omega)(x_4) \end{pmatrix}, \tag{2.6}$$

for all $x_1 \in \mathcal{M}\big(\mathcal{K}(\mathcal{E}) \otimes \mathcal{G}\big)$, $x_2 \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$, $x_3 \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G})$ and $x_4 \in \mathcal{M}(B \otimes \mathcal{G})$.

Several properties of the slice maps $\mathrm{id}_{\mathcal{E}} \otimes \omega$, $\omega \in \mathcal{G}^*$, can be derived from the corresponding properties of $\mathrm{id}_{L(\mathcal{E})} \otimes \omega$.

Proposition 2.4.14. *Let $\mathcal{E}$ be a Hilbert B-module, and let $\omega \in \mathcal{G}^*$. Then the slice map $\mathrm{id}_{\mathcal{E}} \otimes \omega : \mathcal{M}(\mathcal{E} \otimes \mathcal{G}) \to \mathcal{M}(\mathcal{E})$ is a strictly continuous bounded linear map with $\|\mathrm{id}_{\mathcal{E}} \otimes \omega\| \leq \|\omega\|$. It restricts to a bounded linear map $\mathrm{id}_{\mathcal{E}} \otimes \omega : \mathcal{E} \otimes \mathcal{G} \to \mathcal{E}$. Moreover, we have $(\mathrm{id}_{\mathcal{E}} \otimes \omega)\big(\tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G})\big) \subseteq \mathcal{E}$ and hence $\mathrm{id}_{\mathcal{E}} \otimes \omega$ restricts to a map $\mathrm{id}_{\mathcal{E}} \otimes \omega : \tilde{\mathcal{M}}(\mathcal{E} \otimes \mathcal{G}) \to \mathcal{E}$ which is $\mathcal{G}$-strict to norm continuous. The following properties hold:*

(i) *If $\xi \in \mathcal{M}(\mathcal{E})$ and $a \in \mathcal{M}(\mathcal{G})$, then* $(\mathrm{id}_{\mathcal{E}} \otimes \omega)(\xi \otimes a) = \xi\omega(a)$.

(ii) *For $a \in \mathcal{G}$ define $a\omega, \omega a \in \mathcal{G}^*$ by $(a\omega)(b) := \omega(ba)$ and $(\omega a)(b) := \omega(ab)$ for all $b \in \mathcal{G}$. Then we have for all $a \in \mathcal{G}$ and $x \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$*

$$(\mathrm{id}_{\mathcal{E}} \otimes a\omega)(x) = (\mathrm{id}_{\mathcal{E}} \otimes \omega)\big(x(1 \otimes a)\big), \quad (\mathrm{id}_{\mathcal{E}} \otimes a\omega)(x) = (\mathrm{id}_{\mathcal{E}} \otimes \omega)\big((1 \otimes a)x\big).$$

(iii) *Let $\mathcal{F}$ be another Hilbert B-module. If $R \in \mathcal{L}(\mathcal{F}, B)$ and $x \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$, then*

$$(\mathrm{id}_{\mathcal{E}} \otimes \omega)(x)R = (\mathrm{id}_{\mathcal{X}} \otimes \omega)\big(x(R \otimes 1)\big),$$

where $\mathcal{X} := \mathcal{K}(\mathcal{F}, \mathcal{E})$ considered as a Hilbert $\mathcal{K}(\mathcal{E}), \mathcal{K}(\mathcal{F})$-bimodule, and we identify $x(R \otimes 1) \in \mathcal{L}(\mathcal{F} \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G}) \hookrightarrow \mathcal{M}\big(\mathcal{K}(\mathcal{F} \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G})\big) \cong \mathcal{M}(\mathcal{X} \otimes \mathcal{G})$ as in Proposition 2.1.8.

In particular, if $b \in \mathcal{M}(B)$ and $x \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ then

$$(\mathrm{id}_{\mathcal{E}} \otimes \omega)(x)b = (\mathrm{id}_{\mathcal{E}} \otimes \omega)\big(x(b \otimes 1)\big),$$

and if $T \in \mathcal{L}(\mathcal{E}, B)$ and $x \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$, then

$$(\mathrm{id}_{\mathcal{E}} \otimes \omega)(x)T = (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \omega)\big(x(T \otimes 1)\big).$$

(iv) *If $R \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ and $x \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$, then*

$$R(\mathrm{id}_{\mathcal{E}} \otimes \omega)(x) = (\mathrm{id}_{\mathcal{F}} \otimes \omega)\big((R \otimes 1)x\big).$$

In particular, if $S \in \mathcal{L}(\mathcal{E})$ and $x \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ then

$$S(\mathrm{id}_{\mathcal{E}} \otimes \omega)(x) = (\mathrm{id}_{\mathcal{E}} \otimes \omega)\big((S \otimes 1)x\big),$$

and if $T \in \mathcal{L}(\mathcal{E}, B)$ and $x \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$, then

$$T(\mathrm{id}_{\mathcal{E}} \otimes \omega)(x) = (\mathrm{id}_{B} \otimes \omega)\big((T \otimes 1)x\big).$$

(v) *If $\mathcal{E}$ is a right-Hilbert A, B-bimodule, where A is some C^*-algebra, and hence $\mathcal{E} \otimes \mathcal{G}$ is a right-Hilbert $A \otimes \mathcal{G}, B \otimes \mathcal{G}$-bimodule, $\mathcal{M}(\mathcal{E})$ is a right-Hilbert $\mathcal{M}(A), \mathcal{M}(B)$-bimodule, and $\mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ is a right-Hilbert $\mathcal{M}(A \otimes \mathcal{G}), \mathcal{M}(B \otimes \mathcal{G})$-bimodule, and if we denote all the left module actions by $\cdot$, then for all $a \in \mathcal{M}(A)$, and $x \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ we have*

$$a \cdot (\mathrm{id}_{\mathcal{E}} \otimes \omega)(x) = (\mathrm{id}_{\mathcal{E}} \otimes \omega)\big((a \otimes 1) \cdot x\big).$$

Proof. Since the strict topology on $L\big(\mathcal{M}(\mathcal{E} \otimes \mathcal{G})\big)$ $\big($resp. $L\big(\mathcal{M}(\mathcal{E})\big)\big)$, when restricted to $\mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ (resp. $\mathcal{M}(\mathcal{E})$), coincides with the strict topology on $\mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ (resp. $\mathcal{M}(\mathcal{E})$), the strict continuity of $\mathrm{id}_{\mathcal{E}} \otimes \omega$ follows from the strict continuity of $\mathrm{id}_{L(\mathcal{E})} \otimes \omega$. Of course, $\mathrm{id}_{\mathcal{E}} \otimes \omega$ is linear. Since $(\mathrm{id}_{L(\mathcal{E})} \otimes \omega)\big(L(\mathcal{E}) \otimes \mathcal{G}\big) \subseteq L(\mathcal{E})$, it follows that $(\mathrm{id}_{\mathcal{E}} \otimes \omega)(\mathcal{E} \otimes \mathcal{G}) \subseteq \mathcal{E}$. Using that $\mathrm{id}_{L(\mathcal{E})} \otimes \omega : \mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{G}\big) \to \mathcal{M}\big(L(\mathcal{E})\big)$ is bounded with $\|\mathrm{id}_{L(\mathcal{E})} \otimes \omega\| \leq \|\omega\|$, and using that the natural inclusions of $\mathcal{M}(\mathcal{E})$ into $\mathcal{M}\big(L(\mathcal{E})\big)$ and $\mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ into $L\big(\mathcal{M}(\mathcal{E} \otimes \mathcal{G})\big) \cong$

$\mathcal{M}\big(L(\mathcal{E})\otimes\mathcal{G}\big)$ are isometric, it follows that $\mathrm{id}_\mathcal{E}\otimes\omega:\mathcal{M}(\mathcal{E}\otimes\mathcal{G})\to\mathcal{M}(\mathcal{E})$ is bounded with $\|\mathrm{id}_\mathcal{E}\otimes\omega\|\leq\|\omega\|$. It is easy to prove property (i) from the corresponding property for $\mathrm{id}_{L(\mathcal{E})}\otimes\omega$. Since $\mathrm{id}_\mathcal{E}\otimes\omega$ is strictly continuous, to prove property (ii), it is enough to check it for $x\in\mathcal{E}\odot\mathcal{G}$. But this is easy. The other properties are proved using similar arguments, because all the operations involved are strictly continuous. For example, to prove (iii), one only needs to check it for $x\in\mathcal{E}\odot\mathcal{G}$ (and this is easy), and after that one proves that the map $\mathcal{M}(\mathcal{E}\otimes\mathcal{G})\ni x\mapsto x(R\otimes 1)\in\mathcal{M}(\mathcal{X}\otimes\mathcal{G})$ is strictly continuous (this uses the fact that we have a canonical inclusion $\mathcal{K}(\mathcal{X})\subseteq\mathcal{K}(\mathcal{E})$ and hence also $\mathcal{K}(\mathcal{X}\otimes\mathcal{G})\subseteq\mathcal{K}(\mathcal{E}\otimes\mathcal{G})$).

Finally, let $x\in\tilde{\mathcal{M}}(\mathcal{E}\otimes\mathcal{G})$. By Cohen's Factorization Theorem, there is $a\in\mathcal{G}$ and $\theta\in\mathcal{G}$ such that $\omega=\theta a$. Using (ii), we get

$$(\mathrm{id}_\mathcal{E}\otimes\omega)(x)=(\mathrm{id}_\mathcal{E}\otimes\theta)\big((1\otimes a)x\big)\in\mathcal{E},$$

because $(1\otimes a)x\in\mathcal{E}\otimes\mathcal{G}$ and $(\mathrm{id}_\mathcal{E}\otimes\theta)(\mathcal{E}\otimes\mathcal{G})\subseteq\mathcal{E}$. The equation above also shows that the map $\mathrm{id}\otimes\omega:\tilde{\mathcal{M}}(\mathcal{E}\otimes\mathcal{G})\to\mathcal{E}$ is $\mathcal{G}$-strict to norm continuous. □

Remark 2.4.15. Applying Proposition 2.4.14 to $\mathcal{E}^*$ instead of $\mathcal{E}$, we get a slice map $\mathrm{id}_{\mathcal{E}^*}\otimes\omega:\mathcal{M}(\mathcal{E}^*\otimes\mathcal{G})\to\mathcal{M}(\mathcal{E}^*)$ which is a strictly continuous linear map. In fact, it is the strict extension of the slice map $\mathrm{id}_{\mathcal{E}^*}\otimes\omega:\mathcal{E}^*\otimes\mathcal{G}\to\mathcal{E}^*$, which is norm-continuous and satisfies $(\mathrm{id}_{\mathcal{E}^*}\otimes\omega)(\xi^*\otimes a)=\xi^*\omega(a)$ for all $\xi\in\mathcal{E}$ and $a\in\mathcal{G}$. On the other hand we also have defined a slice map $\mathrm{id}_{\mathcal{E}^*}\otimes\omega:\mathcal{L}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})\to\mathcal{L}(\mathcal{E},B)$ as the restriction of $\mathrm{id}_{L(\mathcal{E})}\otimes\omega$ to the corner $\mathcal{L}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})\cong qL\big(\mathcal{M}(\mathcal{E}\otimes\mathcal{G})\big)p\subseteq L\big(\mathcal{M}(\mathcal{E}\otimes\mathcal{G})\big)$ (where p,q are the corner projections). So apparently there is notational problem: we are using the same symbol $\mathrm{id}_{\mathcal{E}^*}\otimes\omega$ for two possibly different maps. But there is no confusion if we identify $\mathcal{L}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})\hookrightarrow\mathcal{M}\big(\mathcal{K}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})\big)\cong\mathcal{M}(\mathcal{E}^*\otimes\mathcal{G})$ and $\mathcal{L}(\mathcal{E},B)\hookrightarrow\mathcal{M}\big(\mathcal{K}(\mathcal{E},B)\big)\cong\mathcal{M}(\mathcal{E}^*)$ as in Remark 2.1.7. In fact, under these identifications $\mathrm{id}_{\mathcal{E}^*}\otimes\omega:\mathcal{L}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})\to\mathcal{L}(\mathcal{E},B)$ is just the restriction of $\mathrm{id}_{\mathcal{E}^*}\otimes\omega:\mathcal{M}(\mathcal{E}^*\otimes\mathcal{G})\to\mathcal{M}(\mathcal{E}^*)$. This can be seen in the following way. Let us denote, for the moment, by S_ω for the map $\mathrm{id}_{\mathcal{E}^*}\otimes\omega:\mathcal{L}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})\to\mathcal{L}(\mathcal{E},B)$. The notation $\mathrm{id}_{\mathcal{E}^*}\otimes\omega$ is used only for the slice map $\mathrm{id}_{\mathcal{E}^*}\otimes\omega:\mathcal{M}(\mathcal{E}^*\otimes\mathcal{G})\to\mathcal{M}(\mathcal{E}^*)$. From the definition of S_ω, it is easy to show that S_ω restricts to a linear bounded map $S_\omega:\mathcal{E}^*\otimes\mathcal{G}\to\mathcal{E}^*$ satisfying $S_\omega(\xi^*\otimes a)=\xi^*\omega(a)$ for all $\xi\in\mathcal{E}^*$ and $a\in\mathcal{G}$. Here we identify $\xi^*\otimes a\in\mathcal{E}^*\otimes\mathcal{G}\cong\mathcal{K}(\mathcal{E},B)\otimes\mathcal{G}\cong\mathcal{K}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})\subseteq\mathcal{L}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})$. In particular, S_ω and $\mathrm{id}_{\mathcal{E}^*}\otimes\omega$ coincide on $\mathcal{E}^*\otimes\mathcal{G}\cong\mathcal{K}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})$. Moreover, since the strict topology on $\mathcal{M}\big(L(\mathcal{E})\otimes\mathcal{G}\big)$, when restricted to the corner $\mathcal{L}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})$, coincides with the bi-strict topology, we see that S_ω is bi-strict continuous. Thus S_ω is the bi-strictly continuous extension of $\mathrm{id}_{\mathcal{E}^*}\otimes\omega:\mathcal{E}^*\otimes\mathcal{G}\to\mathcal{E}^*$ to $\mathcal{L}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})\to\mathcal{L}(\mathcal{E},B)$. Since the bi-strict topology is stronger than the strict topology, we see that S_ω must be the restriction of $\mathrm{id}_{\mathcal{E}^*}\otimes\omega$.

Having the slice maps $\mathrm{id}_\mathcal{E}\otimes\omega$ for all $\omega\in\mathcal{G}^*$, we can now define the set

$$\tilde{\mathcal{M}}_{\mathrm{id}_\mathcal{E}\otimes\varphi}:=\Big\{x\in\mathcal{M}(\mathcal{E}\otimes\mathcal{G}):\text{ the strict limit }\operatorname*{s\text{-}lim}_{\omega\in\mathcal{G}_\varphi}(\mathrm{id}_\mathcal{E}\otimes\omega)(x)\in\mathcal{M}(\mathcal{E})\text{ exists}\Big\}.$$

We also define a slice map $\mathrm{id}_\mathcal{E}\otimes\varphi:\tilde{\mathcal{M}}_{\mathrm{id}_\mathcal{E}\otimes\varphi}\to\mathcal{M}(\mathcal{E})$ by

$$(\mathrm{id}_\mathcal{E}\otimes\varphi)(x):=\operatorname*{s\text{-}lim}_{\omega\in\mathcal{G}_\varphi}(\mathrm{id}_\mathcal{E}\otimes\omega)(x).$$

In order to generalize the case of C^*-algebras and define a set like $\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$ we use the linking algebra of $\mathcal{E}$ as follows.

Definition 2.4.16. For a proper weight φ on $\mathcal{G}$ and a Hilbert B-module $\mathcal{E}$, we define the following sets

$$\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} := \left\{ x \in \mathcal{L}(B\otimes\mathcal{G}, \mathcal{E}\otimes\mathcal{G}) = \mathcal{M}(\mathcal{E}\otimes\mathcal{G}) : \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \right\},$$

$$\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} := \left\{ x \in \mathcal{L}(B\otimes\mathcal{G}, \mathcal{E}\otimes\mathcal{G}) = \mathcal{M}(\mathcal{E}\otimes\mathcal{G}) : \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \right\},$$

$$\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} := \left\{ x \in \mathcal{L}(\mathcal{E}\otimes\mathcal{G}, B\otimes\mathcal{G}) \subseteq \mathcal{M}(\mathcal{E}^*\otimes\mathcal{G}) : \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \right\},$$

$$\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} := \left\{ x \in \mathcal{L}(\mathcal{E}\otimes\mathcal{G}, B\otimes\mathcal{G}) \subseteq \mathcal{M}(\mathcal{E}^*\otimes\mathcal{G}) : \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \right\}.$$

It follows from Equation (2.6) and the fact that the strict topology on $\mathcal{M}\big(L(\mathcal{E})\big)$, when restricted to the corner $\mathcal{M}(\mathcal{E})$, coincides with the strict topology on $\mathcal{M}(\mathcal{E})$, that $\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} \subseteq \tilde{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$. In other words, if $x \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$, then there exists the strict limit $\operatorname*{s\text{-}lim}_{\omega\in\mathcal{G}_\varphi} (\mathrm{id}_{\mathcal{E}}\otimes\omega)(x)$ in $\mathcal{M}(\mathcal{E})$, which is denoted by $(\mathrm{id}_{\mathcal{E}}\otimes\varphi)(x)$. Moreover, we have

$$\begin{pmatrix} 0 & (\mathrm{id}_{\mathcal{E}}\otimes\varphi)(x) \\ 0 & 0 \end{pmatrix} = (\mathrm{id}_{L(\mathcal{E})}\otimes\varphi)\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}.$$

Analogously, since the strict topology on $\mathcal{M}\big(L(\mathcal{E})\big)$, when restricted to the corner $\mathcal{L}(\mathcal{E}, B)$ coincides with the bi-strict topology on $\mathcal{L}(\mathcal{E}, B)$ (see Remark 2.1.10), it follows that for all $x \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ there exists the bi-strict limit $\operatorname*{ss\text{-}lim}_{\omega\in\mathcal{G}_\varphi} (\mathrm{id}_{\mathcal{E}^*}\otimes\omega)(x)$ in $\mathcal{L}(\mathcal{E}, B)$, which we denote by $(\mathrm{id}_{\mathcal{E}^*}\otimes\varphi)(x)$. Moreover, we have

$$\begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\varphi)(x) & 0 \end{pmatrix} = (\mathrm{id}_{L(\mathcal{E})}\otimes\varphi)\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$$

for all $x \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$. Note that, under the canonical embedding $\mathcal{L}(\mathcal{E}, B) \hookrightarrow \mathcal{M}(\mathcal{E}^*)$ (see Remark 2.1.7), the bi-strict topology on $\mathcal{L}(\mathcal{E}, B)$ is stronger than the strict topology, and therefore $\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} \subseteq \tilde{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$.

Remark 2.4.17. Note that considering the dual $\mathcal{E}^* \cong \mathcal{K}(\mathcal{E}, B)$ of $\mathcal{E}$, which is a Hilbert $B, \mathcal{K}(\mathcal{E})$-bimodule, we have

$$L(\mathcal{E}^*) = \begin{pmatrix} B & \mathcal{E}^* \\ \mathcal{E} & \mathcal{K}(\mathcal{E}) \end{pmatrix} \cong \begin{pmatrix} \mathcal{K}(B) & \mathcal{K}(\mathcal{E}, B) \\ \mathcal{K}(B, \mathcal{E}) & \mathcal{K}(\mathcal{E}) \end{pmatrix} \cong \mathcal{K}(B\oplus\mathcal{E}) \cong \mathcal{K}(\mathcal{E}\oplus B) \cong L(\mathcal{E}).$$

This also implies $\mathcal{M}\big(L(\mathcal{E}^*)\big) \cong \mathcal{M}\big(L(\mathcal{E})\big)$. In the same way, if we consider the Hilbert $B\otimes\mathcal{G}, \mathcal{K}(\mathcal{E})\otimes\mathcal{G}$-bimodule $\mathcal{E}^*\otimes\mathcal{G} \cong \mathcal{K}(\mathcal{E}\otimes\mathcal{G}, B\otimes\mathcal{G})$, then $L(\mathcal{E}^*)\otimes\mathcal{G} \cong L(\mathcal{E})\otimes\mathcal{G}$ and (hence) $\mathcal{M}(L(\mathcal{E}^*) \otimes \mathcal{G}) \cong \mathcal{M}(L(\mathcal{E}) \otimes \mathcal{G})$. Under these isomorphisms, an element of the form $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ corresponds to the element $\begin{pmatrix} x_4 & x_3 \\ x_2 & x_1 \end{pmatrix}$. In this way, we have

$$\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} = \left\{ x \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G}) : \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{E}^*)} \otimes \varphi} \right\}$$

and

$$\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} := \left\{ x \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G}) : \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E}^*)} \otimes \varphi} \right\}.$$

So the definitions of $\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ and $\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ in Definition 2.4.16 are redundant. That is, we could have just defined $\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi}$ and $\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi}$, and apply these definitions to the dual $\mathcal{E}^*$ to get $\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ and $\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$. But one has to take care here, because if we consider $\mathcal{E}^*$ only as a right-Hilbert $\mathcal{K}(\mathcal{E})$-module, that is, if we forget the left-Hilbert B-structure of $\mathcal{E}^*$, then we get different definitions in general. For example, if φ is bounded, then $\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} = \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} = \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G})$. But, if we forget the left-Hilbert B-structure of $\mathcal{E}^*$, then we would get $\mathcal{M}(\mathcal{E}^* \otimes \mathcal{G})$ instead, which is strictly bigger than $\mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G})$ in general (see Remark 2.1.12(1)). However, if $\mathcal{E}$ is full, then all these differences disappear.

Definition 2.4.18. Let $\mathcal{E}$ be a Hilbert B-module. Under the identification (see Proposition 2.3.2)

$$\mathcal{M}(L(\mathcal{E}) \otimes H) \cong \begin{pmatrix} \mathcal{L}(\mathcal{E}, \mathcal{E} \otimes H) & \mathcal{L}(B, \mathcal{E} \otimes H) \\ \mathcal{L}(\mathcal{E}, B \otimes H) & \mathcal{L}(B, B \otimes H) \end{pmatrix},$$

we define for all $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi}$ and $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$, the following maps

$$(\mathrm{id}_{\mathcal{E}} \otimes \Lambda)(x) \in \mathcal{L}(B, \mathcal{E} \otimes H), \quad (\mathrm{id}_{\mathcal{E}} \otimes \Lambda)(x) := (\mathrm{id}_{L(\mathcal{E})} \otimes \Lambda) \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}_{12},$$

and

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y) \in \mathcal{L}(\mathcal{E}, B \otimes H), \quad (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y) := (\mathrm{id}_{L(\mathcal{E})} \otimes \Lambda) \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}_{21},$$

where for a matrix m, m_{ij} denotes its (i,j)-entry.

Note that if $\mathcal{E} = B$ is a C^*-algebra, then we have two definitions for the sets $\bar{\mathcal{M}}_{\mathrm{id}_B \otimes \varphi}$ and $\bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}$ and also for the maps $\mathrm{id}_B \otimes \varphi$ and $\mathrm{id}_B \otimes \Lambda$ and hence we have to prove that they coincide. This will follow from the next result.

Proposition 2.4.19. *Let $\mathcal{E}$ be a Hilbert B-module. Then*

(i) $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi} \Longleftrightarrow x^*x \in \bar{\mathcal{M}}^+_{\mathrm{id}_B \otimes \varphi}$ *and* $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} \Longleftrightarrow x^*x \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi}$,

(ii) $\bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{E})} \otimes \varphi} = \begin{pmatrix} \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi} & \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi} \\ \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} & \bar{\mathcal{M}}_{\mathrm{id}_B \otimes \varphi} \end{pmatrix}$ *and* $\bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})} \otimes \varphi} = \begin{pmatrix} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi} & \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi} \\ \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} & \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi} \end{pmatrix}$,

(iii) *for all* $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}$, *we have*

$$(\mathrm{id}_{L(\mathcal{E})} \otimes \varphi) \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(x_1) & (\mathrm{id}_{\mathcal{E}} \otimes \varphi)(x_2) \\ (\mathrm{id}_{\mathcal{E}^*} \otimes \varphi)(x_3) & (\mathrm{id}_B \otimes \varphi)(x_4) \end{pmatrix},$$

and

(iv) *for all* $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}$, *we have*

$$(\mathrm{id}_{L(\mathcal{E})} \otimes \Lambda) \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Lambda)(x_1) & (\mathrm{id}_{\mathcal{E}} \otimes \Lambda)(x_2) \\ (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x_3) & (\mathrm{id}_B \otimes \Lambda)(x_4) \end{pmatrix}.$$

Proof. By definition, $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$ if and only if $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}$. This means that $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x^*x \end{pmatrix} \in \mathcal{M}^+_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}$, that is, there exists the strict limit $\operatorname{s-lim}_{\omega\in\mathcal{G}_\varphi} (\mathrm{id}_{L(\mathcal{E})} \otimes \omega) \begin{pmatrix} 0 & 0 \\ 0 & x^*x \end{pmatrix}$, or what is equivalent, there exists the strict limit $\operatorname{s-lim}_{\omega\in\mathcal{G}_\varphi} (\mathrm{id}_B \otimes \omega)(x^*x)$, that is, $x^*x \in \bar{\mathcal{M}}^+_{\mathrm{id}_B\otimes\varphi}$. Analogously, one proves that $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ if and only if $x^*x \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}$. Now if $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}$, then

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^* \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1^*x_1 + x_3^*x_3 & x_1^*x_2 + x_3^*x_4 \\ x_2^*x_1 + x_4^*x_3 & x_2^*x_2 + x_4^*x_4 \end{pmatrix} \in \bar{\mathcal{M}}^+_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}.$$

It follows that $x_1^*x_1 + x_3^*x_3 \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}$ and $x_2^*x_2 + x_4^*x_4 \in \bar{\mathcal{M}}^+_{\mathrm{id}_B\otimes\varphi}$. Since $\bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}$ and $\bar{\mathcal{M}}^+_{\mathrm{id}_B\otimes\varphi}$ are hereditary cones, we get $x_1^*x_1, x_3^*x_3 \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}$ and $x_2^*x_2, x_4^*x_4 \in \bar{\mathcal{M}}^+_{\mathrm{id}_B\otimes\varphi}$. Thus

$$\bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \subseteq \begin{pmatrix} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} & \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} \\ \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} & \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} \end{pmatrix}.$$

The other inclusion is trivial. From this we get

$$\begin{aligned} \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} &= \operatorname{span} \bar{\mathcal{N}}^*_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \\ &= \operatorname{span} \begin{pmatrix} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} & \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} \\ \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} & \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} \end{pmatrix}^* \begin{pmatrix} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} & \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} \\ \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} & \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} \end{pmatrix} \\ &= \operatorname{span} \begin{pmatrix} \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} & \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} \\ \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_B\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} & \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_B\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} \end{pmatrix}. \end{aligned}$$

Item (i) together with polarization yield the inclusions $\bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} \subseteq \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}$ and

$\bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} \subseteq \bar{\mathcal{M}}_{\mathrm{id}_B\otimes\varphi}$. Note also that

$$\begin{aligned}\begin{pmatrix} 0 & \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} & 0 \\ \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} & 0 \end{pmatrix}^* \begin{pmatrix} 0 & \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} \\ 0 & \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} \end{pmatrix} \\ &\subseteq \bar{\mathcal{N}}^*_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \subseteq \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}.\end{aligned}$$

Thus $\bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} \subseteq \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$. Analogously, $\bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} \subseteq \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$. Therefore

$$\bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \subseteq \begin{pmatrix} \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} & \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi} \\ \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} & \bar{\mathcal{M}}_{\mathrm{id}_B\otimes\varphi} \end{pmatrix}.$$

The other inclusion is trivial. And the formula for the slice map $\mathrm{id}_{L(\mathcal{E})}\otimes\varphi$ follows from Equation (2.6) and the relation (see Remark 2.1.10)

$$\operatorname*{s-lim}_i \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix} = \begin{pmatrix} \operatorname*{s-lim}_i x_i & \operatorname*{s-lim}_i y_i \\ \operatorname*{ss-lim}_i z_i & \operatorname*{s-lim}_i w_i \end{pmatrix} \tag{2.7}$$

for every strictly converging net $\begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix} \in \mathcal{M}\big(L(\mathcal{E})\big)$.

Take any $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}$. Then, for all $\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \in L(\mathcal{E})$ and $s \in \mathcal{G}$, we have

$$\begin{aligned}(\mathrm{id}_{L(\mathcal{E})}\otimes\Lambda)&\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^* \left(\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \otimes \Lambda(s)\right) \\ &= (\mathrm{id}_{L(\mathcal{E})}\otimes\varphi)\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^* \left(\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \otimes s\right)\right) \\ &= (\mathrm{id}_{L(\mathcal{E})}\otimes\varphi)\begin{pmatrix} x_1^*(k\otimes s) + x_3^*(\eta^*\otimes s) & x_1^*(\xi\otimes s) + x_3^*(b\otimes s) \\ x_2^*(k\otimes s) + x_4^*(\eta^*\otimes s) & x_2^*(\xi\otimes s) + x_4^*(b\otimes s) \end{pmatrix} \\ &= \begin{pmatrix} \begin{matrix}(\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi)\big(x_1^*(k\otimes s)\big) + \\ (\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi)(x_3^*(\eta^*\otimes s)\end{matrix} & \begin{matrix}(\mathrm{id}_{\mathcal{E}}\otimes\varphi)\big(x_1^*(\xi\otimes s)\big) + \\ (\mathrm{id}_{\mathcal{E}}\otimes\varphi)\big(x_3^*(b\otimes s)\big)\end{matrix} \\ \\ \begin{matrix}(\mathrm{id}_{\mathcal{E}^*}\otimes\varphi)\big(x_2^*(k\otimes s)\big) + \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\varphi)\big(x_4^*(\eta^*\otimes s)\big)\end{matrix} & \begin{matrix}(\mathrm{id}_B\otimes\varphi)\big(x_2^*(\xi\otimes s)\big) + \\ (\mathrm{id}_B\otimes\varphi)\big(x_4^*(b\otimes s)\big)\end{matrix} \end{pmatrix} =: M.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\begin{pmatrix} (\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\Lambda)(x_1) & (\mathrm{id}_{\mathcal{E}}\otimes\Lambda)(x_2) \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(x_3) & (\mathrm{id}_B\otimes\Lambda)(x_4) \end{pmatrix}^* \left(\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \otimes s\right) \\ &= \begin{pmatrix} \begin{matrix}(\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\Lambda)(x_1)^*\big(k\otimes\Lambda(s)\big) + \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(x_3)^*\big(\eta^*\otimes\Lambda(s)\big)\end{matrix} & \begin{matrix}(\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\Lambda)(x_1)^*\big(\xi\otimes\Lambda(s)\big) + \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(x_3)^*\big(b\otimes\Lambda(s)\big)\end{matrix} \\ \\ \begin{matrix}(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)(x_2)^*\big(k\otimes\Lambda(s)\big) + \\ (\mathrm{id}_B\otimes\Lambda)(x_4)^*\big(\eta^*\otimes\Lambda(s)\big)\end{matrix} & \begin{matrix}(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)(x_2)^*\big(\xi\otimes\Lambda(s)\big) + \\ (\mathrm{id}_B\otimes\Lambda)(x_4)^*\big(b\otimes\Lambda(s)\big)\end{matrix} \end{pmatrix} =: N.\end{aligned}$$

By Proposition 2.4.6, $(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Lambda)(x_1)^*\big(k \otimes \Lambda(s)\big) = (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)\big(x_1^*(k \otimes s)\big)$. We claim that $(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x_3)^*\big(\eta^* \otimes \Lambda(s)\big) = (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(x_3^*\big(\eta^* \otimes s)\big)$. In fact,

$$\begin{aligned}
(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x_3)^*\big(\eta^* \otimes \Lambda(s)\big) &= \left((\mathrm{id}_{L(\mathcal{E})} \otimes \Lambda)\begin{pmatrix} 0 & 0 \\ x_3 & 0 \end{pmatrix}\right)^*_{21}\Big(\eta^* \otimes \Lambda(s)\Big) \\
&= \left((\mathrm{id}_{L(\mathcal{E})} \otimes \Lambda)\begin{pmatrix} 0 & 0 \\ x_3 & 0 \end{pmatrix}^*\begin{pmatrix} 0 & 0 \\ \eta^* \otimes \Lambda(s) & 0 \end{pmatrix}\right)_{11} \\
&= \left((\mathrm{id}_{L(\mathcal{E})} \otimes \varphi)\left(\begin{pmatrix} 0 & x_3^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \eta^* \otimes s & 0 \end{pmatrix}\right)\right)_{11} \\
&= (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)\big(x_3^*(\eta^* \otimes s)\big).
\end{aligned}$$

Thus $M_{11} = N_{11}$. Let us also prove that $M_{12} = N_{12}$. For this, note that writing $\xi = T\zeta$, for some $T \in \mathcal{K}(\mathcal{E})$ and $\zeta \in \mathcal{E}$, we get

$$\begin{aligned}
(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Lambda)(x_1)^*\big(\xi \otimes \Lambda(s)\big) &= (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Lambda)(x_1)^*\big(T \otimes \Lambda(s)\big)\zeta \\
&= (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)\big(x_1^*(T \otimes s)\big)\zeta \\
&= (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(x_1^*(T\zeta \otimes s)\big) \\
&= (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(x_1^*(\xi \otimes s)\big).
\end{aligned}$$

And we also have

$$\begin{aligned}
(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x_3)^*\big(b \otimes \Lambda(s)\big) &= \left((\mathrm{id}_{L(\mathcal{E})} \otimes \Lambda)\begin{pmatrix} 0 & 0 \\ x_3 & 0 \end{pmatrix}\right)^*_{21}\Big(b \otimes \Lambda(s)\Big) \\
&= \left((\mathrm{id}_{L(\mathcal{E})} \otimes \Lambda)\begin{pmatrix} 0 & 0 \\ x_3 & 0 \end{pmatrix}^*\begin{pmatrix} 0 & 0 \\ 0 & b \otimes \Lambda(s) \end{pmatrix}\right)_{11} \\
&= \left((\mathrm{id}_{L(\mathcal{E})} \otimes \varphi)\left(\begin{pmatrix} 0 & x_3^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b \otimes s & 0 \end{pmatrix}\right)\right)_{11} \\
&= (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(x_3^*(b \otimes s)\big).
\end{aligned}$$

Analogously, one proves that $M_{21} = N_{21}$ and $M_{22} = N_{22}$. $\square$

Note that the equality $\bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{E})} \otimes \varphi} = \mathrm{span}\, \bar{\mathcal{N}}^*_{\mathrm{id}_{L(\mathcal{E})} \otimes \varphi} \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})} \otimes \varphi}$ and Proposition 2.4.19(ii) imply the following relations

$$\begin{aligned}
\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi} &= \mathrm{span}\big(\bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}\big), \\
\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi} &= \mathrm{span}\big(\bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}\big), \\
\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} &= \mathrm{span}\big(\bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}} \otimes \varphi} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_B \otimes \varphi} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}\big), \\
\bar{\mathcal{M}}_{\mathrm{id}_B \otimes \varphi} &= \mathrm{span}\big(\bar{\mathcal{N}}^*_{\mathrm{id}_{\mathcal{E}} \otimes \varphi} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi} + \bar{\mathcal{N}}^*_{\mathrm{id}_B \otimes \varphi} \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}\big).
\end{aligned}$$

From Proposition 2.4.19 we can derive several properties for the maps $\mathrm{id}_{\mathcal{E}} \otimes \Lambda$ and $\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda$. We list now some properties for the map $\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda$, which will be more important to us later.

Before we state the properties, we need one more preparation. Suppose that $\pi : \mathcal{G} \to \mathcal{L}(H) = \mathcal{M}\big(\mathcal{K}(H)\big)$ is a nondegenerate $*$-homomorphism, where H is some Hilbert space, and consider the $*$-homomorphism $\mathrm{id}_{L(\mathcal{E})} \otimes \pi : L(\mathcal{E}) \otimes \mathcal{G} \to \mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{K}(H)\big)$, which is also nondegenerate, and therefore has a strictly continuous extension $\mathrm{id}_{L(\mathcal{E})} \otimes \pi : \mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{G}\big) \to \mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{K}(H)\big)$. As we have already noted, there is a canonical identification (see Remark 2.3.3):

$$\mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{G}\big) \cong \begin{pmatrix} \mathcal{L}(\mathcal{E} \otimes \mathcal{G}) & \mathcal{L}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G}) \\ \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G}) & \mathcal{L}(B \otimes \mathcal{G}) \end{pmatrix}.$$

Moreover, note that $L(\mathcal{E}) \otimes \mathcal{K}(H) \cong \mathcal{K}(\mathcal{E} \oplus B) \otimes \mathcal{K}(H) \cong \mathcal{K}\big((\mathcal{E} \otimes H) \oplus (B \otimes H)\big)$, and therefore we also have a canonical identification:

$$\begin{aligned} \mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{K}(H)\big) &\cong \mathcal{L}\big((\mathcal{E} \otimes H) \oplus (B \otimes H)\big) \\ &\cong \begin{pmatrix} \mathcal{L}(\mathcal{E} \otimes H) & \mathcal{L}(B \otimes H, \mathcal{E} \otimes H) \\ \mathcal{L}(\mathcal{E} \otimes H, B \otimes H) & \mathcal{L}(B \otimes H) \end{pmatrix}. \end{aligned}$$

Under these identifications we get a matrix decomposition:

$$(\mathrm{id}_{L(\mathcal{E})} \otimes \pi) \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \pi)(x_1) & (\mathrm{id}_{\mathcal{E}} \otimes \pi)(x_2) \\ (\mathrm{id}_{\mathcal{E}^*} \otimes \pi)(x_3) & (\mathrm{id}_B \otimes \pi)(x_4) \end{pmatrix}. \tag{2.8}$$

To see this, first we observe that the equation above is easily checked for elements of $L(\mathcal{E}) \odot \mathcal{G}$. Since $\mathrm{id}_{L(\mathcal{E})} \otimes \pi$ is strictly continuous, and all the corners in $\mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{G}\big)$ and in $\mathcal{M}\big(L(\mathcal{E}) \otimes \mathcal{K}(H)\big)$ are closed with respect to the bi-strict topologies, it follows that $\mathrm{id}_{L(\mathcal{E})} \otimes \pi$ induces maps on the respective corners, which are necessarily bi-strictly continuous because the strict topologies coincide with the bi-strict topologies on the corners. The assertion now follows because all the involved maps are bi-strictly continuous. Note that, *a priori*, $\mathrm{id}_{\mathcal{E}^*} \otimes \pi$ is a strictly continuous map $\mathcal{M}(\mathcal{E}^* \otimes \mathcal{G}) \to \mathcal{M}\big(\mathcal{E}^* \otimes \mathcal{K}(H)\big)$. And we are considering it above as a bi-strictly continuous map $\mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G}) \to \mathcal{L}(\mathcal{E} \otimes H, B \otimes H)$. But because the bi-strict topology is stronger than the strict topology it follows that the second map is a restriction of the first one. Here we consider the canonical inclusions $\mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G}) \hookrightarrow \mathcal{M}(\mathcal{E}^* \otimes \mathcal{G})$ and $\mathcal{L}(\mathcal{E} \otimes H, B \otimes H) \hookrightarrow \mathcal{M}\big(\mathcal{E}^* \otimes \mathcal{K}(H)\big)$ (as in Remark 2.1.7). Note also that, *a priori*, $\mathrm{id}_{\mathcal{E}} \otimes \pi$ is a strictly continuous map $\mathcal{M}(\mathcal{E} \otimes \mathcal{G}) \to \mathcal{M}\big(\mathcal{E} \otimes \mathcal{K}(H)\big)$, and we are considering it above as a bi-strictly continuous map $\mathcal{L}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G}) \to \mathcal{L}(B \otimes H, \mathcal{E} \otimes H)$. By definition, we have $\mathcal{M}(\mathcal{E} \otimes \mathcal{G}) = \mathcal{L}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G})$ and the strict and bi-strict topologies coincide. Moreover, we also have a canonical inclusion $\mathcal{L}(B \otimes H, \mathcal{E} \otimes H) \hookrightarrow \mathcal{M}\big(\mathcal{E} \otimes \mathcal{K}(H)\big)$. In fact, in this case this inclusion is an isomorphism and the strict and bi-strict topologies coincide (this follows from Proposition 2.1.11). Finally, we mention that $(\mathrm{id}_{\mathcal{E}} \otimes \pi)(x)^* = (\mathrm{id}_{\mathcal{E}^*} \otimes \pi)(x^*)$ for all $x \in \mathcal{L}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G})$.

Proposition 2.4.20. *Let $\mathcal{E}$ be a Hilbert B-module and let φ be a proper weight on $\mathcal{G}$ with a GNS-construction (H, π, Λ).*

(i) *If* $z \in \mathcal{L}(\mathcal{E}, B)$ *and* $s \in \bar{\mathcal{N}}_\varphi$ *then* $z \otimes s \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ *and* $(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(z \otimes s) = z \otimes \Lambda(s)$.

(ii) *If* $x, y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ *then* $y^* x \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y)^* (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x) = (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(y^* x).$$

(iii) *For any* $y \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G})$, *we have*

$$y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} \Longleftrightarrow y(k \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}, \forall k \in \mathcal{K}(\mathcal{E}) \Longleftrightarrow y(\eta \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}, \forall \eta \in \mathcal{E}$$

and in this case

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(y(k \otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y) k \quad \text{and} \quad (\mathrm{id}_B \otimes \Lambda)\big(y(\eta \otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y)\eta.$$

(iv) *For all* $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ *and* $x \in \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}$, *we have* $y^* x \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y^*)(\mathrm{id}_B \otimes \Lambda)(x) = (\mathrm{id}_{\mathcal{E}} \otimes \varphi)(y^* x).$$

In particular, if $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$, $b \in \mathcal{M}(B)$ *and* $s \in \bar{\mathcal{N}}_\varphi$, *then* $y^*(b \otimes s) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y)^* \big(b \otimes \Lambda(s)\big) = (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(y^*(b \otimes s)\big).$$

(v) *If* $x \in \mathcal{M}(B \otimes \mathcal{G})$ *and* $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ *then* $xy \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(xy) = (\mathrm{id}_B \otimes \pi)(x)(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y).$$

(vi) *Let* $\mathcal{F}$ *be another Hilbert* B*-module. If* $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ *and* $T \in \mathcal{L}(\mathcal{F}, \mathcal{E})$, *then* $y(T \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{F}^*} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{F}^*} \otimes \Lambda)\big(y(T \otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y) T.$$

(vii) *If* $x \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G})$ *and* $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi} \subseteq \mathcal{M}\big(\mathcal{K}(\mathcal{E}) \otimes \mathcal{G}\big) \cong \mathcal{L}(\mathcal{E} \otimes \mathcal{G})$ *then* $xy \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(xy) = (\mathrm{id}_{\mathcal{E}^*} \otimes \pi)(x)(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Lambda)(y).$$

(viii) *If* $x \in \mathcal{L}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G})$ *and* $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ *then* $xy \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Lambda)(xy) = (\mathrm{id}_{\mathcal{E}} \otimes \pi)(x)(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y).$$

(ix) *The linear map* $\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda : \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} \subseteq \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G}) \to \mathcal{L}(\mathcal{E}, B \otimes H)$ *is closed for the bi-strict topology of* $\mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G})$ *(see Definition* 2.1.9*) and the* $\mathcal{K}$*-strong topology of* $\mathcal{L}(\mathcal{E}, B \otimes H)$ *(see Definition* 2.1.13*).*

Proof. Essentially, all the properties follow by combining Proposition 2.4.6 (applied to $A = L(\mathcal{E})$, the linking algebra of $\mathcal{E}$) with Proposition 2.4.19. For example, (i) follows from Propositions 2.4.6(ii) and 2.4.19(ii),(iv). And (ii) follows from Polarization and Propositions 2.4.6(i) and 2.4.19(i),(iii),(iv). Let us prove (iii) with more details. Propositions 2.4.6(iv) and 2.4.19(ii) imply that

$$\begin{aligned} y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} &\Longrightarrow \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi} \\ &\Longrightarrow \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \left(\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \otimes 1 \right) = \begin{pmatrix} 0 & 0 \\ y(k\otimes 1) & y(\xi \otimes 1) \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}, \end{aligned}$$

for all $k \in \mathcal{K}(\mathcal{E})$, $\xi, \eta \in \mathcal{E}$ and $b \in B$. Thus $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ implies that $y(k\otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ and $y(\xi\otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ for all $k \in \mathcal{K}(\mathcal{E})$ and $\xi \in \mathcal{E}$. Applying this again to $y(k\otimes 1)$ we get that if $y(k\otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ for all $k \in \mathcal{K}(\mathcal{E})$, then $y(k\xi \otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ for all $k \in \mathcal{K}(\mathcal{E})$ and $\xi \in \mathcal{E}$. Since $\mathcal{K}(\mathcal{E})\mathcal{E} = \mathcal{E}$ we get that $y(\eta\otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ for all $\eta \in \mathcal{E}$. Thus

$$y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} \Longrightarrow y(k\otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}, \forall k \in \mathcal{K}(\mathcal{E}) \Longrightarrow y(\eta \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}, \forall \eta \in \mathcal{E}.$$

Suppose now that $y(\eta\otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ for all $\eta \in \mathcal{E}$. This means that $(\eta^*\otimes 1)y^*y(\eta\otimes 1) \in \bar{\mathcal{M}}^+_{\mathrm{id}_B\otimes\varphi}$ for all $\eta \in \mathcal{E}$, that is, there exists the limit

$$\operatorname*{s\text{-}lim}_{\omega\in\mathcal{G}_\varphi} (\mathrm{id}_B\otimes\omega)\big((\eta^*\otimes 1)y^*y(\eta\otimes 1)\big) = \operatorname*{s\text{-}lim}_{\omega\in\mathcal{G}_\varphi} \langle \eta | (\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\omega)(y^*y)\eta\rangle_B \in B$$

for all $\eta \in \mathcal{E}$. Since $\mathcal{E} = \mathcal{E}\cdot B$, the strict convergence above is equivalent to norm convergence. It follows from Lemma 2.4.4 that $y^*y \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}$, that is, $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$. Finally, by Propositions 2.4.6(iv) and 2.4.19(iv), we obtain, for all $k \in \mathcal{K}(\mathcal{E})$, $\xi, \eta \in \mathcal{E}$ and $b \in B$,

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(y)k & (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(y)\xi \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(y) & 0 \end{pmatrix} \begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \\ &= (\mathrm{id}_{L(\mathcal{E})}\otimes\Lambda) \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \\ &= (\mathrm{id}_{L(\mathcal{E})}\otimes\Lambda) \left(\begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix} \otimes 1 \right) \\ &= (\mathrm{id}_{L(\mathcal{E})}\otimes\Lambda) \begin{pmatrix} 0 & 0 \\ y(k\otimes 1) & y(\xi\otimes 1) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)\big(y(k\otimes 1)\big) & (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)\big(y(\xi\otimes 1)\big) \end{pmatrix}. \end{aligned}$$

Therefore $(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(y)k = (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)\big(y(k\otimes 1)\big)$ and $(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(y)\xi = (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)\big(y(\xi\otimes 1)\big)$. This proves (iii). The proofs of (iv) and (v) are left to the reader. Both are proved using Propositions 2.4.6 and 2.4.19 together. Let us prove (vi). Since $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$, using (iii), we get $y(T\otimes 1_{\mathcal{G}})(\eta\otimes 1_{\mathcal{G}}) = y(T\eta\otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ for all $\eta \in \mathcal{E}$. Again by (iii), $y(T\otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{F}^*}\otimes\varphi}$ and

$$(\mathrm{id}_{\mathcal{F}^*}\otimes\Lambda)\big(y(T\otimes 1_{\mathcal{G}})\big)\eta = (\mathrm{id}_B\otimes\Lambda)\big(y(T\eta\otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(y)T\eta.$$

In order to prove (vii), note that it follows from Proposition 2.4.19(i) and the inequality $(xy)^*(xy) = y^*x^*xy \leq \|x\|^2 y^*y \in \mathcal{M}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}$, that $xy \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$. Now by Proposition 2.4.19(iv) we have

$$\begin{aligned}\begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(xy) & 0 \end{pmatrix} &= (\mathrm{id}_{L(\mathcal{E})}\otimes\Lambda)\begin{pmatrix} 0 & 0 \\ xy & 0 \end{pmatrix} \\ &= (\mathrm{id}_{L(\mathcal{E})}\otimes\pi)\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}(\mathrm{id}_{L(\mathcal{E})}\otimes\Lambda)\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\pi)(x) & 0 \end{pmatrix}\begin{pmatrix} (\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\Lambda)(y) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{E}^*}\otimes\pi)(x)(\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\Lambda)(y) & 0 \end{pmatrix}.\end{aligned}$$

Therefore $(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(xy) = (\mathrm{id}_{\mathcal{E}^*}\otimes\pi)(x)(\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\Lambda)(y)$. Property (viii) is proved similarly. Finally, we prove (ix). Let $\{x_i\}$ be a net in $\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ and $x \in \mathcal{L}(\mathcal{E}\otimes\mathcal{G}, B\otimes\mathcal{G})$ such that $x_i \to x$ in the bi-strict topology and let $R \in \mathcal{L}(\mathcal{E}, B\otimes H)$ such that $R_i := (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(x_i) \to R$ in the $\mathcal{K}$-strong topology. We have to show that $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ and $(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(x) = R$. It follows from Equation (2.7) that

$$X_i := \begin{pmatrix} 0 & 0 \\ x_i & 0 \end{pmatrix} \to X := \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \text{ strictly in } \mathcal{M}(L(\mathcal{E})\otimes\mathcal{G}),$$

and we claim that

$$\tilde{R}_i := \begin{pmatrix} 0 & 0 \\ R_i & 0 \end{pmatrix} \to \tilde{R} := \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} \text{ strongly in } \mathcal{L}(L(\mathcal{E}), L(\mathcal{E})\otimes\mathcal{G}).$$

In fact, for $Y := \begin{pmatrix} T & \xi \\ \eta^* & b \end{pmatrix} \in \begin{pmatrix} \mathcal{K}(\mathcal{E}) & \mathcal{E} \\ \mathcal{E}^* & B \end{pmatrix} = L(\mathcal{E})$, we have

$$\tilde{R}_i Y = \begin{pmatrix} 0 & 0 \\ R_i T & R_i\xi \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ RT & R\xi \end{pmatrix} = \tilde{R}Y.$$

Therefore $\tilde{R}_i \to \tilde{R}$ strongly. Since $x_i \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ we have $X_i \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}$. And from 2.4.19(iv) we have $(\mathrm{id}_{L(\mathcal{E})}\otimes\Lambda)(X_i) = \tilde{R}_i$. By Proposition 2.4.6(v) applied to the C^*-algebra $L(\mathcal{E})$ we obtain that $X \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{E})}\otimes\varphi}$ and $(\mathrm{id}_{L(\mathcal{E})}\otimes\Lambda)(X) = \tilde{R}$. It follows that $x \in \mathcal{N}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ and $(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(x) = R$. $\square$

As already mentioned, we also get several properties for the map $\mathrm{id}_{\mathcal{E}}\otimes\Lambda$. For example, for any $x \in \mathcal{L}(B\otimes\mathcal{G}, \mathcal{E}\otimes\mathcal{G}) = \mathcal{M}(\mathcal{E}\otimes\mathcal{G})$, we have $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$ if and only if $x^*x \in \bar{\mathcal{M}}^+_{\mathrm{id}_B\otimes\varphi}$, and for all $x, y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$,

$$(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)(y)^*(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)(x) = (\mathrm{id}_B\otimes\varphi)(y^*x).$$

Moreover, $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$ if and only if $x(b\otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$ for all $b \in B$, and in this case

$$(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)\big(x(b\otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{E}}\otimes\Lambda)(x)b.$$

In fact, in what follows we generalize the constructions above and prove some properties that can be applied to both KSGNS-maps $\mathrm{id}_{\mathcal{E}} \otimes \Lambda$ and $\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda$.

Let $\mathcal{E}$, $\mathcal{F}$ be Hilbert B-modules, and consider the Hilbert $\mathcal{K}(\mathcal{E}),\mathcal{K}(\mathcal{F})$-bimodule $\mathcal{X} := \mathcal{K}(\mathcal{F},\mathcal{E})$. Since $\mathcal{X}^* \cong \mathcal{K}(\mathcal{E},\mathcal{F})$, we have

$$L(\mathcal{X}) \cong \begin{pmatrix} \mathcal{K}(\mathcal{E}) & \mathcal{K}(\mathcal{F},\mathcal{E}) \\ \mathcal{K}(\mathcal{E},\mathcal{F}) & \mathcal{K}(\mathcal{F}) \end{pmatrix} \cong \mathcal{K}(\mathcal{E} \oplus \mathcal{F}).$$

Thus

$$\mathcal{M}\big(L(\mathcal{X})\big) \cong \begin{pmatrix} \mathcal{L}(\mathcal{E}) & \mathcal{L}(\mathcal{F},\mathcal{E}) \\ \mathcal{L}(\mathcal{E},\mathcal{F}) & \mathcal{L}(\mathcal{F}) \end{pmatrix} \cong \mathcal{L}(\mathcal{E} \oplus \mathcal{F}).$$

Similarly, if we consider the Hilbert $\mathcal{K}(\mathcal{E})\otimes\mathcal{G},\mathcal{K}(\mathcal{F})\otimes\mathcal{G}$-bimodule $\mathcal{X}\otimes\mathcal{G} \cong \mathcal{K}(\mathcal{F}\otimes\mathcal{G},\mathcal{E}\otimes\mathcal{G})$, then we have

$$L(\mathcal{X}\otimes\mathcal{G}) \cong \begin{pmatrix} \mathcal{K}(\mathcal{E}\otimes\mathcal{G}) & \mathcal{K}(\mathcal{F}\otimes\mathcal{G},\mathcal{E}\otimes\mathcal{G}) \\ \mathcal{K}(\mathcal{E}\otimes\mathcal{G},\mathcal{F}\otimes\mathcal{G}) & \mathcal{K}(\mathcal{F}\otimes\mathcal{G}) \end{pmatrix} \cong \mathcal{K}\big((\mathcal{E}\oplus\mathcal{F})\otimes\mathcal{G}\big)$$

and

$$\mathcal{M}\big(L(\mathcal{X}\otimes\mathcal{G})\big) \cong \begin{pmatrix} \mathcal{L}(\mathcal{E}\otimes\mathcal{G}) & \mathcal{L}(\mathcal{F}\otimes\mathcal{G},\mathcal{E}\otimes\mathcal{G}) \\ \mathcal{L}(\mathcal{E}\otimes\mathcal{G},\mathcal{F}\otimes\mathcal{G}) & \mathcal{L}(\mathcal{F}\otimes\mathcal{G}) \end{pmatrix} \cong \mathcal{L}\big((\mathcal{E}\oplus\mathcal{F})\otimes\mathcal{G}\big)$$

Note also that

$$\begin{aligned} L(\mathcal{X})\otimes H &\cong \mathcal{K}(\mathcal{E}\oplus\mathcal{F})\otimes H \\ &\cong \mathcal{K}\big(\mathcal{E}\oplus\mathcal{F},(\mathcal{E}\otimes H)\oplus(\mathcal{F}\otimes H)\big) \\ &\cong \begin{pmatrix} \mathcal{K}(\mathcal{E},\mathcal{E}\otimes H) & \mathcal{K}(\mathcal{F},\mathcal{E}\otimes H) \\ \mathcal{K}(\mathcal{E},\mathcal{F}\otimes H) & \mathcal{K}(\mathcal{F},\mathcal{F}\otimes H) \end{pmatrix}. \end{aligned}$$

And because $\mathcal{K}\big(\mathcal{K}(\mathcal{E}\oplus\mathcal{F})\otimes H\big) \cong \mathcal{K}(\mathcal{E}\oplus\mathcal{F})\otimes\mathcal{K}(H) \cong \mathcal{K}\big((\mathcal{E}\oplus\mathcal{F})\otimes H\big)$ it follows from Proposition 2.1.11 that

$$\begin{aligned} \mathcal{M}\big(L(\mathcal{X})\otimes H\big) &\cong \mathcal{M}\Big(\mathcal{K}\big(\mathcal{E}\oplus\mathcal{F},(\mathcal{E}\otimes H)\oplus(\mathcal{F}\otimes H)\big)\Big) \\ &\cong \begin{pmatrix} \mathcal{L}(\mathcal{E},\mathcal{E}\otimes H) & \mathcal{L}(\mathcal{F},\mathcal{E}\otimes H) \\ \mathcal{L}(\mathcal{E},\mathcal{F}\otimes H) & \mathcal{L}(\mathcal{F},\mathcal{F}\otimes H) \end{pmatrix}. \end{aligned}$$

As above, we can define for $\mathcal{X}$ the slice map $\mathrm{id}_{\mathcal{X}} \otimes \varphi$ and the generalized KSGNS-construction $\mathrm{id}_{\mathcal{X}} \otimes \Lambda$, which are the restrictions to the upper right corner of the corresponding maps for $L(\mathcal{X})$. More precisely, we can define the space

$$\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi} = \{x \in \mathcal{L}(\mathcal{F}\otimes\mathcal{G},\mathcal{E}\otimes\mathcal{G}) : \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{X})}\otimes\varphi}\},$$

and the slice map $\mathrm{id}_{\mathcal{X}} \otimes \varphi : \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi} \subseteq \mathcal{L}(\mathcal{F}\otimes\mathcal{G},\mathcal{E}\otimes\mathcal{G}) \to \mathcal{L}(\mathcal{F},\mathcal{E})$ by the relation

$$(\mathrm{id}_{L(\mathcal{X})}\otimes\varphi)\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\mathrm{id}_{\mathcal{X}}\otimes\varphi)(x) \\ 0 & 0 \end{pmatrix}.$$

And we can also define the space

$$\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi} = \{x \in \mathcal{L}(\mathcal{F}\otimes\mathcal{G}, \mathcal{E}\otimes\mathcal{G}) : \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{X})}\otimes\varphi}\}$$

and the KSGNS-construction $\mathrm{id}_{\mathcal{X}}\otimes\Lambda : \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi} \subseteq \mathcal{L}(\mathcal{F}\otimes\mathcal{G}, \mathcal{E}\otimes\mathcal{G}) \to \mathcal{L}(\mathcal{F}, \mathcal{E}\otimes H)$ by the relation

$$(\mathrm{id}_{L(\mathcal{X})}\otimes\Lambda)\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(x) \\ 0 & 0 \end{pmatrix}.$$

Analogously, we can define the slice map $\mathrm{id}_{\mathcal{X}^*}\otimes\varphi : \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}^*}\otimes\varphi} \subseteq \mathcal{L}(\mathcal{E}\otimes\mathcal{G}, \mathcal{F}\otimes\mathcal{G}) \to \mathcal{L}(\mathcal{E}, \mathcal{F})$ and the KSGNS-map $\mathrm{id}_{\mathcal{X}^*}\otimes\Lambda : \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}^*}\otimes\varphi} \subseteq \mathcal{L}(\mathcal{E}\otimes\mathcal{G}, \mathcal{F}\otimes\mathcal{G}) \to \mathcal{L}(\mathcal{E}, \mathcal{F}\otimes H)$ which are the restrictions to the lower left corner of the corresponding maps for $L(\mathcal{X})$. And in this way we get (as in Proposition 2.4.19)

$$\bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{X})}\otimes\varphi} = \begin{pmatrix} \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} & \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi} \\ \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}^*}\otimes\varphi} & \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{F})}\otimes\varphi} \end{pmatrix}, \tag{2.9}$$

$$\bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{X})}\otimes\varphi} = \begin{pmatrix} \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi} & \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi} \\ \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}^*}\otimes\varphi} & \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{K}(\mathcal{F})}\otimes\varphi} \end{pmatrix}, \tag{2.10}$$

and

$$(\mathrm{id}_{L(\mathcal{X})}\otimes\varphi)\begin{pmatrix} x_1 & x_2 \\ x_2 & x_4 \end{pmatrix} = \begin{pmatrix} (\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi)(x_1) & (\mathrm{id}_{\mathcal{X}}\otimes\varphi)(x_2) \\ (\mathrm{id}_{\mathcal{X}^*}\otimes\varphi)(x_3) & (\mathrm{id}_{\mathcal{K}(\mathcal{F})}\otimes\varphi)(x_4) \end{pmatrix}, \tag{2.11}$$

$$(\mathrm{id}_{L(\mathcal{X})}\otimes\Lambda)\begin{pmatrix} x_1 & x_2 \\ x_2 & x_4 \end{pmatrix} = \begin{pmatrix} (\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\Lambda)(x_1) & (\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(x_2) \\ (\mathrm{id}_{\mathcal{X}^*}\otimes\Lambda)(x_3) & (\mathrm{id}_{\mathcal{K}(\mathcal{F})}\otimes\Lambda)(x_4) \end{pmatrix}. \tag{2.12}$$

This is the most general situation we are going to need. Note that if $\mathcal{F} = B$, then we have $\mathcal{X} = \mathcal{K}(B, \mathcal{E}) \cong \mathcal{E}$, and we get back $\mathrm{id}_{\mathcal{X}}\otimes\varphi = \mathrm{id}_{\mathcal{E}}\otimes\varphi$ and $\mathrm{id}_{\mathcal{X}}\otimes\Lambda = \mathrm{id}_{\mathcal{E}}\otimes\Lambda$. And if $\mathcal{E} = B$, then $\mathcal{X} = \mathcal{K}(\mathcal{F}, B) \cong \mathcal{F}^*$ and we get back $\mathrm{id}_{\mathcal{X}}\otimes\varphi = \mathrm{id}_{\mathcal{F}^*}\otimes\varphi$ and $\mathrm{id}_{\mathcal{X}}\otimes\Lambda = \mathrm{id}_{\mathcal{F}^*}\otimes\Lambda$.

We collect some properties of the maps defined above in the following result.

Proposition 2.4.21. *Suppose that $\mathcal{E}$ and $\mathcal{F}$ are Hilbert B-modules and consider the Hilbert $\mathcal{K}(\mathcal{E}), \mathcal{K}(\mathcal{F})$-bimodule $\mathcal{X} := \mathcal{K}(\mathcal{F}, \mathcal{E})$ as above. Let φ be a proper weight on a C^*-algebra $\mathcal{G}$ with a GNS-construction (H, π, Λ).*

(i) *If $z \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ and $s \in \bar{\mathcal{N}}_\varphi$ then $z \otimes s \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ and*

$$(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(z\otimes s) = z\otimes\Lambda(s).$$

(ii) *Let $x \in \mathcal{L}(\mathcal{F}\otimes\mathcal{G}, \mathcal{E}\otimes\mathcal{G})$. Then $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ if and only if $x^*x \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{F})}\otimes\varphi}$. Moreover, for $x, y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ we have $x^*y \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{F})}\otimes\varphi}$ and*

$$(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(x)^*(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(y) = (\mathrm{id}_{\mathcal{K}(\mathcal{F})}\otimes\varphi)(y^*x).$$

(iii) *For all* $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}} \otimes \varphi}$, $k \in \mathcal{K}(\mathcal{E})$ *and* $s \in \mathcal{N}_\varphi$ *we have* $x^*(k \otimes s) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}^*} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(x)^*\big(k \otimes \Lambda(s)\big) = (\mathrm{id}_{\mathcal{X}^*} \otimes \varphi)\big(x^*(k \otimes s)\big).$$

For all $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}} \otimes \varphi}$, $\xi \in \mathcal{E}$ *and* $s \in \mathcal{N}_\varphi$ *we have* $x^*(\xi \otimes s) \in \tilde{\mathcal{M}}_{\mathrm{id}_{\mathcal{F}} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(x)^*\big(\xi \otimes \Lambda(s)\big) = (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(x^*(\xi \otimes s)\big).$$

(iv) *For any* $x \in \mathcal{L}(\mathcal{F} \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G})$, *we have*

$$x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}} \otimes \varphi} \iff x(k \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}} \otimes \varphi}, \forall k \in \mathcal{K}(\mathcal{F}) \iff x(\eta \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi}, \forall \eta \in \mathcal{F}$$

and in this case

$$(\mathrm{id}_{\mathcal{X}} \otimes \Lambda)\big(x(k \otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(x)k \quad \text{and} \quad (\mathrm{id}_{\mathcal{E}} \otimes \Lambda)\big(x(\eta \otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(x)\eta.$$

(v) *If* $x \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G})$ *and* $y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}} \otimes \varphi}$ *then* $xy \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{F}^*} \otimes \varphi}$ *and*

$$(\mathrm{id}_{\mathcal{F}^*} \otimes \Lambda)(xy) = (\mathrm{id}_{\mathcal{E}^*} \otimes \pi)(x)(\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(y).$$

Proof. Statements (i) and (ii) are analogous to Proposition 2.4.20 (i) and (ii), respectively, and they are proved using Equations (2.11), (2.12) and Proposition 2.4.6(i),(ii).

We prove (iii). Since $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} k \otimes s & 0 \\ 0 & 0 \end{pmatrix}$ belong to $\bar{\mathcal{N}}_{\mathrm{id}_{L(\mathcal{X})} \otimes \varphi}$, we have

$$\begin{pmatrix} 0 & 0 \\ x^*(k \otimes s) & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} k \otimes s & 0 \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{X})} \otimes \varphi},$$

which implies that $x^*(k \otimes s) \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{X})} \otimes \varphi}$ and

$$\begin{aligned}
\begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(x)^*\big(k \otimes \Lambda(s)\big) & 0 \end{pmatrix} &= \begin{pmatrix} 0 & (\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(x) \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} k \otimes \Lambda(s) & 0 \\ 0 & 0 \end{pmatrix} \\
&= (\mathrm{id}_{L(\mathcal{X})} \otimes \Lambda) \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} k \otimes \Lambda(s) & 0 \\ 0 & 0 \end{pmatrix} \\
&= (\mathrm{id}_{L(\mathcal{X})} \otimes \varphi) \left(\begin{pmatrix} 0 & 0 \\ x^* & 0 \end{pmatrix} \begin{pmatrix} k \otimes s & 0 \\ 0 & 0 \end{pmatrix} \right) \\
&= (\mathrm{id}_{L(\mathcal{X})} \otimes \varphi) \begin{pmatrix} 0 & 0 \\ x^*(k \otimes s) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{X}^*} \otimes \varphi)\big(x^*(k \otimes s)\big) & 0 \end{pmatrix}.
\end{aligned}$$

Thus $(\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(x)^*\big(k \otimes \Lambda(s)\big) = (\mathrm{id}_{\mathcal{X}^*} \otimes \varphi)\big(x^*(k \otimes s)\big)$. To prove the second part we may suppose that $\xi = k\eta$, where $k \in \mathcal{K}(\mathcal{E})$ and $\eta \in \mathcal{E}$. It follows from the first part that $x^*(k \otimes s) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}^*} \otimes \varphi}$ and

$$(\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(x)^*\big(\xi \otimes \Lambda(s)\big) = (\mathrm{id}_{\mathcal{X}} \otimes \Lambda)(x)^*\big(k \otimes \Lambda(s)\big)\eta = (\mathrm{id}_{\mathcal{X}^*} \otimes \varphi)\big(x^*(k \otimes s)\big)\eta.$$

So all we have to prove is that, for any $y \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}^*}\otimes\varphi}$ and $\eta \in \mathcal{E}$, we have $y(\eta \otimes 1_{\mathcal{G}}) \in \tilde{\mathcal{M}}_{\mathrm{id}_{\mathcal{F}}\otimes\varphi}$ and $(\mathrm{id}_{\mathcal{X}^*} \otimes \varphi)(y)\eta = (\mathrm{id}_{\mathcal{F}} \otimes \varphi)\big(y(\eta \otimes 1_{\mathcal{G}})\big)$. By definition, $y \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}^*}\otimes\varphi}$ if and only if $\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{X})}\otimes\varphi}$ and

$$\begin{pmatrix} 0 & 0 \\ (\mathrm{id}_{\mathcal{X}^*} \otimes \varphi)(y) & 0 \end{pmatrix} = (\mathrm{id}_{L(\mathcal{X})} \otimes \varphi) \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \in \mathcal{M}\big(L(\mathcal{X})\big) \cong \mathcal{L}(\mathcal{E} \oplus \mathcal{F}).$$

Now note that if $\mathcal{Y}$ is any Hilbert B-module, and $z \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{Y})}\otimes\varphi}$ and $\zeta \in \mathcal{Y}$, then $z(\zeta \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{Y}}\otimes\varphi}$ and $(\mathrm{id}_{\mathcal{Y}} \otimes \varphi)\big(z(\zeta \otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{K}(\mathcal{Y})} \otimes \varphi)(z)\zeta$. In fact, $z \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{Y})}\otimes\varphi}$ if and only if $\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{Y})}\otimes\varphi}$ and hence

$$\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix} \otimes 1_{\mathcal{G}} \right) = \begin{pmatrix} 0 & z(\zeta \otimes 1_{\mathcal{G}}) \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{L(\mathcal{Y})}\otimes\varphi}.$$

Thus $z(\zeta \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{Y}}\otimes\varphi}$ and

$$\begin{aligned}
\begin{pmatrix} 0 & (\mathrm{id}_{\mathcal{Y}} \otimes \varphi)\big(z(\zeta \otimes 1_{\mathcal{G}})\big) \\ 0 & 0 \end{pmatrix} &= (\mathrm{id}_{L(\mathcal{Y})} \otimes \varphi)\left(\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix} \otimes 1_{\mathcal{G}} \right) \right) \\
&= \begin{pmatrix} (\mathrm{id}_{\mathcal{K}(\mathcal{Y})} \otimes \varphi)(z) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (\mathrm{id}_{\mathcal{K}(\mathcal{Y})} \otimes \varphi)(y)\zeta \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore $(\mathrm{id}_{\mathcal{Y}} \otimes \varphi)\big(z(\zeta \otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{K}(\mathcal{Y})} \otimes \varphi)(y)\zeta$ as claimed. Applying this now to $\mathcal{Y} := \mathcal{E} \oplus \mathcal{F}$ and observing that $\mathcal{K}(\mathcal{Y}) \cong L(\mathcal{X})$ we get that

$$\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} \eta \otimes 1_{\mathcal{G}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y(\eta \otimes 1_{\mathcal{G}}) \end{pmatrix} \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}\oplus\mathcal{F}}\otimes\varphi} \subseteq \tilde{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}\oplus\mathcal{F}}\otimes\varphi},$$

which implies that $y(\eta \otimes 1_{\mathcal{G}}) \in \tilde{\mathcal{M}}_{\mathrm{id}_{\mathcal{F}}\otimes\varphi}$ and

$$\begin{aligned}
\begin{pmatrix} 0 \\ (\mathrm{id}_{\mathcal{X}^*} \otimes \varphi)(y)\eta \end{pmatrix} &= (\mathrm{id}_{L(\mathcal{X})} \otimes \varphi) \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} \eta \\ 0 \end{pmatrix} \\
&= (\mathrm{id}_{\mathcal{E}\oplus\mathcal{F}} \otimes \varphi)\left(\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} \eta \otimes 1_{\mathcal{G}} \\ 0 \end{pmatrix} \right) \\
&= (\mathrm{id}_{\mathcal{E}\oplus\mathcal{F}} \otimes \varphi) \begin{pmatrix} 0 \\ y(\eta \otimes 1_{\mathcal{G}}) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ (\mathrm{id}_{\mathcal{F}} \otimes \varphi)\big(y(\eta \otimes 1_{\mathcal{G}})\big) \end{pmatrix}.
\end{aligned}$$

Now we prove (iv). By (ii), $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ if and only if $x^*x \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{F})}\otimes\varphi}$. By Lemma 2.4.4, this is equivalent to $x(k\otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ for all $k \in \mathcal{K}(\mathcal{F})$ (see also Remark 2.4.7). An analogous argument also shows that $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ if and only if $x(\eta\otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$ for all $\eta \in \mathcal{F}$. Applying now (iii), we get, for all $\xi \in \mathcal{E}$ and $s \in \mathcal{N}_\varphi$,

$$\begin{aligned}(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)\big(x(\eta\otimes 1_{\mathcal{G}})\big)^*\big(\xi\otimes\Lambda(s)\big) &= (\mathrm{id}_B\otimes\varphi)\big((\eta^*\otimes 1_{\mathcal{G}})x^*(\xi\otimes s)\big)\\ &= \eta^*(\mathrm{id}_{\mathcal{F}}\otimes\varphi)\big(x^*(\xi\otimes s)\big)\\ &= \eta^*(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(x)^*\big(\xi\otimes\Lambda(s)\big).\end{aligned}$$

Thus $(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)\big(x(\eta\otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(x)\eta$. It follows that

$$(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)\big(x(k\otimes 1_{\mathcal{G}})\big)\eta = (\mathrm{id}_{\mathcal{E}}\otimes\Lambda)\big(x(k\eta\otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(x)k\eta$$

for all $\eta \in \mathcal{F}$ and $k \in \mathcal{K}(\mathcal{F})$, whence $(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)\big(x(k\otimes 1_{\mathcal{G}})\big) = (\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(x)k$.

Finally we prove (v). We have

$$(xy)^*(xy) = y^*x^*xy \le \|x\|^2 y^*y \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{F})}\otimes\varphi},$$

and hence $xy \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{F}^*}\otimes\varphi}$. Thus $xy(\eta\otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ for all $\eta\in\mathcal{F}$, and

$$(\mathrm{id}_{\mathcal{F}^*}\otimes\Lambda)(xy)\eta = (\mathrm{id}_B\otimes\Lambda)\big(xy(\eta\otimes 1_{\mathcal{G}})\big).$$

Now note that for all $x \in \mathcal{L}(\mathcal{E}\otimes\mathcal{G}, B\otimes\mathcal{G})$ and $z \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$ we have $xz \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ and $(\mathrm{id}_B\otimes\Lambda)(xz) = (\mathrm{id}_{\mathcal{E}^*}\otimes\pi)(x)(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)(z)$. In fact, this follows by considering x and z embedded in $\mathcal{M}\big(L(\mathcal{E})\otimes\mathcal{G}\big) \cong L\big(\mathcal{M}(\mathcal{E}\otimes\mathcal{G})\big)$ and working with $\mathrm{id}_{L(\mathcal{E})}\otimes\Lambda$ and $\mathrm{id}_{L(\mathcal{E})}\otimes\pi$. Thus, by (iv), we get

$$\begin{aligned}(\mathrm{id}_{\mathcal{F}^*}\otimes\Lambda)(xy)\eta &= (\mathrm{id}_B\otimes\Lambda)\big(xy(\eta\otimes 1_{\mathcal{G}})\big)\\ &= (\mathrm{id}_{\mathcal{E}^*}\otimes\pi)(x)(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)\big(y(\eta\otimes 1_{\mathcal{G}})\big)\\ &= (\mathrm{id}_{\mathcal{E}^*}\otimes\pi)(x)(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(y)\eta.\end{aligned}$$

We conclude that $(\mathrm{id}_{\mathcal{F}^*}\otimes\Lambda)(xy) = (\mathrm{id}_{\mathcal{E}^*}\otimes\pi)(x)(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(y)$. □

Finally, we mention some generalized KMS-properties. Suppose that φ is a KMS-weight and let σ be a modular group for φ. Then we can generalize Proposition 2.4.13 in the following way.

Proposition 2.4.22. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules and define $\mathcal{X} := \mathcal{K}(\mathcal{F},\mathcal{E})$. Let J be the modular conjugation of φ in the GNS-construction (H,π,Λ).*

(i) *If $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ and $a \in \mathcal{D}(\sigma_{\frac{\mathrm{i}}{2}})$, then $x(1\otimes a) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ and*

$$(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)\big(x(1\otimes a)\big) = \big(1\otimes J\pi\big(\sigma_{\frac{\mathrm{i}}{2}}(a)\big)^*J\big)(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(x).$$

(ii) *If $x \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ and $a \in \mathcal{D}(\sigma_{-\mathrm{i}})$, then $(1\otimes a)x$ and $x\big(1\otimes\sigma_{-\mathrm{i}}(a)\big)$ belong to $\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$ and*

$$(\mathrm{id}_{\mathcal{X}}\otimes\varphi)\big((1\otimes a)x\big) = (\mathrm{id}_{\mathcal{X}}\otimes\varphi)\big(x\big(1\otimes\sigma_{-\mathrm{i}}(a)\big)\big).$$

Proof. Using Equations (2.9), (2.10), (2.11) and (2.12), this follows from Proposition 2.4.13 applied to the linking algebra of $\mathcal{X}$. □

2.5 Locally compact quantum groups

In this section we give a short overview of some basic concepts from locally compact quantum group theory following [41].

Let $\mathcal{G}$ be a C^*-algebra. A *comultiplication* on $\mathcal{G}$ is a nondegenerate $*$-homomorphism $\Delta : \mathcal{G} \to \mathcal{M}(\mathcal{G} \otimes \mathcal{G})$ satisfying

$$(\Delta \otimes \mathrm{id}_{\mathcal{G}})\Delta = (\mathrm{id}_{\mathcal{G}} \otimes \Delta)\Delta.$$

A C^*-algebra with a comultiplication is called a *bi-C^*-algebra*.

Fix a bi-C^*-algebra $\mathcal{G}$ and let φ be a proper weight on $\mathcal{G}$. We say that φ is *left invariant* if

$$\varphi\big((\omega \otimes \mathrm{id}_{\mathcal{G}})\Delta(b)\big) = \omega(1)\varphi(b) \quad \text{for all } b \in \mathcal{M}_\varphi^+ \text{ and } \omega \in \mathcal{G}_+^*.$$

The weight φ is called *right invariant* if

$$\varphi\big((\mathrm{id}_{\mathcal{G}} \otimes \omega)\Delta(b)\big) = \omega(1)\varphi(b) \quad \text{for all } b \in \mathcal{M}_\varphi^+ \text{ and } \omega \in \mathcal{G}_+^*.$$

Notice that we use the extension of φ to $\mathcal{M}(\mathcal{G})^+$ in the equations above, because we only know that $(\omega \otimes \mathrm{id}_{\mathcal{G}})\Delta(b), (\mathrm{id}_{\mathcal{G}} \otimes \omega)\Delta(b) \in \mathcal{M}(\mathcal{G})^+$.

Definition 2.5.1. Let $\mathcal{G}$ be a bi-C^*-algebra with a comultiplication Δ satisfying

$$\mathcal{G} \otimes \mathcal{G} = \overline{\mathrm{span}}\big(\Delta(\mathcal{G})(1_{\mathcal{G}} \otimes \mathcal{G})\big) = \overline{\mathrm{span}}\big(\Delta(\mathcal{G})(\mathcal{G} \otimes 1_{\mathcal{G}})\big).$$

Assume there is a faithful left invariant KMS-weight φ on $\mathcal{G}$ (*left Haar weight*) and a faithful right invariant KMS-weight ψ on $\mathcal{G}$ (*right Haar weight*). Then $\mathcal{G}$ is called a (*reduced*) *locally compact quantum group*.

The original definition in [41] only assumes that the weights φ and ψ are "approximate KMS" and only φ is faithful, but this turns out to be equivalent to the requirement that both are faithful KMS-weights. Moreover, any proper left (resp. right) invariant weight on a locally compact quantum group is automatically faithful and KMS and is a positive scalar multiple of φ (resp. ψ).

Fix a GNS-construction (H, π, Λ) for the left Haar weight φ. Since φ is faithful, $\pi : \mathcal{G} \to \mathcal{L}(H)$ implements a faithful representation of $\mathcal{G}$ on H. So whenever we want, we can (and we will) assume that $\mathcal{G} \subseteq \mathcal{L}(H)$ and $\pi = \iota$ is the inclusion map $\mathcal{G} \hookrightarrow \mathcal{L}(H)$. Sometimes, we shall also use the notation $L^2(\mathcal{G}) = H$. Therefore whenever $\mathcal{G}$ is a locally compact quantum group, then (we assume that) $\mathcal{G}$ is a nondegenerate C^*-subalgebra of $\mathcal{L}\big(L^2(\mathcal{G})\big)$, where $(L^2(\mathcal{G}), \iota, \Lambda)$ is a GNS-construction for the left Haar weight φ. We denote by J and ∇ the modular conjugation and the modular operator, respectively, of φ in the GNS-construction $(L^2(\mathcal{G}), \iota, \Lambda)$.

Many objects associated to a locally compact quantum group can be constructed from the definition: the *antipode* of $\mathcal{G}$ will be denoted by S, the *scaling group* by τ, the *unitary antipode* by R, whereas the *scaling constant* will be denoted by ν. The *modular element* of $\mathcal{G}$ will be denoted by δ. For details see [41].

Given a left Haar weight φ, we have a natural choice for the right Haar weight ψ by setting $\psi = \varphi R$. We always choose ψ in this way. With this choice, we have a natural GNS-construction of the form $(L^2(\mathcal{G}), \iota, \Gamma)$ for right Haar weight ψ.

Let $\tilde{\mathcal{G}} := \mathcal{G}'' \subseteq \mathcal{L}\big(L^2(\mathcal{G})\big)$ be the von Neumann algebra generated by $\mathcal{G}$. The Haar weights φ and ψ can be extended to $\tilde{\mathcal{G}}$. We denote these extensions by $\tilde{\varphi}$ and $\tilde{\psi}$, respectively. Moreover, the comultiplication Δ extends to a comultiplication $\tilde{\Delta}$ on $\tilde{\mathcal{G}}$. In this way, $(\tilde{\mathcal{G}}, \tilde{\Delta})$ is a *von Neumann algebraic quantum group* with left and right Haar weights $\tilde{\varphi}$ and $\tilde{\psi}$, respectively. Finally, we remark that the von Neumann algebra $\tilde{\mathcal{G}}$ is in *standard form* ([67, 10.15]). As a consequence all the normal functionals on $\tilde{\mathcal{G}}$ are vector functionals ([68, Theorem V.3.15]). In other words, we have

$$\tilde{\mathcal{G}}_* = \{\omega_{u,v} : u, v \in H\}, \tag{2.13}$$

where $\omega_{u,v}(x) := \langle u \,|\, xv\rangle$ for all $x \in \tilde{\mathcal{G}}$.

2.5.1 The multiplicative unitaries

The *left regular corepresentation* of $\mathcal{G}$ is the unitary $W \in \mathcal{L}(H \otimes H)$ defined by the equation

$$W^*\big(\Lambda(a) \otimes \Lambda(b)\big) = (\Lambda \otimes \Lambda)\big(\Delta(b)(a \otimes 1)\big) \quad \text{for all } a, b \in \mathcal{N}_\varphi. \tag{2.14}$$

It is a *multiplicative unitary*, meaning that it satisfies the pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.\text{[5]}$$

One can recover $\mathcal{G}$ from W by $\mathcal{G} = \overline{\operatorname{span}}\{(\mathrm{id} \otimes \omega)(W) : \omega \in \mathcal{L}(H)_*\}$ and the comultiplication by $\Delta(x) = W^*(1 \otimes x)W$ for all $x \in \mathcal{G}$. Once we have W, we can define

$$\widehat{\mathcal{G}} := \overline{\operatorname{span}}\{(\omega \otimes \mathrm{id})(W) : \omega \in \mathcal{L}(H)_*\}.$$

Defining a comultiplication $\hat{\Delta}$ on $\widehat{\mathcal{G}}$ by $\hat{\Delta} = \hat{W}^*(1 \otimes x)\hat{W}$, where $\hat{W} := \Sigma W^* \Sigma$ and $\Sigma : H \otimes H \to H \otimes H$ is the flip operator, one proves that $(\widehat{\mathcal{G}}, \hat{\Delta})$ is again a locally compact quantum group, called the *dual* of $\mathcal{G}$. Objects associated to $\widehat{\mathcal{G}}$ are denoted by adding the symbol $\hat{}$ on the corresponding object of $\mathcal{G}$. Thus the Haar weights of $\widehat{\mathcal{G}}$ are denoted by $\hat{\varphi}$ and $\hat{\psi}$. If we take this process once again we get the generalized Pontrjagin duality $(\widehat{\widehat{\mathcal{G}}}, \hat{\hat{\Delta}}) = (\mathcal{G}, \Delta)$. Finally, we remark that $W \in \mathcal{M}(\mathcal{G} \otimes \widehat{\mathcal{G}})$, and we have the relations $(\Delta \otimes \mathrm{id})(W) = W_{13}W_{23}$ and $(\mathrm{id} \otimes \hat{\Delta})(W) = W_{13}W_{12}$.

Analogously, one defines the *right regular corepresentation* of $\mathcal{G}$. It is a multiplicative unitary $V \in \mathcal{L}(H \otimes H)$ defined in terms of the right Haar weight ψ by

$$V\big(\Gamma(a) \otimes \Gamma(b)\big) = (\Gamma \otimes \Gamma)\big(\Delta(a)(1 \otimes b)\big), \quad a, b \in \mathcal{N}_\psi.$$

[5] Here and throughout the rest of this work we shall use the standard leg numbering notation. For example, W_{12} and W_{23} are simply $1 \otimes W$ and $W \otimes 1$, respectively, and W_{13} stands for W sitting on the first and third factor. Precise definitions can be found in [6].

We have $V \in \mathcal{M}(\hat{J}\widehat{\mathcal{G}}\hat{J} \otimes \mathcal{G})$, $\Delta(x) = V(x \otimes 1)V^*$ for all $x \in \mathcal{G}$ and $(\mathrm{id} \otimes \Delta)(V) = V_{12}V_{13}$. We also mention the relations

$$V = (\hat{J} \otimes \hat{J})\hat{W}(\hat{J} \otimes \hat{J}) \tag{2.15}$$

and

$$\hat{W} = (J \otimes \hat{J})\hat{W}^*(J \otimes \hat{J}). \tag{2.16}$$

Given a locally compact quantum group $\mathcal{G}$, we define $\mathcal{G}^{\mathrm{c}} := J\mathcal{G}J$. Then $\mathcal{G}^{\mathrm{c}}$ is also a locally compact quantum group, called the *C^*-commutant* of $\mathcal{G}$. The comultiplication of $\mathcal{G}^{\mathrm{c}}$ is defined by $\Delta^{\mathrm{c}}(x) := (J \otimes J)\Delta(JxJ)(J \otimes J)$, for all $x \in \mathcal{G}^{\mathrm{c}}$. Canonical choices for left and right Haar weights on $\mathcal{G}^{\mathrm{c}}$ are $\varphi^c(x) = \varphi(JxJ)$ and $\psi^c(x) = \psi(JxJ)$ for all $x \in (\mathcal{G}^{\mathrm{c}})^+$. The von Neumann algebraic quantum group associated to $\mathcal{G}^{\mathrm{c}}$ is $\mathcal{G}' = J\tilde{\mathcal{G}}J$, the commutant of $\mathcal{G}$. In particular, $\mathcal{G}^{\mathrm{c}} \subseteq \mathcal{G}'$.

We also define the *opposite* quantum group $\mathcal{G}^{\mathrm{op}}$ of $\mathcal{G}$. The underlying C^*-algebra is $\mathcal{G}$ itself, and the comultiplication is defined by flipping the comultiplication of $\mathcal{G}$: $\Delta^{\mathrm{op}}(x) := \sigma\Delta(x)$, where $\sigma : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ is the flip homomorphism. A left (right) Haar weight of $\mathcal{G}^{\mathrm{op}}$ is a right (left) Haar weight of $\mathcal{G}$. The von Neumann algebraic quantum group of $\mathcal{G}^{\mathrm{op}}$ is the opposite von Neumann algebraic quantum group of $\tilde{\mathcal{G}}$.

Finally, we remark that $\widehat{\mathcal{G}^{\mathrm{op}}} = \hat{J}\widehat{\mathcal{G}}\hat{J}$ and $\widehat{\mathcal{G}^{\mathrm{c}}} = \widehat{\mathcal{G}}^{\mathrm{op}}$, that is, the dual of the opposite is the C^*-commutant of the dual and the dual of the C^*-commutant is the opposite of the dual. We also remark that the C^*-commutant of the opposite is equal to the opposite of the C^*-commutant, that is, we have $\mathcal{G}^{\mathrm{op,c}} = \mathcal{G}^{\mathrm{c,op}}$. Moreover, we have $\mathcal{G}^{\mathrm{op,c}} \cong \mathcal{G}$ as locally compact quantum groups. The isomorphism is given by $\mathrm{Ad}_U : \mathcal{G}^{\mathrm{op,c}} \to \mathcal{G}$, where $\mathrm{Ad}_U(x) = UxU^*$ and $U := \hat{J}J$. This follows from the fact that the unitary antipode R of $\mathcal{G}$ satisfies $R(x) = \hat{J}x^*\hat{J}$ and $(R \otimes R)\Delta(x) = \Delta^{\mathrm{op}}\big(R(x)\big)$ for all $x \in \mathcal{G}$. The same relations are true on the von Neumann algebraic level (see [73, Proposition 1.14.10] for details).

Example 2.5.2. **(1)** Let G be a locally compact group. Then the commutative C^*-algebra $\mathcal{G} = \mathcal{C}_0(G)$ has a natural structure of locally compact quantum group. The comultiplication is given by $\Delta(f)(s,t) = f(st)$ for all $f \in \mathcal{C}_0(G)$ and $s,t \in G$, where we identify $\mathcal{M}\big(\mathcal{C}_0(G) \otimes \mathcal{C}_0(G)\big) \cong \mathcal{C}_b(G)$ in the usual way. Any left Haar weight on $\mathcal{G}$ is given by $\varphi(f) = \int_G f(t)\,\mathrm{d}t$, where $\mathrm{d}t$ is some left Haar measure on G. Analogously, right Haar weights correspond to right Haar measures on G. Fix a left Haar measure $\mathrm{d}t$ on G and let $L^2(G)$ be the space of (equivalence classes of) square-integrable functions on G. There is a canonical GNS-construction for φ of the form $(L^2(G), M, \Lambda)$, where $M : \mathcal{C}_0(G) \to \mathcal{L}\big(L^2(G)\big)$ is the multiplication representation and Λ denotes the inclusion map from $\mathcal{N}_\varphi = \mathcal{C}_0(G) \cap L^2(G)$ into $L^2(G)$. We always use this GNS-construction. The left regular corepresentation of $\mathcal{G}$ is the unitary $W \in \mathcal{L}\big(L^2(G) \otimes L^2(G)\big)$ given by $W\zeta(s,t) = \zeta(s, s^{-1}t)$ for all $\zeta \in L^2(G \times G) \cong L^2(G) \otimes L^2(G)$ and $s,t \in G$. The right regular corepresentation of $\mathcal{G}$ is the unitary V on $L^2(G \times G)$ given by $V\zeta(s,t) = \delta_G(t)^{\frac{1}{2}}\xi(st,t)$, where δ_G denotes the modular function of G. Since $\mathcal{G}$ is commutative, the C^*-commutant of $\mathcal{G}$ coincides with $\mathcal{G}$. The opposite of $\mathcal{G}$ corresponds to $\mathcal{C}_0(G^{\mathrm{op}})$, where G^{op} denotes G with the opposite multiplication. This example describes all commutative locally compact quantum groups.

(2) The dual of the first example is $\widehat{\mathcal{G}} = C^*_{\mathrm{r}}(G)$, the reduced group C^*-algebra of G. Recall that $C^*_{\mathrm{r}}(G)$ is the C^*-subalgebra of $\mathcal{L}\big(L^2(G)\big)$ generated by the operators $\lambda(f)$, $f \in$

$\mathcal{C}_c(G)$, where $\lambda : G \to \mathcal{L}\big(L^2(G)\big)$ is the left regular representation of G: $\lambda_t(\xi)|_s = \xi(t^{-1}s)$ for all $\xi \in L^2(G)$ and $s, t \in G$. The comultiplication on $\widehat{\mathcal{G}}$ is characterized by the equation $\hat{\Delta}(\lambda_t) = \lambda_t \otimes \lambda_t$ for all $t \in G$. The quantum group $\widehat{\mathcal{G}}$ is cocommutative in the sense that it is equal to its opposite or, equivalent, its dual is commutative. All cocommutative locally compact quantum groups are of this form. As a consequence of the cocommutativity, any left Haar weight on $\widehat{\mathcal{G}}$ is also a right Haar weight, that is, $\widehat{\mathcal{G}}$ is unimodular (in other words, the modular element is trivial). A canonical choice for a (left and right) Haar weight on $\widehat{\mathcal{G}}$ is the so-called *Plancherel weight* which we are going to describe later in details (see Section 6.1).

A locally compact quantum group $\mathcal{G}$ is called *compact* if it is unital as a C^*-algebra. Compact quantum groups are unimodular and the Haar weight is bounded. Conversely, if the (left or right) Haar weight is bounded, then $\mathcal{G}$ is compact.

Dually, $\mathcal{G}$ is called *discrete* if the dual $\widehat{\mathcal{G}}$ is compact. In the case of groups, that is, for $\mathcal{G} = \mathcal{C}_0(G)$, we have that $\mathcal{G}$ is compact (resp. discrete) if and only if G is compact (resp. discrete). Therefore, the dual $\widehat{\mathcal{G}} = C^*_{\mathrm{r}}(G)$ is compact (resp. discrete) if and only if G is discrete (resp. compact).

2.5.2 The L^1-algebra of $\mathcal{G}$

For a locally compact quantum group $\mathcal{G}$, we define

$$L^1(\mathcal{G}) := \overline{\mathrm{span}}\{a\varphi b^* : a, b \in \mathcal{N}_\varphi\} = \overline{\mathrm{span}}\{\omega_{\xi,\eta} : \xi, \eta \in H\} \subseteq \mathcal{G}^*,$$

where $a\varphi b^* \in \mathcal{G}^*$ is defined by $(a\varphi b^*)(x) := \varphi(b^* x a) = \langle \Lambda(b) | x\Lambda(a)\rangle$ for all $a, b \in \mathcal{N}_\varphi$ and $\omega_{\xi,\eta}(x) := \langle \xi \mid x\eta\rangle$ for all $\xi, \eta \in H$ and $x \in \mathcal{G}$. The restriction map $\tilde{\mathcal{G}}_* \to L^1(\mathcal{G})$ is an isomorphism (of Banach spaces) between the predual $\tilde{\mathcal{G}}_*$ of $\tilde{\mathcal{G}}$ and $L^1(\mathcal{G})$. In particular, it follows from Equation (2.13) that

$$L^1(\mathcal{G}) = \{\omega_{\xi,\eta} : \xi, \eta \in H\}. \tag{2.17}$$

The dual Banach space $\mathcal{G}^*$ can be turned into a Banach algebra by defining the multiplication $(\omega \cdot \theta)(x) = (\omega \otimes \theta)\Delta(x)$. Moreover, it can be proved that $L^1(\mathcal{G})$ is a two-sided ideal of $\mathcal{G}^*$ and for every $a \in \mathcal{M}(\mathcal{G})$ and $\omega \in L^1(\mathcal{G})$ we have $a\omega, \omega a \in L^1(\mathcal{G})$, where $(a\omega)(x) := \omega(xa)$ and $(\omega a)(x) := \omega(ax)$.

The equation $\lambda(\omega) = (\omega \otimes \mathrm{id})(W)$ defines an injective contractive algebra homomorphism $\lambda : \mathcal{G}^* \to \mathcal{M}(\widehat{\mathcal{G}})$ such that $\lambda(L^1(\mathcal{G}))$ is a dense subalgebra of $\widehat{\mathcal{G}}$.

Example 2.5.3. Let G be a locally compact group and consider the quantum group $\mathcal{G} = \mathcal{C}_0(G)$. Here one can identify the dual space $\mathcal{G}^*$ with the space of all bounded complex measures $\mathcal{M}(G)$ on G. The product above becomes the usual convolution product of measures. The L^1-algebra of $\mathcal{G}$ is the usual L^1-algebra $L^1(G)$ of G (identified as a subalgebra of $\mathcal{M}(G)$ in the usual way) with convolution product of functions.

And for the dual of $\mathcal{G}$, that is, for $\widehat{\mathcal{G}} = C^*_{\mathrm{r}}(G)$, the dual space is (identified with) the (*reduced*) *Fourier–Stieltjes algebra* $B_{\mathrm{r}}(G)$ consisting of all bounded continuous functions of the form $t \mapsto \omega(\lambda_t)$, where $\omega \in C^*_{\mathrm{r}}(G)^*$ and λ denotes the left regular representation of G. The product of $B_{\mathrm{r}}(G)$ is simply the pointwise product of functions. The L^1-algebra of $\widehat{\mathcal{G}}$ is the *Fourier algebra* $A(G)$, that is, the subalgebra of $B_{\mathrm{r}}(G)$ consisting of the continuous functions of the form $t \mapsto \omega_{\xi,\eta}(\lambda_t) = \langle \xi \,|\, \lambda_t \eta \rangle$, where $\xi, \eta \in L^2(G)$. Note that by Equation (2.17) $A(G)$ is, in fact, a closed subspace of $B_{\mathrm{r}}(G)$ (and therefore a Banach algebra). For more details on $A(G)$ we refer to [22].

In general, the L^1-algebra of a locally compact quantum group has no bounded approximate unit. For instance, the Fourier algebra of a locally compact group G has a bounded approximate unit if and only if G is amenable (this is Leptin's Theorem [45]; see also [77, Theorem 7.1.3]). On the other hand, the L^1-algebra of G always has a bounded approximate unit. This leads us to:

Definition 2.5.4. A locally compact quantum group $\mathcal{G}$ is called *co-amenable*, if $L^1(\mathcal{G})$ has a bounded approximate unit.

By the discussion above, $\mathcal{C}_0(G)$ is always co-amenable, for any locally compact group G, and the dual $C^*_{\mathrm{r}}(G)$ is co-amenable if and only if G is amenable. There are several characterizations of co-amenability. For example, $\mathcal{G}$ is co-amenable if and only if the Banach algebra $\mathcal{G}^*$ is unital. In this case, the unit ϵ of $\mathcal{G}^*$ is the so-called *counit* of $\mathcal{G}$. It is, in fact, a $*$-homomorphism $\epsilon : \mathcal{G} \to \mathbb{C}$ and satisfies $(\mathrm{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$. All this is done in [8], where the notion of *amenability* is also defined. This is essentially the dual notion of co-amenability. For example, $\mathcal{C}_0(G)$ is amenable if and only if G is amenable and, on the other hand, $C^*_{\mathrm{r}}(G)$ is always amenable. It is shown in [8] that co-amenability of a locally compact quantum group $\mathcal{G}$ implies amenability of the dual $\widehat{\mathcal{G}}$. The converse, however, is an open problem.

Note that if $\mathcal{G}$ is co-amenable, then, in particular, $L^1(\mathcal{G})$ is a *nondegenerate* Banach algebra in the sense that the closed linear space of $L^1(\mathcal{G}) \cdot L^1(\mathcal{G})$ is dense in $L^1(\mathcal{G})$ (in fact, by Cohen's Factorization Theorem, we have $L^1(\mathcal{G}) \cdot L^1(\mathcal{G}) = L^1(\mathcal{G})$). However, nondegeneracy of $L^1(\mathcal{G})$ is weaker than co-amenability of $\mathcal{G}$. For example, the Fourier algebra $A(G)$ is always nondegenerate. This can be seen from the fact that the subspace $A_c(G)$ of all functions in $A(G)$ with compact support is dense in $A(G)$ (which follows from the fact that $\mathcal{C}_c(G)$ is dense in $L^2(G)$) and the fact that for any compact subset $K \subseteq G$, there is $\omega \in A_c(G)$ such that $\omega|_K = 1$ (see [22, Lemme 3.2]). This implies that $A_c(G) \cdot A_c(G) = A_c(G)$ and therefore $A(G)$ is nondegenerate. In fact, this is true for any locally compact quantum group:

Proposition 2.5.5. *Let $\mathcal{G}$ be a locally compact quantum group. Then $L^1(\mathcal{G})$ is a nondegenerate Banach algebra.*

Proof. Let W be the left regular corepresentation of $\mathcal{G}$. We know that the comultiplication of $\mathcal{G}$ is given by $\Delta(x) = W^*(1 \otimes x)W$ for all $x \in \mathcal{G}$. Thus, for all $\xi, \eta, f, g \in H = L^2(\mathcal{G})$ and $x \in \mathcal{G}$, we have

$$\begin{aligned}\omega_{\xi,\eta} \cdot \omega_{f,g}(x) &= (\omega_{\xi,\eta} \otimes \omega_{f,g})(W^*(1 \otimes x)W) \\ &= \langle W(\xi \otimes f) \,|\, (1 \otimes x) W(\eta \otimes g) \rangle.\end{aligned}$$

The assertion now follows from the fact that W is a unitary operator on $H \otimes H$. $\square$

2.5.3 The universal companion of $\mathcal{G}$

We mainly use locally compact groups in the reduced form in this work. However, we shall also sometimes need the universal form. In this section we give a short overview of locally compact quantum groups in the universal setting. We refer to [39] for details.

Let $\mathcal{G}$ be a locally compact quantum group. In general, since the antipode S of $\mathcal{G}$ is unbounded, the algebra $L^1(\mathcal{G})$ does not carry an appropriate $*$-structure. We define the following subspace of $L^1(\mathcal{G})$ (where $\mathcal{D}(S)$ denotes the domain of S):

$$L^1_*(\mathcal{G}) := \{\omega \in L^1(\mathcal{G}) : \text{ exists } \theta \in L^1(\mathcal{G}) \text{ such that } \theta(x) = \overline{\omega(S(x)^*)} \text{ for all } x \in \mathcal{D}(S)\}.$$

Then $L^1_*(\mathcal{G})$ is a dense subalgebra of $L^1(\mathcal{G})$, and it has an involution given by $\omega^*(x) := \overline{\omega(S(x)^*)}$ for all $\omega \in L^1_*(\mathcal{G})$ and $x \in \mathcal{D}(S)$. Moreover, $L^1_*(\mathcal{G})$ is a Banach $*$-algebra with the norm $\|\omega\|_* := \max\{\|\omega\|, \|\omega^*\|\}$. We denote the C^*-enveloping algebra of $L^1_*(\mathcal{G})$ by $\widehat{\mathcal{G}}_{\mathrm{u}}$. Therefore $\widehat{\mathcal{G}}_{\mathrm{u}}$ is, by definition, the completion of $L^1_*(\mathcal{G})$ with respect to the universal C^*-norm

$$\|\omega\|_{\mathrm{u}} := \sup\{\pi(\omega) : \pi \text{ is a } *\text{-representation of } L^1_*(\mathcal{G})\}.$$

The map $\lambda : L^1_*(\mathcal{G}) \to \widehat{\mathcal{G}} \subseteq \mathcal{L}(H)$, $\omega \mapsto (\omega \otimes \mathrm{id})(W)$ is an injective $*$-representation, and therefore $\|\cdot\|_{\mathrm{u}}$ is really a norm and not just a semi-norm. Thus one has an embedding $L^1_*(\mathcal{G}) \hookrightarrow \widehat{\mathcal{G}}_{\mathrm{u}}$, and we identify $L^1_*(\mathcal{G}) \subseteq \widehat{\mathcal{G}}_{\mathrm{u}}$ via this embedding. The following universal property holds: whenever A is a C^*-algebra and $\pi : L^1_*(\mathcal{G}) \to A$ is a $*$-homomorphism, there is a unique $*$-homomorphism $\pi_{\mathrm{u}} : \widehat{\mathcal{G}}_{\mathrm{u}} \to A$ which extends π. Abusing the notation we write $\pi_{\mathrm{u}} = \pi$. In particular, we denote by $\lambda : \widehat{\mathcal{G}}_{\mathrm{u}} \to \widehat{\mathcal{G}}$ the extension of $\lambda : L^1_*(\mathcal{G}) \to \widehat{\mathcal{G}}$. Since $\lambda(L^1_*(\mathcal{G}))$ is dense in $\widehat{\mathcal{G}}$, the map $\lambda : \widehat{\mathcal{G}}_{\mathrm{u}} \to \widehat{\mathcal{G}}$ is a surjective $*$-homomorphism.

There is a unitary $\hat{\mathcal{W}} \in \mathcal{M}(\mathcal{G} \otimes \widehat{\mathcal{G}}_{\mathrm{u}})$, called the *universal corepresentation* of $\mathcal{G}$, such that $\omega = (\omega \otimes \mathrm{id})(\hat{\mathcal{W}})$ for all $\omega \in L^1_*(\mathcal{G})$. It satisfies $(\Delta \otimes \mathrm{id})(\hat{\mathcal{W}}) = \hat{\mathcal{W}}_{13}\hat{\mathcal{W}}_{23}$. The *universal locally compact quantum group* $(\widehat{\mathcal{G}}_{\mathrm{u}}, \hat{\Delta}_{\mathrm{u}})$ of $(\widehat{\mathcal{G}}, \hat{\Delta})$ is defined in such a way that the comultiplication $\hat{\Delta}_{\mathrm{u}} : \widehat{\mathcal{G}}_{\mathrm{u}} \to \mathcal{M}(\widehat{\mathcal{G}}_{\mathrm{u}} \otimes \widehat{\mathcal{G}}_{\mathrm{u}})$ satisfies $(\mathrm{id} \otimes \hat{\Delta}_{\mathrm{u}})(\hat{\mathcal{W}}) = \hat{\mathcal{W}}_{13}\hat{\mathcal{W}}_{12}$. Moreover, one has that

$$(\mathrm{id} \otimes \lambda)(\hat{\mathcal{W}}) = W, \quad (\lambda \otimes \lambda)\hat{\Delta}_{\mathrm{u}} = \hat{\Delta}\lambda, \quad \text{and} \quad (\lambda \otimes \mathrm{id})\sigma\hat{\Delta}_{\mathrm{u}}(x) = \hat{\mathcal{W}}(\lambda(x) \otimes 1)\hat{\mathcal{W}}^*$$

for all $x \in \widehat{\mathcal{G}}_{\mathrm{u}}$, where $\sigma : \widehat{\mathcal{G}}_{\mathrm{u}} \otimes \widehat{\mathcal{G}}_{\mathrm{u}} \to \widehat{\mathcal{G}}_{\mathrm{u}} \otimes \widehat{\mathcal{G}}_{\mathrm{u}}$ is the flip map. There is a *counit* $\hat{\epsilon}_{\mathrm{u}}$ for $\widehat{\mathcal{G}}_{\mathrm{u}}$. This means that $\hat{\epsilon}_{\mathrm{u}} : \widehat{\mathcal{G}}_{\mathrm{u}} \to \mathbb{C}$ is a $*$-homomorphism such that $(\hat{\epsilon}_{\mathrm{u}} \otimes \mathrm{id})\hat{\Delta}_{\mathrm{u}} = (\mathrm{id} \otimes \hat{\epsilon}_{\mathrm{u}})\hat{\Delta}_{\mathrm{u}} = \mathrm{id}$. It satisfies $(\mathrm{id} \otimes \hat{\epsilon}_{\mathrm{u}})(\hat{\mathcal{W}}) = 1$. There are left and right Haar weights $\hat{\varphi}_{\mathrm{u}}$ and $\hat{\psi}_{\mathrm{u}}$ on $(\widehat{\mathcal{G}}_{\mathrm{u}}, \hat{\Delta}_{\mathrm{u}})$, defined by $\hat{\varphi}_{\mathrm{u}} = \hat{\varphi}\lambda$ and $\hat{\psi}_{\mathrm{u}} = \hat{\psi}\lambda$, with GNS-constructions $(L^2(\mathcal{G}), \lambda, \hat{\Lambda}_{\mathrm{u}})$, and $(L^2(\mathcal{G}), \lambda, \hat{\Gamma}_{\mathrm{u}})$, respectively, where $\hat{\Lambda}_{\mathrm{u}} = \hat{\Lambda} \circ \lambda$ and $\hat{\Gamma}_{\mathrm{u}} = \hat{\Gamma} \circ \lambda$. The Haar weights are KMS-weights and are unique up to positive scalars.

Similarly, considering the dual $(\widehat{\mathcal{G}}, \hat{\Delta})$, one defines the Banach $*$-algebra $L^1_*(\widehat{\mathcal{G}})$ and an injective $*$-homomorphism $\hat{\lambda} : L^1_*(\widehat{\mathcal{G}}) \to \mathcal{G}$ by $\hat{\lambda}(\omega) = (\mathrm{id} \otimes \omega)(W^*)$ for all $\omega \in \widehat{\mathcal{G}}^*$. Let $\mathcal{G}_{\mathrm{u}}$ be the enveloping C^*-algebra of $L^1_*(\widehat{\mathcal{G}})$. Then $\hat{\lambda}$ extends to a surjective $*$-homomorphism $\hat{\lambda} : \mathcal{G}_{\mathrm{u}} \to \mathcal{G}$. There is a unique unitary $\mathcal{W} \in \mathcal{M}(\mathcal{G}_{\mathrm{u}} \otimes \widehat{\mathcal{G}})$, called the *left regular corepresentation* of $\mathcal{G}_{\mathrm{u}}$, such that $\omega = (\mathrm{id} \otimes \omega)(\mathcal{W}^*)$ for all $\omega \in L^1_*(\widehat{\mathcal{G}})$. It satisfies $(\mathrm{id} \otimes \hat{\Delta})(\mathcal{W}) = \mathcal{W}_{13}\mathcal{W}_{12}$. The *universal locally compact quantum group* $(\mathcal{G}_{\mathrm{u}}, \Delta_{\mathrm{u}})$ of $(\mathcal{G}, \Delta)$ is defined in such a way that the comultiplication $\Delta_u : \mathcal{G}_u \to \mathcal{M}(\mathcal{G}_u \otimes \mathcal{G}_u)$ satisfies $(\Delta_u \otimes \mathrm{id})(\mathcal{W}) = \mathcal{W}_{13}\mathcal{W}_{23}$. Moreover, one has that

$$(\hat{\lambda} \otimes \mathrm{id})(\mathcal{W}) = W, \quad (\hat{\lambda} \otimes \hat{\lambda})\Delta_u = \Delta\hat{\lambda}, \quad \text{and} \quad (\mathrm{id} \otimes \hat{\lambda})\hat{\Delta}_u(x) = \mathcal{W}^*(1 \otimes \hat{\lambda}(x))\mathcal{W}$$

for all $x \in \mathcal{G}_u$. The counit of $\mathcal{G}_u$ is denoted by ϵ_u. It satisfies $(\epsilon_u \otimes \mathrm{id})(\mathcal{W}) = 1$. The left and right Haar weights on $(\mathcal{G}_u, \Delta_u)$ will be denoted by $\varphi_u = \varphi\hat{\lambda}$ and $\psi_u = \psi\hat{\lambda}$, respectively, with GNS-constructions $(L^2(\mathcal{G}), \hat{\lambda}, \Lambda_u)$, and $(L^2(\mathcal{G}), \hat{\lambda}, \Gamma_u)$ respectively, where $\Lambda_u = \Lambda \circ \hat{\lambda}$ and $\Gamma_u = \Gamma \circ \hat{\lambda}$.

There is a unique unitary $\mathcal{U} \in \mathcal{M}(\mathcal{G}_u \otimes \widehat{\mathcal{G}}_u)$, called the *universal corepresentation* of $\mathcal{G}_u$, such that $\mathcal{U}_{13} = \mathcal{W}_{12}^*\hat{\mathcal{W}}_{23}\mathcal{W}_{12}\hat{\mathcal{W}}_{23}^*$. Moreover, we have $(\Delta_u \otimes \mathrm{id})(\mathcal{U}) = \mathcal{U}_{13}\mathcal{U}_{23}$, $(\mathrm{id} \otimes \hat{\Delta}_u)(\mathcal{U}) = \mathcal{U}_{13}\mathcal{U}_{12}$ and

$$(\mathrm{id} \otimes \lambda)(\mathcal{U}) = \mathcal{W}, \quad (\hat{\lambda} \otimes \mathrm{id})(\mathcal{U}) = \hat{\mathcal{W}}, \quad (\hat{\lambda} \otimes \lambda)(\mathcal{U}) = W.$$

Finally, we mention that $\mathcal{G}$ is co-amenable if and only if $\hat{\lambda} : \mathcal{G}_u \to \mathcal{G}$ is injective (and therefore an isomorphism).

2.5.4 Corepresentations

Definition 2.5.6. Let $(\mathcal{G}, \Delta)$ be a locally compact quantum group and let $\mathcal{E}$ be a Hilbert B-module. A *left corepresentation* of $\mathcal{G}$ on $\mathcal{E}$ is a unitary $u \in \mathcal{L}(\mathcal{G} \otimes \mathcal{E})$ satisfying the relation $(\Delta \otimes \mathrm{id})(u) = u_{13}u_{23}$. A *right corepresentation* of $\mathcal{G}$ on $\mathcal{E}$ is a unitary $u \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G})$ satisfying $(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}$.

Let $u \in \mathcal{L}(\mathcal{G} \otimes \mathcal{E})$ be a left corepresentation of $\mathcal{G}$ on $\mathcal{E}$. Then $u^{\mathrm{op}} := \sigma(u)^*$ is a right corepresentation of the opposite quantum group $\mathcal{G}^{\mathrm{op}}$ on $\mathcal{E}$, where $\sigma : \mathcal{G} \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{G}$ is the flip map. This gives a bijective correspondence between left corepresentations of $\mathcal{G}$ on $\mathcal{E}$ and right corepresentations of $\mathcal{G}^{\mathrm{op}}$ on $\mathcal{E}$.

The universal dual $\widehat{\mathcal{G}}_u$ of $\mathcal{G}$ encodes the corepresentation theory of $\mathcal{G}$ in the sense that left corepresentations u of $\mathcal{G}$ on $\mathcal{E}$ correspond to nondegenerate representations (that is, $*$-homomorphisms) $\mu : \widehat{\mathcal{G}}_u \to \mathcal{L}(\mathcal{E})$ satisfying $(\mathrm{id} \otimes \mu)(\hat{\mathcal{W}}) = u$, where $\hat{\mathcal{W}} \in \mathcal{M}(\mathcal{G} \otimes \widehat{\mathcal{G}}_u)$ is the universal corepresentation of $\mathcal{G}$.

Using the correspondence between right and left corepresentations of $\mathcal{G}$ and $\mathcal{G}^{\mathrm{op}}$ we see that right corepresentations of $\mathcal{G}$ correspond to nondegenerate representations of $\widehat{\mathcal{G}}_u^{\mathrm{c}}$, the universal locally compact quantum group of $\widehat{\mathcal{G}^{\mathrm{op}}} = \widehat{\mathcal{G}}^{\mathrm{c}}$. More precisely, there is a universal corepresentation $\hat{\mathcal{V}} \in \mathcal{M}(\widehat{\mathcal{G}}_u^{\mathrm{c}} \otimes \mathcal{G})$ of $\mathcal{G}$ such that the formula $(\mu \otimes \mathrm{id}_{\mathcal{G}})(\hat{\mathcal{V}}) = u$ gives a bijective correspondence between right corepresentations $u \in \mathcal{M}(B \otimes \mathcal{G})$ of $\mathcal{G}$ and nondegenerate $*$-homomorphisms $\mu : \widehat{\mathcal{G}}_u^{\mathrm{c}} \to \mathcal{M}(B)$. In fact, the (right) universal corepresentation $\hat{\mathcal{V}} \in \mathcal{M}(\widehat{\mathcal{G}}_u^{\mathrm{c}} \otimes \mathcal{G})$ of $\mathcal{G}$ is the opposite of the (left) universal corepresentation $\hat{\mathcal{W}}^{\mathrm{op}} \in \mathcal{M}(\mathcal{G} \otimes \widehat{\mathcal{G}}_u^{\mathrm{c}}) = \mathcal{M}(\mathcal{G}^{\mathrm{op}} \otimes (\widehat{\mathcal{G}^{\mathrm{op}}})_u)$ of $\mathcal{G}^{\mathrm{op}}$. In other words, we have $\hat{\mathcal{V}} := \sigma(\hat{\mathcal{W}}^{\mathrm{op}})^*$, where $\sigma : \mathcal{G} \otimes \widehat{\mathcal{G}}_u^{\mathrm{c}} \to \widehat{\mathcal{G}}_u^{\mathrm{c}} \otimes \mathcal{G}$ is the flip map. Recall that $\hat{\mathcal{W}}^{\mathrm{op}}$ satisfies $(\Delta^{\mathrm{op}} \otimes \mathrm{id})(\hat{\mathcal{W}}) = \hat{\mathcal{W}}_{13}^{\mathrm{op}}\hat{\mathcal{W}}_{23}^{\mathrm{op}}$ (that is, $\hat{\mathcal{W}}^{\mathrm{op}}$ is a left corepresentation of $\mathcal{G}^{\mathrm{op}}$) and $(\mathrm{id} \otimes \lambda^{\mathrm{op}})(\hat{\mathcal{W}}^{\mathrm{op}}) = W^{\mathrm{op}} = \Sigma V^* \Sigma$, where $\lambda^{\mathrm{op}} : \widehat{\mathcal{G}}_u^{\mathrm{c}} \to \widehat{\mathcal{G}}^{\mathrm{c}}$ is the canonical surjection and V is the right regular corepresentation of $\mathcal{G}$. It follows that $\hat{\mathcal{V}}$ is, in fact, a (right) corepresentation of $\mathcal{G}$ and it satisfies $(\lambda^{\mathrm{op}} \otimes \mathrm{id}_{\mathcal{G}})(\hat{\mathcal{V}}) = V$. If we denote the comultiplication of $\widehat{\mathcal{G}}_u^{\mathrm{c}}$ by $\hat{\Delta}_u^{\mathrm{c}}$, then we have $(\mathrm{id} \otimes \hat{\Delta}_u^{\mathrm{c}})(\hat{\mathcal{W}}^{\mathrm{op}}) = \hat{\mathcal{W}}_{13}^{\mathrm{op}}\hat{\mathcal{W}}_{12}^{\mathrm{op}}$. It follows that $(\hat{\Delta}_u^{\mathrm{c}} \otimes \mathrm{id})(\hat{\mathcal{V}}) = \hat{\mathcal{V}}_{13}\hat{\mathcal{V}}_{23}$, that is, $\hat{\mathcal{V}}$ is also a left corepresentation of $(\widehat{\mathcal{G}}_u^{\mathrm{c}}, \hat{\Delta}_u^{\mathrm{c}})$.

2.6 Coactions of quantum groups

2.6.1 Coactions on C^*-algebras

Definition 2.6.1. Let $\mathcal{G}$ be a locally compact quantum group and let A be a C^*-algebra. A *coaction* of $\mathcal{G}$ on A is a nondegenerate $*$-homomorphism

$$\gamma_A : A \to \mathcal{M}(A \otimes \mathcal{G})$$

satisfying $(\gamma_A \otimes \mathrm{id}_{\mathcal{G}})\gamma_A = (\mathrm{id}_A \otimes \Delta)\gamma_A$. We write (A, γ_A) to indicate all the data. A coaction γ_A is called

(i) *continuous* if $\overline{\mathrm{span}}\big((1_A \otimes \mathcal{G})\gamma_A(A)\big) = A \otimes \mathcal{G}$,

(ii) *weakly continuous* if $\overline{\mathrm{span}}\big\{(\mathrm{id} \otimes \omega)\big(\gamma_A(\xi)\big) : \omega \in L^1(\mathcal{G})\big\} = A$.

A C^*-algebra with a continuous coaction of $\mathcal{G}$ is also called a *$\mathcal{G}$-equivariant C^*-algebra*, or simply, a *$\mathcal{G}$-C^*-algebra*.

Notice that we do *not* assume coactions to be injective. If the coaction γ_A is injective we say it is *reduced*. If, in addition, γ_A is continuous we also say that A is a *reduced $\mathcal{G}$-C^*-algebra*.

Remark 2.6.2. (1) Note that a coaction γ_A is continuous if and only if

$$\overline{\mathrm{span}}\big(\gamma_A(A)(1_A \otimes \mathcal{G})\big) = A \otimes \mathcal{G}.$$

In particular, for continuous coactions we have $\gamma_A(A) \subseteq \tilde{\mathcal{M}}(A \otimes \mathcal{G})$. A coaction satisfying this last condition will be called *admissible*. Most authors include admissibility in the definition of coactions. However, for general locally compact quantum groups this turns out to be too strong. More precisely, if one assumes that the canonical coaction of $\mathcal{G}$ on the algebra of compact operators $\mathcal{K} = \mathcal{K}(H)$ (see Example 2.6.18(3) below) is admissible, then one is already assuming that $\mathcal{G}$ is regular (see Proposition 2.7.14 for details). We recall the notion of regularity of quantum groups in Section 2.7.4. The definition of coactions we use here (without the admissibility condition) has also been used in [7, 75].

(2) Given $\xi \in \mathcal{M}(A)$ and $\omega \in \mathcal{G}^*$, we define $\omega * \xi := (\mathrm{id}_A \otimes \omega)\gamma_A(\xi)$. This defines a left action of $\mathcal{G}^*$ on $\mathcal{M}(A)$ turning $\mathcal{M}(A)$ into a Banach left $\mathcal{G}^*$-module (recall that $\mathcal{G}^*$ is a Banach algebra with $\omega_1 \cdot \omega_2 := (\omega_1 \otimes \omega_2) \circ \Delta$). In particular, if we restrict the action to $L^1(\mathcal{G})$, then $\mathcal{M}(A)$ is also a Banach left $L^1(\mathcal{G})$-module. Note that, even if $\omega \in L^1(\mathcal{G})$ and $\xi \in A$, the element $\omega * \xi$ is, *a priori*, only in $\mathcal{M}(A)$. It will be in A, for example, if γ_A is admissible (this follows from Proposition 2.4.14). In general, we say that γ_A is *weakly admissible* if $\omega * \xi \in A$ for all $\omega \in L^1(\mathcal{G})$ and $\xi \in A$. Thus for weakly admissible coactions, the operation $*$ turns A into a Banach left $L^1(\mathcal{G})$-module. By definition, this action will be nondegenerate (meaning that the closed linear span of $L^1(\mathcal{G}) * A$ is dense in A) if and only if γ_A is weakly continuous. Since $L^1(\mathcal{G}) = \{\omega|_{\mathcal{G}} : \omega \in \mathcal{L}(H)_*\}$, a coaction (A, γ_A) is weakly continuous if and only if the linear span of $(\mathrm{id} \otimes \omega)\big(\gamma_A(\xi)\big)$, $\omega \in \mathcal{L}(H)_*$ is dense in A. Thus our definition of weak continuity coincides with the definition given in [7] (where the terminology "continuity in the weak sense" is used instead).

Proposition 5.8 in [7] shows that if $\mathcal{G}$ is regular, then weak continuity and continuity are equivalent notions and if $\mathcal{G}$ is semi-regular, but not regular, then there are weakly continuous coactions of $\mathcal{G}$ which are not continuous. Moreover, the same example given in [7, Proposition 5.8] also provides an example of a weakly continuous coaction which is not admissible (see Remark 2.7.15 for details). In particular, weak admissibility does not imply admissibility in general. Finally, we mention that, although weak admissibility seems to be a very weak condition, if one assumes that the canonical coaction of $\mathcal{G}$ on $\mathcal{K}$ is weakly admissible, then $\mathcal{G}$ must be regular (see Proposition 2.7.14).

(3) Strictly speaking, what we have defined in Definition 2.6.1 is a *right coaction* of $\mathcal{G}$ on A. A *left coaction* of $\mathcal{G}$ on a A is a nondegenerate $*$-homomorphism $\gamma_A : A \to \mathcal{M}(\mathcal{G} \otimes A)$ satisfying $(\mathrm{id}_{\mathcal{G}} \otimes \gamma_A)\gamma_A = (\Delta \otimes \mathrm{id}_A)\gamma_A$. These two concepts are equivalent in the sense that given a left coaction γ_A of $\mathcal{G}$ on A, the map $\gamma_A^{\mathrm{op}} := \sigma \circ \gamma_A : A \to \mathcal{M}(A \otimes \mathcal{G})$ is a right coaction of the opposite quantum group $\mathcal{G}^{\mathrm{op}}$ on A, where $\sigma : \mathcal{G} \otimes A \to A \otimes \mathcal{G}$ is the flip map. This gives a bijective correspondence between left coactions of $\mathcal{G}$ and right coactions of $\mathcal{G}^{\mathrm{op}}$.

(4) Observe that one can define coactions of arbitrary bi-C^*-algebras. In particular, one can also define coactions of the universal locally compact quantum group $\mathcal{G}_{\mathrm{u}}$ of $\mathcal{G}$. Coactions of $\mathcal{G}_{\mathrm{u}}$ are also called *full coactions*. Note that if $\gamma_A^{\mathrm{u}} : A \to \mathcal{M}(A \otimes \mathcal{G}_{\mathrm{u}})$ is a full coaction of $\mathcal{G}_{\mathrm{u}}$ on a C^*-algebra A, then the map

$$\gamma_A := (\mathrm{id}_A \otimes \hat{\lambda}) \circ \gamma_A^{\mathrm{u}} : A \to \mathcal{M}(A \otimes \mathcal{G})$$

is a coaction of $\mathcal{G}$ on A. Recall that $\hat{\lambda} : \mathcal{G}_{\mathrm{u}} \to \mathcal{G}$ denotes the canonical surjection. We call the coaction γ_A the *reduced form* of γ_A^{u}. In this work we shall only consider coactions in the reduced form, that is, coactions of reduced locally compact quantum groups. However, one can apply our definitions and results to full coactions just by considering their reduced forms.

Example 2.6.3. **(1)** Any C^*-algebra A can be turned into a $\mathcal{G}$-C^*-algebra by considering on A the *trivial coaction* of $\mathcal{G}$: $\gamma_{tr} : A \to \mathcal{M}(A \otimes \mathcal{G})$, $\gamma_{tr}(a) = a \otimes 1$.

(2) In the group case, that is, for $\mathcal{G} = \mathcal{C}_0(G)$, where G is some locally compact group, continuous coactions of $\mathcal{G}$ correspond to (strongly) continuous actions of G. If α is a continuous action of G on a C^*-algebra A, then the corresponding coaction of $\mathcal{G}$ on A is given by $\gamma_A(a)|_t := \alpha_t(a)$, where we identify $[t \mapsto \alpha_t(a)] \in \mathcal{C}_b(G, A) \cong \tilde{\mathcal{M}}(A \otimes \mathcal{G}) \subseteq \mathcal{M}(A \otimes \mathcal{G})$. A C^*-algebra with a continuous action of G is also called a *G-C^*-algebra*.

(3) Again if G is a locally compact group, a coaction of the quantum group $\mathcal{G} = C_{\mathrm{r}}^*(G)$ is, by definition, a coaction of the group G. A C^*-algebra with a continuous coaction of G will also be called a *$\widehat{G}$-C^*-algebra*. If G is Abelian, then the Fourier transform gives an isomorphism $C_{\mathrm{r}}^*(G) \cong \mathcal{C}_0(\widehat{G})$ of locally compact quantum groups, where $\widehat{G}$ is the Pontrjagin dual of G. So, by (2), continuous coactions of G correspond to continuous actions of $\widehat{G}$. This explains the terminology.

Definition 2.6.4. Let (A, γ_A) and (B, γ_B) be coactions of $\mathcal{G}$. A nondegenerate $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ is called *$\mathcal{G}$-equivariant* if

$$\gamma_B\big(\pi(a)\big) = (\pi \otimes \mathrm{id}_{\mathcal{G}})\big(\gamma_A(a)\big) \quad \text{for all } a \in A.$$

Remark 2.6.5. In the above definition we are using the strictly continuous extension $\pi \otimes \mathrm{id}_\mathcal{G} : \mathcal{M}(A \otimes \mathcal{G}) \to \mathcal{M}(B \otimes \mathcal{G})$ which exists because π is assumed to be nondegenerate. It is sometimes useful to work with degenerate homomorphisms. The same definition above makes sense if we suppose that the coaction γ_A is admissible, that is, if $\gamma_A(A) \subseteq \tilde{\mathcal{M}}(A \otimes \mathcal{G})$ (in particular, if it is continuous), because then we can use the $\mathcal{G}$-strict continuous extension $\pi \otimes \mathrm{id}_\mathcal{G} : \tilde{\mathcal{M}}(A \otimes \mathcal{G}) \to \mathcal{M}(B \otimes \mathcal{G})$ as in Proposition 2.2.7.

2.6.2 Coactions on Hilbert modules

Let B be a C^*-algebra with a coaction γ_B of a locally compact quantum group $(\mathcal{G}, \Delta)$. Let $\mathcal{E}$ be a Hilbert B-module. Recall that $\mathcal{M}(\mathcal{E}) = \mathcal{L}(B, \mathcal{E})$ denotes the multiplier Hilbert $\mathcal{M}(B)$-module of $\mathcal{E}$.

Definition 2.6.6. A *coaction* of $\mathcal{G}$ on $\mathcal{E}$ (compatible with the coaction γ_B on B) is a linear map

$$\gamma_\mathcal{E} : \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes \mathcal{G}) = \mathcal{L}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G})$$

satisfying the following conditions:

(i) $\gamma_\mathcal{E}(\xi \cdot b) = \gamma_\mathcal{E}(\xi)\gamma_B(b)$ for all $\xi \in \mathcal{E}$, $b \in B$,

(ii) $\langle \gamma_\mathcal{E}(\xi) \,|\, \gamma_\mathcal{E}(\eta) \rangle_{\mathcal{M}(B \otimes \mathcal{G})} = \gamma_B(\langle \xi \,|\, \eta \rangle_B)$ for all $\xi, \eta \in \mathcal{E}$,

(iii) $\mathrm{span}\big(\gamma_\mathcal{E}(\mathcal{E})(B \otimes \mathcal{G})\big)$ is dense in $\mathcal{E} \otimes \mathcal{G}$,

(iv) $(\gamma_\mathcal{E} \otimes \mathrm{id}_\mathcal{G})\gamma_\mathcal{E} = (\mathrm{id}_\mathcal{E} \otimes \Delta)\gamma_\mathcal{E}$.

Remark 2.6.7. The condition (iv) above makes sense if (i), (ii) and (iii) hold (see [5] for details). Moreover, if we have a coaction $\gamma_\mathcal{E}$ of $\mathcal{G}$ on $\mathcal{E}$, then there is an induced coaction $\gamma_{\mathcal{K}(\mathcal{E})}$ of $\mathcal{G}$ on the algebra of compact operators $\mathcal{K}(\mathcal{E})$ determined by the equation

$$\gamma_{\mathcal{K}(\mathcal{E})}(|\xi\rangle\langle\eta|) = \gamma_\mathcal{E}(\xi)\gamma_\mathcal{E}(\eta)^* \text{ for all } \xi, \eta \in \mathcal{E}.$$

We have $\gamma_\mathcal{E}(T\xi) = \gamma_{\mathcal{K}(\mathcal{E})}(T)\gamma_\mathcal{E}(\xi)$ for all $T \in \mathcal{L}(\mathcal{E}) \cong \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$ and $\xi \in \mathcal{E}$. Thus, if we consider $\mathcal{E}$ as a Hilbert $\mathcal{K}(\mathcal{E}), B$-bimodule, a coaction $\gamma_\mathcal{E}$ of $\mathcal{G}$ on $\mathcal{E}$ is nothing but a nondegenerate Hilbert bimodule homomorphism

$$\gamma_\mathcal{E} : {}_{\mathcal{K}(\mathcal{E})}\mathcal{E}_B \longrightarrow {}_{\mathcal{M}(\mathcal{K}(\mathcal{E}) \otimes \mathcal{G})}\mathcal{M}(\mathcal{E} \otimes \mathcal{G})_{\mathcal{M}(B \otimes \mathcal{G})}$$

with coefficient maps $\gamma_{\mathcal{K}(\mathcal{E})} : \mathcal{K}(\mathcal{E}) \to \mathcal{M}\big(\mathcal{K}(\mathcal{E}) \otimes \mathcal{G}\big)$ and $\gamma_B : B \to \mathcal{M}(B \otimes \mathcal{G})$ (see Definition 2.1.6). In particular, the map $\gamma_\mathcal{E}$ has a strict continuous extension to the multiplier bimodule $\mathcal{M}(\mathcal{E})$ which is compatible with the extensions of the coefficient coactions $\gamma_{\mathcal{K}(\mathcal{E})}$ and γ_B. By Proposition 2.2.4, the map $\gamma_\mathcal{E} \otimes \mathrm{id}_\mathcal{G} : \mathcal{E} \otimes \mathcal{G} \to \mathcal{M}(\mathcal{E} \otimes \mathcal{G} \otimes \mathcal{G})$ is a nondegenerate $\gamma_{\mathcal{K}(\mathcal{E})} \otimes \mathrm{id}_\mathcal{G}, \gamma_B \otimes \mathrm{id}_\mathcal{G}$-compatible Hilbert bimodule homomorphism, so that it also has a strict continuous extension to the multiplier Hilbert bimodule $\mathcal{M}(\mathcal{E} \otimes \mathcal{G})$.

Definition 2.6.8. If the coaction γ_B on B is continuous, then a Hilbert B-module with a γ_B-compatible coaction of $\mathcal{G}$ is called a *$\mathcal{G}$-equivariant Hilbert B-module*, or simply, a *Hilbert $B, \mathcal{G}$-module*.

Remark 2.6.9. **(1)** Let $\mathcal{E}$ be a Hilbert B-module and let $\gamma_{\mathcal{E}}$ be a γ_B-compatible coaction of $\mathcal{G}$ on $\mathcal{E}$. If γ_B is continuous, that is, if $\mathcal{E}$ is a Hilbert $B,\mathcal{G}$-module, then we have (using that $\mathcal{E} = \mathcal{E}\cdot B$)

$$\begin{aligned}\overline{\operatorname{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})(1_B\otimes\mathcal{G})\big) &= \overline{\operatorname{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E}\cdot B)(1_B\otimes\mathcal{G})\big)\\ &= \overline{\operatorname{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})\gamma_B(B)(1_B\otimes\mathcal{G})\big)\\ &= \overline{\operatorname{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})(B\otimes\mathcal{G})\big) = \mathcal{E}\otimes B.\end{aligned}$$

(2) We say that a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$ on a Hilbert B-module $\mathcal{E}$ is *continuous* if the underlying coaction γ_B of $\mathcal{G}$ on B is continuous and $\overline{\operatorname{span}}\big((1_{\mathcal{E}}\otimes\mathcal{G})\gamma_{\mathcal{E}}(\mathcal{E})\big) = \mathcal{E}\otimes\mathcal{G}$. If $\gamma_{\mathcal{E}}$ is continuous, then it is easy to see that the corresponding coaction $\gamma_{\mathcal{K}(\mathcal{E})}$ on the algebra of compact operators $\mathcal{K}(\mathcal{E})$ is continuous as well.

Notice that if $\gamma_{\mathcal{E}}$ is continuous, then $\gamma_{\mathcal{E}}(\mathcal{E}) \subseteq \tilde{\mathcal{M}}(\mathcal{E}\otimes\mathcal{G})$. In general, we say that a γ_B-compatible coaction $\gamma_{\mathcal{E}}$ is *admissible* if $\gamma_B(B) \subseteq \tilde{\mathcal{M}}(B\otimes\mathcal{G})$ (that is, γ_B is admissible) and $\gamma_{\mathcal{E}}(\mathcal{E}) \subseteq \tilde{\mathcal{M}}(\mathcal{E}\otimes\mathcal{G})$.

Most authors include admissibility in the definition of coactions. However, as already noted for C^*-algebras, this turns out to be too strong for general locally compact quantum groups. A detailed discussion will be given in Section 2.7.4.

(3) Suppose that $\mathcal{G}$ is regular (see Section 2.7.4 below). If $\mathcal{E}$ is a Hilbert $B,\mathcal{G}$-module, then the coaction on $\mathcal{E}$ is automatically continuous. This follows from Proposition 5.8 in [7]. See also Remark 12.5 in [75]. In general, this is not true.

Let $\mathcal{E}$ be a Hilbert B-module with a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$. Given $\omega\in\mathcal{G}^*$ and $\xi\in\mathcal{M}(\mathcal{E})$ we define

$$\omega * \xi := (\mathrm{id}_{\mathcal{E}}\otimes\omega)\big(\gamma_{\mathcal{E}}(\xi)\big). \tag{2.18}$$

This gives $\mathcal{M}(\mathcal{E})$ the structure of a Banach left $\mathcal{G}^*$-module. In particular, it is also a Banach left $L^1(\mathcal{G})$-module. But even if $\xi\in\mathcal{E}$ and $\omega\in L^1(\mathcal{G})$, it is not true, in general, that $\omega*\xi\in\mathcal{E}$. This will happen, for example, if $\gamma_{\mathcal{E}}$ is admissible, and in particular if $\gamma_{\mathcal{E}}$ is continuous. The best situation is when we have a Hilbert $B,\mathcal{G}$-module:

Proposition 2.6.10. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. Then the left action* (2.18) *turns $\mathcal{E}$ into a nondegenerate Banach left $L^1(\mathcal{G})$-module, that is, $\overline{\operatorname{span}}(L^1(\mathcal{G})*\mathcal{E}) = \mathcal{E}$.*

Proof. Since γ_B is continuous, we have $\overline{\operatorname{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})(1\otimes\mathcal{G})\big) = \mathcal{E}\otimes\mathcal{G}$ (see Remark 2.6.9(1)). Any element of $L^1(\mathcal{G})$ can be written in the form $a\omega$, where $a\in\mathcal{G}$ and $\omega\in L^1(\mathcal{G})$ (recall that $(a\omega)(x) = \omega(xa)$ for all $x\in\mathcal{G}$). Therefore

$$\begin{aligned}\overline{\operatorname{span}}\big(L^1(\mathcal{G})*\mathcal{E}\big) &= \overline{\operatorname{span}}\big\{(\mathrm{id}_{\mathcal{E}}\otimes\omega)\big(\gamma_{\mathcal{E}}(\mathcal{E})(1\otimes\mathcal{G})\big) : \omega\in L^1(\mathcal{G})\big\}\\ &= \overline{\operatorname{span}}\big\{(\mathrm{id}_{\mathcal{E}}\otimes\omega)(\mathcal{E}\otimes\mathcal{G}) : \omega\in L^1(\mathcal{G})\big\} = \mathcal{E}.\end{aligned}$$ □

In general, we say that a γ_B-compatible coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$ on $\mathcal{E}$ is *weakly continuous* if γ_B is weakly continuous and the linear span of $L^1(\mathcal{G})*\mathcal{E}$ is dense in $\mathcal{E}$. As already noted for C^*-algebras, if $\mathcal{G}$ is regular, then weak continuity implies continuity, but, in general, this is not true. Moreover, in general, weak continuity of $\gamma_{\mathcal{E}}$ does not imply weak continuity of the induced coaction on $\mathcal{K}(\mathcal{E})$ (see Example 2.6.18(3) below). We discuss these problems with more details in Section 2.7.4.

Corollary 2.6.11. *Suppose that $\mathcal{G}$ is co-amenable, that is, suppose that $L^1(\mathcal{G})$ has a bounded approximate unit. If $\gamma_{\mathcal{E}}$ is a weakly continuous coaction of $\mathcal{G}$ on a Hilbert B-module $\mathcal{E}$ (in particular if $\mathcal{E}$ is a Hilbert $B,\mathcal{G}$-module), then $\gamma_{\mathcal{E}}$ is injective.*

Proof. Let $\gamma_{\mathcal{E}} : \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ be the coaction of $\mathcal{G}$ on $\mathcal{E}$. Suppose that $\gamma_{\mathcal{E}}(\xi) = 0$ for some $\xi \in \mathcal{E}$. Then $\omega * \xi = (\mathrm{id}_{\mathcal{E}} \otimes \omega)\big(\gamma_{\mathcal{E}}(\xi)\big) = 0$ for all $\omega \in L^1(\mathcal{G})$. Thus, if (ω_i) is any bounded approximate unit for $L^1(\mathcal{G})$, then it follows from Proposition 2.6.10 that $0 = \omega_i * \xi \to \xi$ and therefore $\xi = 0$. $\square$

Given a Hilbert B-module $\mathcal{E}$ and a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$ on $\mathcal{E}$ (compatible with γ_B), one can define a unitary $V_{\mathcal{E}} \in \mathcal{L}(\mathcal{E} \underset{\gamma_B}{\otimes} (B \otimes \mathcal{G}), \mathcal{E} \otimes \mathcal{G})$ by

$$V_{\mathcal{E}}(\xi \underset{\gamma_B}{\otimes} x) := \gamma_{\mathcal{E}}(\xi)x \quad \text{for all } \xi \in \mathcal{E} \text{ and } x \in B \otimes \mathcal{G}. \tag{2.19}$$

It satisfies the relation

$$(V_{\mathcal{E}} \underset{\mathbb{C}}{\otimes} 1)(V_{\mathcal{E}} \underset{\gamma_B \otimes \mathrm{id}}{\otimes} 1) = V_{\mathcal{E}} \underset{\mathrm{id} \otimes \Delta}{\otimes} 1,$$

where one uses the following identifications:

$$\begin{tikzcd}
\mathcal{E} \underset{(\gamma_B \otimes \mathrm{id})\gamma_B}{\otimes} (B \otimes \mathcal{G} \otimes \mathcal{G}) \arrow[r, equal] \arrow[d, "\cong"'] & \mathcal{E} \underset{(\mathrm{id} \otimes \Delta)\gamma_B}{\otimes} (B \otimes \mathcal{G} \otimes \mathcal{G}) \arrow[d, "\cong"] \\
\big(\mathcal{E} \underset{\gamma_B}{\otimes} (B \otimes \mathcal{G})\big) \underset{\gamma_B \otimes \mathrm{id}}{\otimes} (B \otimes \mathcal{G} \otimes \mathcal{G}) \arrow[d, "V_{\mathcal{E}} \underset{\gamma_B \otimes \mathrm{id}}{\otimes} 1"'] & \big(\mathcal{E} \underset{\gamma_B}{\otimes} (B \otimes \mathcal{G})\big) \underset{\mathrm{id} \otimes \Delta}{\otimes} (B \otimes \mathcal{G} \otimes \mathcal{G}) \arrow[ddd, "V_{\mathcal{E}} \underset{\mathrm{id} \otimes \Delta}{\otimes} 1"] \\
(\mathcal{E} \otimes \mathcal{G}) \underset{\gamma_B \otimes \mathrm{id}}{\otimes} (B \otimes \mathcal{G} \otimes \mathcal{G}) \arrow[d, "\cong"'] & \\
\big(\mathcal{E} \underset{\gamma_B}{\otimes} (B \otimes \mathcal{G})\big) \otimes \mathcal{G} \arrow[dd, "V_{\mathcal{E}} \underset{\mathbb{C}}{\otimes} 1"'] & \\
 & (\mathcal{E} \otimes \mathcal{G}) \underset{\mathrm{id} \otimes \Delta}{\otimes} (B \otimes \mathcal{G} \otimes \mathcal{G}) \arrow[d, "\cong"] \\
\mathcal{E} \otimes \mathcal{G} \otimes \mathcal{G} \arrow[r, equal] & \mathcal{E} \otimes \mathcal{G} \otimes \mathcal{G}
\end{tikzcd}$$

If γ_B is trivial, then $V_{\mathcal{E}} \in \mathcal{L}(\mathcal{E} \underset{\gamma_B}{\otimes} (B \otimes \mathcal{G}), \mathcal{E} \otimes \mathcal{G}) \cong \mathcal{L}(\mathcal{E} \otimes \mathcal{G})$ and we have

$$\gamma_{\mathcal{E}}(\xi) = V_{\mathcal{E}}(\xi \otimes 1) \quad \text{for all } \xi \in \mathcal{E}.$$

In this case $V_{\mathcal{E}}$ is a (right) corepresentation of $\mathcal{G}$ on $\mathcal{E}$. Conversely, if $V_{\mathcal{E}} \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G})$ is any unitary (right) corepresentation of $\mathcal{G}$ on $\mathcal{E}$, then the formula $\gamma_{\mathcal{E}}(\xi) = V_{\mathcal{E}}(\xi \otimes 1)$ defines a coaction of $\mathcal{G}$ on $\mathcal{E}$ which is compatible with the trivial coaction on B. Thus, for trivial coefficients, coactions of $\mathcal{G}$ correspond to corepresentations of $\mathcal{G}$.

Definition 2.6.12. An operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are Hilbert B-modules with coactions $\gamma_{\mathcal{E}}$ and $\gamma_{\mathcal{F}}$ of $\mathcal{G}$, is called $\mathcal{G}$-equivariant if

$$\gamma_{\mathcal{F}}(T\xi) = (T \otimes 1)\gamma_{\mathcal{E}}(\xi), \quad \text{for all } \xi \in \mathcal{E}.$$

It is easy to see that this is equivalent to

$$V_{\mathcal{F}}(T \underset{\gamma_B}{\otimes} 1)V_{\mathcal{E}}^* = T \otimes 1,$$

where $V_{\mathcal{E}} \in \mathcal{L}\big(\mathcal{E} \otimes_{\gamma_B} (B \otimes \mathcal{G}), \mathcal{E} \otimes \mathcal{G}\big)$ and $V_{\mathcal{F}} \in \mathcal{L}\big(\mathcal{F} \otimes_{\gamma_B} (B \otimes \mathcal{G}), \mathcal{F} \otimes \mathcal{G}\big)$ are the unitaries associated to the coactions $\gamma_{\mathcal{E}}$ and $\gamma_{\mathcal{F}}$, respectively.

We say that two coactions $\gamma_{\mathcal{E}}$ and $\gamma_{\mathcal{F}}$ are *equivalent* if there is a $\mathcal{G}$-equivariant unitary $U \in \mathcal{L}(\mathcal{E}, \mathcal{F})$. Thus $\gamma_{\mathcal{E}}$ and $\gamma_{\mathcal{F}}$ are equivalent if and only if there is a unitary operator $U : \mathcal{E} \to \mathcal{F}$ such that

$$V_{\mathcal{F}}(U \underset{\gamma_B}{\otimes} 1)V_{\mathcal{E}}^* = (U \otimes 1). \tag{2.20}$$

We denote the set of $\mathcal{G}$-equivariant operators in $\mathcal{L}(\mathcal{E}, \mathcal{F})$ by $\mathcal{L}^{\mathcal{G}}(\mathcal{E}, \mathcal{F})$. One has that $T \in \mathcal{L}^{\mathcal{G}}(\mathcal{E}, \mathcal{F})$ if and only if $T^* \in \mathcal{L}^{\mathcal{G}}(\mathcal{F}, \mathcal{E})$, and if $T \in \mathcal{L}^{\mathcal{G}}(\mathcal{E}, \mathcal{F})$ and $R \in \mathcal{L}^{\mathcal{G}}(\mathcal{D}, \mathcal{E})$, where $\mathcal{D}$ is another $\mathcal{G}$-equivariant Hilbert B-module, then $RT \in \mathcal{L}^{\mathcal{G}}(\mathcal{D}, \mathcal{F})$. The space $\mathcal{L}^{\mathcal{G}}(\mathcal{E}) := \mathcal{L}^{\mathcal{G}}(\mathcal{E}, \mathcal{E})$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{E})$.

Given a C^*-algebra A with a coaction γ_A of $\mathcal{G}$, we define the *fixed point algebra*

$$\mathcal{M}_1(A) := \{a \in \mathcal{M}(A) : \gamma_A(a) = a \otimes 1\}. \tag{2.21}$$

Proposition 2.6.13. *Let $\mathcal{E}$ be a Hilbert $B, \mathcal{G}$-module. Then, under the canonical identification $\mathcal{L}(\mathcal{E}) \cong \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$, we have $\mathcal{L}^{\mathcal{G}}(\mathcal{E}) \cong \mathcal{M}_1\big(\mathcal{K}(\mathcal{E})\big)$.*

Proof. If $\gamma_{\mathcal{K}(\mathcal{E})}(T) = T \otimes 1$, then $\gamma_{\mathcal{E}}(T\xi) = \gamma_{\mathcal{K}(\mathcal{E})}(T)\gamma_{\mathcal{E}}(\xi) = (T \otimes 1)\gamma_{\mathcal{E}}(\xi)$, that is, T is in $\mathcal{L}^{\mathcal{G}}(\mathcal{E})$. Conversely, if $T \in \mathcal{L}^{\mathcal{G}}(\mathcal{E})$, then $\gamma_{\mathcal{K}(\mathcal{E})}(T)\gamma_{\mathcal{E}}(\xi)x = \gamma_{\mathcal{E}}(T\xi)x = (T \otimes 1)\gamma_{\mathcal{E}}(\xi)x$ for all $\xi \in \mathcal{E}$ and $x \in B \otimes \mathcal{G}$. Since $\overline{\operatorname{span}}\,\gamma_{\mathcal{E}}(\mathcal{E})(B \otimes \mathcal{G}) = \mathcal{E} \otimes \mathcal{G}$, it follows that $T \in \mathcal{M}_1\big(\mathcal{K}(\mathcal{E})\big)$. □

Proposition 2.6.14. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules with coactions $\gamma_{\mathcal{E}}$ and $\gamma_{\mathcal{F}}$ of a locally compact quantum group $\mathcal{G}$. For an operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ the following assertions are equivalent:*

(i) *T is $\mathcal{G}$-equivariant, that is, $\gamma_{\mathcal{F}}(T\xi) = (T \otimes 1)\gamma_{\mathcal{E}}(\xi)$ for all $\xi \in \mathcal{E}$,*

(ii) *T is $L^1(\mathcal{G})$-invariant, that is, $T(\omega * \xi) = \omega * (T\xi)$ for all $\xi \in \mathcal{E}$, $\omega \in L^1(\mathcal{G})$.*

Proof. If (i) is true, then we get

$$T(\omega * \xi) = T\big((\mathrm{id}_{\mathcal{E}} \otimes \omega)\gamma_{\mathcal{E}}(\xi)\big) = (\mathrm{id}_{\mathcal{F}} \otimes \omega)\big((T \otimes 1)\gamma_{\mathcal{E}}(\xi)\big) = \omega * (T\xi)$$

for all $\xi \in \mathcal{E}$ and $\omega \in L^1(\mathcal{G})$. Hence (i) implies (ii). Assume now that (ii) is true. Then for all $\xi \in \mathcal{E}$ and $\omega \in L^1(\mathcal{G})$ we have

$$\begin{aligned}(\mathrm{id}_{\mathcal{F}} \otimes \omega)\big(\gamma_{\mathcal{F}}(T\xi)\big) &= \omega * (T\xi) = T(\omega * \xi) \\ &= T\big((\mathrm{id}_{\mathcal{E}} \otimes \omega)\gamma_{\mathcal{E}}(\xi)\big) = (\mathrm{id}_{\mathcal{F}} \otimes \omega)\big((T \otimes 1)\gamma_{\mathcal{E}}(\xi)\big).\end{aligned}$$

Since $\omega \in L^1(\mathcal{G})$ is arbitrary, and since $L^1(\mathcal{G})$ contains elements of the form $\omega_{u,v}$, $u, v \in H$, this implies that $\gamma_{\mathcal{F}}(T\xi) = (T \otimes 1)\gamma_{\mathcal{E}}(\xi)$, that is, T is $\mathcal{G}$-equivariant. □

Definition 2.6.15. Let $\mathcal{E}_1$ be a Hilbert B-module and let $\mathcal{E}_2$ be a Hilbert C-module with coactions $\gamma_{\mathcal{E}_1}$ and $\gamma_{\mathcal{E}_2}$ of $\mathcal{G}$, respectively, and suppose that $\pi : B \to \mathcal{L}(\mathcal{E}_2)$ is a $\mathcal{G}$-equivariant nondegenerate $*$-homomorphism. Consider $\mathcal{E} := \mathcal{E}_1 \underset{\pi}{\otimes} \mathcal{E}_2$. The *balanced tensor product* of $\gamma_{\mathcal{E}_1}$ and $\gamma_{\mathcal{E}_2}$ is the coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$ on $\mathcal{E}$ defined by:

$$\gamma_{\mathcal{E}}(\xi \underset{\pi}{\otimes} \eta) = (\gamma_{\mathcal{E}_1}(\xi) \underset{\pi\otimes\mathrm{id}_{\mathcal{G}}}{\otimes} 1) \circ \gamma_{\mathcal{E}_2}(\eta)$$

for all $\xi \in \mathcal{E}_1$ and $\eta \in \mathcal{E}_2$, where $1 = 1_{\mathcal{E}_2} \otimes 1_{\mathcal{G}}$ and one identifies

$$\gamma_{\mathcal{E}_1}(\xi) \underset{\pi\otimes\mathrm{id}_{\mathcal{G}}}{\otimes} 1 \in \mathcal{L}\big((B \otimes \mathcal{G}) \underset{\pi\otimes\mathrm{id}_{\mathcal{G}}}{\otimes} (\mathcal{E}_2 \otimes \mathcal{G}), (\mathcal{E}_1 \otimes \mathcal{G}) \underset{\pi\otimes\mathrm{id}_{\mathcal{G}}}{\otimes} (\mathcal{E}_2 \otimes \mathcal{G})\big) \cong \mathcal{L}(\mathcal{E}_2 \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G}).$$

See [5, Proposition 2.10] for details. See also Proposition 2.13 in [15].

Definition 2.6.16. Let $\mathcal{E}$ be a Hilbert B-module and suppose that $\gamma_{\mathcal{E}}$ is a coaction of $\mathcal{G}$ on $\mathcal{E}$. A *cocycle* for $(\mathcal{E}, \gamma_{\mathcal{E}})$, or shortly, a $\gamma_{\mathcal{E}}$*-cocycle* is a unitary $u \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G})$ satisfying

$$(\mathrm{id} \otimes \Delta)(u) = u_{12}(\gamma_{\mathcal{K}(\mathcal{E})} \otimes \mathrm{id}_{\mathcal{G}})(u).$$

Remark 2.6.17. We recall some well-known facts about cocycles. Details can be found in [25].

(1) Given a unitary $u \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G})$, it is easy to see that it is a cocycle for $(\mathcal{E}, \gamma_{\mathcal{E}})$ if and only if the map $u \cdot \gamma_{\mathcal{E}} : \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ given by $(u \cdot \gamma_{\mathcal{E}})(\xi) := u \circ \gamma_{\mathcal{E}}(\xi)$ defines a coaction of $\mathcal{G}$ on $\mathcal{E}$ (with the same coefficient coaction of $\mathcal{G}$ on B).

Given a cocycle u for $(\mathcal{E}, \gamma_{\mathcal{E}})$, the coaction $u \cdot \gamma_{\mathcal{E}}$ is continuous if and only if the linear span of $(1 \otimes \mathcal{G})u\gamma_{\mathcal{E}}(\mathcal{E})$ is dense in $\mathcal{E} \otimes \mathcal{G}$. In this case we say that u is *continuous*.

Note that the coaction induced by $u \cdot \gamma_{\mathcal{E}}$ on $\mathcal{K}(\mathcal{E})$ is $\gamma^u_{\mathcal{K}(\mathcal{E})} := \mathrm{Ad}_u \circ \gamma_{\mathcal{K}(\mathcal{E})}$. In particular, if γ_A is a coaction of $\mathcal{G}$ on a C^*-algebra A and $u \in \mathcal{M}(A \otimes \mathcal{G})$ is a γ_A-cocycle, then the map $\gamma^u_A = \mathrm{Ad}_u \circ \gamma_A$ is a coaction of $\mathcal{G}$ on A.

Two coactions γ_A and $\tilde{\gamma}_A$ of $\mathcal{G}$ on A are called *exterior equivalent* if there is a cocycle u for (A, γ_A) such that $\tilde{\gamma}_A = \gamma^u_A$.

(2) If $\mathcal{E}$ (and also B) is considered with the trivial coaction γ_{tr}, then a cocycle for $(\mathcal{E}, \gamma_{tr})$ is just a (right) corepresentation of $\mathcal{G}$ on $\mathcal{E}$, that is, a unitary $u \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G})$ satisfying $(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}$. Note that in this case u is continuous if and only if the linear span of $(1 \otimes \mathcal{G})u(\mathcal{E} \otimes 1_{\mathcal{G}})$ is dense in $\mathcal{E} \otimes \mathcal{G}$. In this case, we also say that u is a *continuous corepresentation*. Suppose that $\mathcal{G}$ is regular. Then any corepresentation of $\mathcal{G}$ is continuous. This follows from Remark 2.6.9(3). On the other hand, if $\mathcal{G}$ is not regular, then the right regular corepresentation $V \in \mathcal{L}(H \otimes \mathcal{G})$ is not continuous. In fact, V is continuous if and only if $\mathcal{G}$ is regular. See Proposition 2.7.11 below.

(3) Let (A, γ_A) and (B, γ_B) be coactions of $\mathcal{G}$ and suppose that $\pi : A \to \mathcal{M}(B)$ is a nondegenerate $\mathcal{G}$-equivariant $*$-homomorphism. If u is a cocycle for (A, γ_A), then $v := (\pi \otimes \mathrm{id}_{\mathcal{G}})(u)$ is a cocycle for (B, γ_B) and π is also equivariant with respect to the coactions (A, γ^u_A) and (B, γ^v_B). If γ_B is continuous and u is continuous, then so is v. In fact, this follows from the calculation (using that π is nondegenerate, so that $\pi(A)B = B$):

$$\begin{aligned}\overline{\mathrm{span}}((1\otimes\mathcal{G})v\gamma_B(B)) &= \overline{\mathrm{span}}\big((\pi\otimes\mathrm{id}_{\mathcal{G}})\big((1\otimes\mathcal{G})u\big)\gamma_B(B)\big)\\ &= \overline{\mathrm{span}}\big((\pi\otimes\mathrm{id}_{\mathcal{G}})\big((1\otimes\mathcal{G})u\gamma_A(A)\big)\gamma_B(B)\big)\\ &= \overline{\mathrm{span}}\big((\pi(A)\otimes\mathcal{G})\gamma_B(B)\big)\\ &= \overline{\mathrm{span}}\big((\pi(A)\otimes 1)(B\otimes\mathcal{G})\big) = B\otimes\mathcal{G}.\end{aligned}$$

Example 2.6.18. **(1)** Let (A,γ_A) be a coaction of $\mathcal{G}$, and suppose that B is an arbitrary C^*-algebra considered with the trivial coaction γ_{tr} of $\mathcal{G}$. Then the map

$$\gamma_A\otimes_*\mathrm{id}_B := (1_A\otimes\sigma)\circ(\gamma_A\otimes\mathrm{id}_B) : A\otimes B\to\mathcal{M}(A\otimes B\otimes\mathcal{G})$$

defines a coaction of $\mathcal{G}$ on $A\otimes B$, where $\sigma:\mathcal{G}\otimes B\to B\otimes\mathcal{G}$ is the flip map. Moreover, note that the canonical map $\pi: B\to\mathcal{M}(A\otimes B)$, $b\mapsto 1_A\otimes b$ is equivariant. Thus, if u is a cocycle for (B,γ_{tr}), that is, a corepresentation $u\in\mathcal{M}(B\otimes\mathcal{G})$ of $\mathcal{G}$ on B, then $u_{23} = 1_A\otimes u = (\pi\otimes\mathrm{id}_{\mathcal{G}})(u)$ is a cocycle for $(A\otimes B,\gamma_A\otimes_*\mathrm{id}_B)$. In particular, the maps

$$\gamma_B := [b\mapsto u(b\otimes 1)u^*] \quad\text{and}\quad \gamma_{A\otimes B} := [x\mapsto u_{23}(\gamma_A\otimes_*\mathrm{id}_B)(x)u_{23}^*]$$

define coactions of $\mathcal{G}$ on B and $A\otimes B$, respectively, and π is equivariant with respect these coactions. If u is continuous, and γ_A is continuous, then it follows from Remark 2.6.17(3) that $\gamma_{A\otimes B}$ is continuous.

(2) The example above can be applied to the following situation. Let $\mathcal{E}$ be a Hilbert B-module with a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$ and let K be a Hilbert space. Then the map

$$\gamma_{\mathcal{E}}\otimes_*\mathrm{id}_K : \mathcal{E}\otimes K\to\mathcal{M}(\mathcal{E}\otimes K\otimes\mathcal{G}),\quad \zeta\mapsto\Sigma_{23}(\gamma_{\mathcal{E}}\otimes\mathrm{id}_K)(\zeta)$$

defines a coaction of $\mathcal{G}$ on $\mathcal{E}\otimes K$, where $\Sigma:\mathcal{G}\otimes K\to K\otimes\mathcal{G}$ is the flip map (and $\Sigma_{23} = 1_{\mathcal{E}}\otimes\Sigma$). The corresponding coaction of $\mathcal{G}$ on $\mathcal{K}(\mathcal{E}\otimes K)\cong\mathcal{K}(\mathcal{E})\otimes\mathcal{K}(K)$ is the coaction $\gamma_{\mathcal{K}(\mathcal{E})}\otimes_*\mathrm{id}_{\mathcal{K}(K)}$ considered in (1). Thus, if $u\in\mathcal{L}(K\otimes\mathcal{G})$ is a corepresentation of $\mathcal{G}$, then u_{23} is a cocycle for $(\mathcal{E}\otimes K,\gamma_{\mathcal{E}}\otimes_*\mathrm{id}_K)$, and therefore, by Remark 2.6.17(1), the map

$$\gamma_{\mathcal{E}\otimes H} : \mathcal{E}\otimes K\to\mathcal{M}(\mathcal{E}\otimes K\otimes\mathcal{G}),\quad \zeta\mapsto u_{23}\Sigma_{23}(\gamma_{\mathcal{E}}\otimes\mathrm{id}_K)(\zeta)$$

is a coaction of $\mathcal{G}$ on $\mathcal{E}\otimes K$. Note that the corresponding coaction on the algebra of compact operators $\mathcal{K}(\mathcal{E}\otimes K)\cong\mathcal{K}(\mathcal{E})\otimes\mathcal{K}(K)$ is the one considered in (1):

$$\gamma_{\mathcal{K}(\mathcal{E}\otimes K)}(x) = u_{23}(\gamma_{\mathcal{K}(\mathcal{E})}\otimes_*\mathrm{id})(x)u_{23}^* = u_{23}\Sigma_{23}(\gamma_{\mathcal{K}(\mathcal{E})}\otimes\mathrm{id})(x)\Sigma_{23}^*u_{23}^*.$$

If u is continuous and $\gamma_{\mathcal{E}}$ is continuous, then so is $\gamma_{\mathcal{E}\otimes K}$. This follows from the calculation:

$$\begin{aligned}\overline{\mathrm{span}}\big((1\otimes 1\otimes\mathcal{G})\gamma_{\mathcal{E}\otimes K}(\mathcal{E}\otimes K)\big) &= \overline{\mathrm{span}}\Big(\big(1\otimes(1\otimes\mathcal{G})u\big)\Sigma_{23}\big(\gamma_{\mathcal{E}}(\mathcal{E})\otimes K\big)\Big)\\ &= \overline{\mathrm{span}}\Big(\big(1\otimes(1\otimes\mathcal{G})u(\mathcal{K}(K)\otimes 1)\big)\Sigma_{23}\big(\gamma_{\mathcal{E}}(\mathcal{E})\otimes K\big)\Big)\\ &= \overline{\mathrm{span}}\Big(\big(1\otimes\mathcal{K}(H)\otimes\mathcal{G}\big)\Sigma_{23}\big(\gamma_{\mathcal{E}}(\mathcal{E})\otimes K\big)\Big)\\ &= \overline{\mathrm{span}}\Big(\Sigma_{23}\big((1\otimes\mathcal{G})\gamma_{\mathcal{E}}(\mathcal{E})\otimes K\big)\Big) = \mathcal{E}\otimes K\otimes\mathcal{G}.\end{aligned}$$

(3) As a special case of (2) one can consider the right regular corepresentation $V \in \mathcal{L}(H \otimes \mathcal{G})$ to get the coaction

$$\tilde{\gamma}_{\mathcal{E}\otimes H} := V_{23}\Sigma_{23}(\gamma_{\mathcal{E}} \otimes \mathrm{id}_H) : \mathcal{E} \otimes H \to \mathcal{M}(\mathcal{E} \otimes H),$$

where $\mathcal{E}$ is any Hilbert B-module with a coaction of $\gamma_{\mathcal{E}}$ of $\mathcal{G}$. Another possibility is to consider the left regular corepresentation $\hat{W} \in \mathcal{M}(\widehat{\mathcal{G}} \otimes \mathcal{G}) \subseteq \mathcal{L}(H \otimes \mathcal{G})$. We have $(\mathrm{id} \otimes \Delta)(\hat{W}) = \hat{W}_{13}\hat{W}_{12}$, which implies that $\hat{W}^*$ is a (right) corepresentation of $\mathcal{G}$ on H. Therefore the map

$$\gamma_{\mathcal{E}\otimes H} := \hat{W}^*_{23}\Sigma_{23}(\gamma_{\mathcal{E}} \otimes \mathrm{id}_H) : \mathcal{E} \otimes H \to \mathcal{M}(\mathcal{E} \otimes H),$$

also defines a coaction of $\mathcal{G}$ on $\mathcal{E} \otimes H$. In particular, if $\mathcal{E} = \mathbb{C}$, then we obtain two coactions of $\mathcal{G}$ on H:

$$\tilde{\gamma}_H(\xi) = V(\xi \otimes 1_{\mathcal{G}}) \quad \text{and} \quad \gamma_H(\xi) = \hat{W}^*(\xi \otimes 1_{\mathcal{G}}), \quad \xi \in H.$$

In fact, these coactions are equivalent, that is, there is an equivariant unitary $U \in \mathcal{L}(H)$, which means that $(U \otimes 1)V = \hat{W}^*(U \otimes 1)$ (that is, the corepresentations V and $\hat{W}^*$ are equivalent). By Equations (2.15) and (2.16), the unitary $U := J\hat{J}$ satisfies this relation. Therefore γ_H and $\tilde{\gamma}_H$ are equivalent and, as a consequence, the corresponding coactions $\tilde{\gamma}_{\mathcal{K}}$ and $\gamma_{\mathcal{K}}$ on $\mathcal{K} = \mathcal{K}(H)$ are isomorphic via the map $x \mapsto \mathrm{Ad}_U(x) = UxU^*$. Note that the coactions $\tilde{\gamma}_{\mathcal{K}}$ and $\gamma_{\mathcal{K}}$ are given by the formulas:

$$\tilde{\gamma}_{\mathcal{K}}(x) = V(x \otimes 1)V^* \quad \text{and} \quad \gamma_{\mathcal{K}}(x) = \hat{W}^*(x \otimes 1)\hat{W}, \quad x \in \mathcal{K}.$$

More generally, the coactions $\tilde{\gamma}_{\mathcal{E}\otimes H}$ and $\gamma_{\mathcal{E}\otimes H}$ are equivalent. The unitary $1 \otimes U \in \mathcal{L}(\mathcal{E} \otimes H)$ implements this equivalence. Hence the corresponding coactions $\tilde{\gamma}_{\mathcal{K}(\mathcal{E}\otimes H)}$ and $\gamma_{\mathcal{K}(\mathcal{E}\otimes H)}$ on the algebra of compact operators $\mathcal{K}(\mathcal{E} \otimes H)$ are isomorphic via $x \mapsto \mathrm{Ad}_{(1\otimes U)}(x) = (1 \otimes U)x(1 \otimes U^*)$.

2.6.3 Invariant direct summands

Let $\mathcal{F}$ be a Hilbert B-module and suppose that we have a *direct summand* $\mathcal{E}$ of $\mathcal{F}$. This means that $\mathcal{E}$ is a B-submodule of $\mathcal{F}$ which is *complementable*, that is, $\mathcal{E} \oplus \mathcal{E}^\perp = \mathcal{F}$. This yields a canonical isomorphism $\mathcal{M}(\mathcal{E}) \oplus \mathcal{M}(\mathcal{E}^\perp) \cong \mathcal{M}(\mathcal{F})$, so that $\mathcal{M}(\mathcal{E})$ is also a direct summand of $\mathcal{M}(\mathcal{F})$. We denote by $P_{\mathcal{E}}$ the projection of $\mathcal{F}$ onto $\mathcal{E}$. There are canonical isomorphisms $\mathcal{F} \otimes \mathcal{G} \cong (\mathcal{E} \otimes \mathcal{G}) \oplus (\mathcal{E}^\perp \otimes \mathcal{G})$ and $\mathcal{M}(\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{M}(\mathcal{E} \otimes \mathcal{G}) \oplus \mathcal{M}(\mathcal{E}^\perp \otimes \mathcal{G})$. Under these identifications, $\mathcal{E} \otimes \mathcal{G}$ becomes a direct summand of $\mathcal{F} \otimes \mathcal{G}$ and $\mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ a direct summand of $\mathcal{M}(\mathcal{F} \otimes \mathcal{G})$. The projection of $\mathcal{F} \otimes \mathcal{G}$ onto $\mathcal{E} \otimes \mathcal{G}$ is $P_{\mathcal{E}} \otimes 1_{\mathcal{G}}$. We shall use these identifications in what follows.

Proposition 2.6.19. *Let $\mathcal{F}$ be a Hilbert B-module with a coaction $\gamma_{\mathcal{F}}$ of $\mathcal{G}$ and suppose that $\mathcal{E}$ is a direct summand of $\mathcal{F}$. The following statements are equivalent:*

(i) $(P_{\mathcal{E}} \otimes 1_{\mathcal{G}})\gamma_{\mathcal{F}}(\eta) = \gamma_{\mathcal{F}}\big(P_{\mathcal{E}}(\eta)\big)$ *for all* $\eta \in \mathcal{F}$.

(ii) $\gamma_{\mathcal{F}}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ *and* $\overline{\mathrm{span}}\big(\gamma_{\mathcal{F}}(\mathcal{E})(B \otimes \mathcal{G})\big) = \mathcal{E} \otimes \mathcal{G}$.

(iii) $\gamma_{\mathcal{F}}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ *and* $\gamma_{\mathcal{F}}(\mathcal{E}^{\perp}) \subseteq \mathcal{M}(\mathcal{E}^{\perp} \otimes \mathcal{G})$.

Proof. First we prove that (i) implies (ii). Of course, from (i) it follows that $\gamma_{\mathcal{F}}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$. So we only have to prove that $\overline{\operatorname{span}}\big(\gamma_{\mathcal{F}}(\mathcal{E})(B \otimes \mathcal{G})\big) = \mathcal{E} \otimes B$. Let $x \in \mathcal{E} \otimes \mathcal{G} \subseteq \mathcal{F} \otimes \mathcal{G}$. Since $\gamma_{\mathcal{F}}$ is a coaction, for every $\epsilon > 0$, there exist $\eta_1, \ldots, \eta_n \in \mathcal{F}$ and $y_1, \ldots, y_n \in B \otimes \mathcal{G}$ such that

$$\left\| \sum_{i=1}^{n} \gamma_{\mathcal{F}}(\eta_i) y_i - x \right\| < \epsilon.$$

Take $\xi_i := P_{\mathcal{E}}(\eta_i)$, $i = 1, \ldots, n$. Then

$$\begin{aligned}
\left\| \sum_{i=1}^{n} \gamma_{\mathcal{F}}(\xi_i) y_i - x \right\| &= \left\| \sum_{i=1}^{n} \gamma_{\mathcal{F}}\big(P_{\mathcal{E}}(\eta_i)\big) y_i - x \right\| \\
&= \left\| (P_{\mathcal{E}} \otimes 1_{\mathcal{G}}) \left(\sum_{i=1}^{n} \gamma_{\mathcal{F}}(\eta_i) y_i - x \right) \right\| \\
&\leq \left\| \sum_{i=1}^{n} \gamma_{\mathcal{F}}(\eta_i) y_i - x \right\| < \epsilon.
\end{aligned}$$

It is easy to prove that (ii) is equivalent to (iii). So it remains to prove that (ii) implies (i). We claim that $\gamma_{\mathcal{F}}(\mathcal{E}^{\perp}) \subseteq \mathcal{M}(\mathcal{E}^{\perp} \otimes \mathcal{G})$. Take $\xi^{\perp} \in \mathcal{E}^{\perp}$. Then, for all $\xi \in \mathcal{E}$ and $y \in B \otimes \mathcal{G}$, we have

$$\langle \gamma_{\mathcal{F}}(\xi^{\perp}) | \gamma_{\mathcal{F}}(\xi) y \rangle = \gamma_B(\langle \xi^{\perp} | \xi \rangle) y = 0.$$

Since $\overline{\operatorname{span}}\big(\gamma_{\mathcal{F}}(\mathcal{E})(B \otimes \mathcal{G})\big) = \mathcal{E} \otimes \mathcal{G}$, we get $\langle \gamma_F(\xi^{\perp}) | z \rangle = 0$ for all $z \in \mathcal{E} \otimes \mathcal{G}$. Thus, for all $T \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ and $y \in B \otimes \mathcal{G}$, we have

$$\langle \gamma_{\mathcal{F}}(\xi^{\perp}) | T \rangle y = \langle \gamma_{\mathcal{F}}(\xi^{\perp}) | T y \rangle = 0.$$

That is, $\langle \gamma_{\mathcal{F}}(\xi^{\perp}) | T \rangle = 0$. Therefore $\gamma_{\mathcal{F}}(\xi^{\perp}) \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})^{\perp} = \mathcal{M}(\mathcal{E}^{\perp} \otimes \mathcal{G})$, proving our claim. Now, if $\eta \in \mathcal{F}$ and $\eta = \xi + \xi^{\perp}$, with $\xi \in \mathcal{E}$ and $\xi^{\perp} \in \mathcal{E}^{\perp}$, then

$$(P_{\mathcal{E}} \otimes 1_{\mathcal{G}})\big(\gamma_{\mathcal{F}}(\eta)\big) = (P_{\mathcal{E}} \otimes 1_{\mathcal{G}})\big(\gamma_{\mathcal{F}}(\xi) + \gamma_{\mathcal{F}}(\xi^{\perp})\big) = \gamma_{\mathcal{F}}(\xi) = \gamma_{\mathcal{F}}\big(P_{\mathcal{E}}(\eta)\big). \qquad \square$$

Definition 2.6.20. If the equivalent conditions of Proposition 2.6.19 are satisfied, then we say that $\mathcal{E}$ is a *$\mathcal{G}$-invariant direct summand* of $\mathcal{F}$.

Note that $\mathcal{E}$ is a $\mathcal{G}$-invariant direct summand of $\mathcal{F}$ if and only if the formula $\gamma_{\mathcal{E}}(\xi) := \gamma_{\mathcal{F}}(\xi)$ defines a coaction $\gamma_{\mathcal{E}} : \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ of $\mathcal{G}$ on $\mathcal{E}$. In this case, the projection $P_{\mathcal{E}} : \mathcal{F} \to \mathcal{E}$ is a $\mathcal{G}$-equivariant operator, that is, $P_{\mathcal{E}} \in \mathcal{L}^{\mathcal{G}}(\mathcal{F}, \mathcal{E})$. Of course, $\mathcal{E}^{\perp}$ is also a $\mathcal{G}$-invariant direct summand of $\mathcal{E}$ and hence we also have a coaction $\gamma_{\mathcal{E}^{\perp}}$ of $\mathcal{G}$ on $\mathcal{E}^{\perp}$ which is the restriction of $\gamma_{\mathcal{F}}$ to $\mathcal{E}^{\perp}$ and such that the projection $P_{\mathcal{E}^{\perp}} : \mathcal{F} \to \mathcal{E}^{\perp}$ is $\mathcal{G}$-equivariant. Moreover, we have

$$\gamma_{\mathcal{F}}(\eta) = \gamma_{\mathcal{E}}(\xi) + \gamma_{\mathcal{E}^{\perp}}(\xi^{\perp}), \quad \text{for all } \eta = \xi + \xi^{\perp} \in \mathcal{F} = \mathcal{E} \oplus \mathcal{E}^{\perp}.$$

Proposition 2.6.21. *Let $\mathcal{F}$ be a Hilbert B-module with a coaction $\gamma_{\mathcal{F}}$ of $\mathcal{G}$, and let $\mathcal{E}$ be a $\mathcal{G}$-invariant direct summand of $\mathcal{F}$. If $\gamma_{\mathcal{F}}$ is continuous, then so is $\gamma_{\mathcal{E}}$.*

Proof. Let $x \in \mathcal{E} \otimes \mathcal{G}$ and $\epsilon > 0$. By assumption, we have $\overline{\operatorname{span}}\big((1_{\mathcal{F}} \otimes \mathcal{G})\gamma_{\mathcal{F}}(\mathcal{F})\big) = \mathcal{F} \otimes \mathcal{G}$. So there are $x_1, \ldots, x_n \in \mathcal{G}$ and $\eta_1, \ldots, \eta_n \in \mathcal{F}$ such that

$$\|x - \sum_{i=1}^{n} (1_{\mathcal{F}} \otimes x_i)\gamma_{\mathcal{F}}(\eta_i)\| < \epsilon.$$

For each i, we define $\xi_i := P_{\mathcal{E}}(\eta_i) \in \mathcal{E}$. Then we get

$$\begin{aligned} \left\| x - \sum_{i=1}^{n} (1_{\mathcal{E}} \otimes x_i)\gamma_{\mathcal{E}}(\xi_i) \right\| &= \left\| (P_{\mathcal{E}} \otimes 1_{\mathcal{G}}) \left(x - \sum_{i=1}^{n} (1_{\mathcal{F}} \otimes x_i)\gamma_{\mathcal{F}}(\eta_i) \right) \right\| \\ &\leq \left\| x - \sum_{i=1}^{n} (1_{\mathcal{F}} \otimes x_i)\gamma_{\mathcal{F}}(\eta_i) \right\| < \epsilon. \end{aligned}$$ □

2.6.4 Invariant ideals

Let B be a C^*-algebra and $I \subseteq B$ a closed ideal. By Proposition 2.2.8, we may identify

$$\tilde{\mathcal{M}}(I \otimes \mathcal{G}) \cong \{m \in \tilde{\mathcal{M}}(B \otimes \mathcal{G}) : m(1 \otimes \mathcal{G}), (1 \otimes \mathcal{G})m \subseteq I \otimes \mathcal{G}\}.$$

We shall use this identification in what follows.

Definition 2.6.22. Let B be a $\mathcal{G}$-C^*-algebra and let $I \subseteq B$ be a closed ideal. We say that I is *$\mathcal{G}$-invariant* if $\gamma_B(I) \subseteq \tilde{\mathcal{M}}(I \otimes \mathcal{G})$ and the restriction map

$$\gamma_I := \gamma_B|_I : I \to \tilde{\mathcal{M}}(I \otimes \mathcal{G})$$

defines a coaction of $\mathcal{G}$ on I.

Note that we do not assume that the restriction γ_I is a continuous coaction. But the following result shows that this is, in fact, automatic.

Proposition 2.6.23. *Let B be a $\mathcal{G}$-C^*-algebra and let $I \subseteq B$ be a closed ideal. Then I is $\mathcal{G}$-invariant if and only if $\overline{\operatorname{span}}\, \gamma_B(I)(1 \otimes \mathcal{G}) = I \otimes B$.*

Proof. Assume that I is $\mathcal{G}$-invariant and take $x \in I \otimes B$. Since γ_B is continuous, we can approximate x by a sum of the form $\sum \gamma_B(b_i)(1 \otimes y_i)$, where $b_i \in B$ and $y_i \in \mathcal{G}$. Since the restriction $\gamma_I = \gamma_B|_I : I \to \tilde{\mathcal{M}}(I \otimes B)$ is a coaction, it is in particular a nondegenerate $*$-homomorphism. Thus, if (e_j) is an approximate unit for I, then $\gamma_I(e_j)x$ approximates x. We conclude that x is approximately $\gamma_I(e_j)\left(\sum \gamma_B(b_i)(1 \otimes y_i)\right) = \sum \gamma_B(e_j b_i)(1 \otimes y_i)$. Thus $\overline{\operatorname{span}}\, \gamma_B(I)(1 \otimes \mathcal{G}) = I \otimes B$. Conversely, if this is true, then $\gamma_B(I) \subseteq \tilde{\mathcal{M}}(I \otimes B)$ and the restriction $\gamma_I = \gamma_B|_I$ is a nondegenerate $*$-homomorphism. The coaction identity for γ_I follows directly from that of γ_B. □

Suppose that I is a $\mathcal{G}$-invariant ideal of a $\mathcal{G}$-C^*-algebra B. Let $q : B \to B/I$ be the quotient map. If $b \in I$ then $\gamma_B(b)(1 \otimes x) \in I \otimes \mathcal{G}$ for all $x \in \mathcal{G}$, and hence

$$(q \otimes \mathrm{id}_{\mathcal{G}})\big(\gamma_B(b)\big)(1 \otimes x) = (q \otimes \mathrm{id}_{\mathcal{G}})\big(\gamma_B(b)(1 \otimes x)\big) = 0$$

It follows that the map $\gamma_{B/I} : B/I \to \mathcal{M}(B/I \otimes \mathcal{G})$ given by

$$\gamma_{B/I}\big(q(b)\big) := (q \otimes \mathrm{id}_{\mathcal{G}})\gamma_B(b)$$

well-defines a coaction of $\mathcal{G}$ on B/I. Since γ_B is continuous, so is $\gamma_{B/I}$. Note that, by the definition of $\gamma_{B/I}$, the quotient map q is $\mathcal{G}$-equivariant.

Proposition 2.6.24. *Let $\mathcal{E}$ be a Hilbert $B, \mathcal{G}$-module with a continuous coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$. Then $I := \overline{\operatorname{span}}\langle \mathcal{E}, \mathcal{E}\rangle_B$ is a $\mathcal{G}$-invariant ideal of B.*

Proof. Since $\gamma_{\mathcal{E}}$ is continuous, we have

$$\begin{aligned}\overline{\operatorname{span}}\,\gamma_B(I)(1 \otimes \mathcal{G}) &= \overline{\operatorname{span}}\,\gamma_{\mathcal{E}}(\mathcal{E})^*\gamma_{\mathcal{E}}(\mathcal{E})(1 \otimes \mathcal{G}) \\ &= \overline{\operatorname{span}}\,\gamma_{\mathcal{E}}(\mathcal{E})^*(\mathcal{E} \otimes \mathcal{G}) = \overline{\operatorname{span}}\,\gamma_{\mathcal{E}}(\mathcal{E})^*(1 \otimes \mathcal{G})(\mathcal{E} \otimes \mathcal{G}) \\ &= \overline{\operatorname{span}}((1 \otimes \mathcal{G})\gamma_{\mathcal{E}}(\mathcal{E}))^*(\mathcal{E} \otimes \mathcal{G}) = \overline{\operatorname{span}}(\mathcal{E} \otimes \mathcal{G})^*(\mathcal{E} \otimes \mathcal{G}) = I \otimes \mathcal{G}. \qquad \square\end{aligned}$$

2.7 Crossed products

2.7.1 Reduced crossed products

Let $\gamma_A : A \to \mathcal{M}(A \otimes \mathcal{G})$ be a continuous coaction of $\mathcal{G}$. Recall that $\mathcal{G}$ and $\widehat{\mathcal{G}}$ are C^*-subalgebras of $\mathcal{L}(H)$, where $H = L^2(\mathcal{G})$. Thus we may view $\mathcal{M}(A \otimes \mathcal{G})$ as a C^*-subalgebra of $\mathcal{M}\big(A \otimes \mathcal{K}(H)\big) \cong \mathcal{L}(A \otimes H)$. The *reduced crossed product* of the coaction (A, γ_A) is by definition

$$A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} := \overline{\operatorname{span}}\big(\gamma_A(A)(1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big) \subseteq \mathcal{L}(A \otimes H),$$

where $\widehat{\mathcal{G}}^{\mathrm{c}} = \hat{J}\widehat{\mathcal{G}}\hat{J}$ is the C^*-commutant of $\widehat{\mathcal{G}}$ (see Section 2.5.1). One proves that $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\operatorname{span}}\big((1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\gamma_A(A)\big)$ and therefore $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is a C^*-subalgebra of $\mathcal{L}(A \otimes H)$.

There are canonical nondegenerate $*$-homomorphisms

$$j^{\mathrm{r}}_A : A \to \mathcal{M}(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}), \quad j^{\mathrm{r}}_{\widehat{\mathcal{G}}^{\mathrm{c}}} : \widehat{\mathcal{G}}^{\mathrm{c}} \to \mathcal{M}(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})$$

given by $j^{\mathrm{r}}_A(a) = \gamma_A(a)$ and $j^{\mathrm{r}}_{\widehat{\mathcal{G}}^{\mathrm{c}}}(x) = 1 \otimes x$. The *dual coaction* of γ_A is the coaction $\widehat{\gamma}^{\mathrm{c}}_A : A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \to \mathcal{M}(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \otimes \widehat{\mathcal{G}}^{\mathrm{c}})$ of $\widehat{\mathcal{G}}^{\mathrm{c}}$ on $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ satisfying

$$\widehat{\gamma}^{\mathrm{c}}_A\big(\gamma_A(a)(1 \otimes x)\big) = \big(j^{\mathrm{r}}_A(a) \otimes 1\big)\big((j^{\mathrm{r}}_{\widehat{\mathcal{G}}^{\mathrm{c}}} \otimes \mathrm{id})\hat{\Delta}^{\mathrm{c}}(x)\big) = \big(\gamma_A(a) \otimes 1\big)\big(1 \otimes \hat{\Delta}^{\mathrm{c}}(x)\big)$$

for all $a \in A$ and $x \in \widehat{\mathcal{G}}^{\mathrm{c}}$, where $\hat{\Delta}^{\mathrm{c}}$ is the comultiplication of the locally compact quantum group $\widehat{\mathcal{G}}^{\mathrm{c}}$. Thus, if we "identify" A and $\widehat{\mathcal{G}}^{\mathrm{c}}$ inside of $\mathcal{M}(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})$ via the maps j^{r}_A and $j^{\mathrm{r}}_{\widehat{\mathcal{G}}^{\mathrm{c}}}$, the dual coaction acts trivially on A and by the comultiplication $\hat{\Delta}^{\mathrm{c}}$ on $\widehat{\mathcal{G}}^{\mathrm{c}}$. If $A = \mathbb{C}$ with trivial coaction γ_{tr} of $\mathcal{G}$ then $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \widehat{\mathcal{G}}^{\mathrm{c}}$ and $\widehat{\gamma}^{\mathrm{c}}_{tr}$ is $\hat{\Delta}^{\mathrm{c}}$, the comultiplication of $\widehat{\mathcal{G}}^{\mathrm{c}}$.

If we start with a coaction $\gamma_A : A \to \mathcal{M}(A \otimes \mathcal{G}^{\mathrm{op}})$ of the opposite quantum group $\mathcal{G}^{\mathrm{op}}$ (or, equivalently, with a left coaction of $\mathcal{G}$; see Remark 2.6.1(3)), then, because $\widehat{\mathcal{G}^{\mathrm{op}}} = \widehat{\mathcal{G}}^{\mathrm{c}}$, we get

$$A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}^{\mathrm{op}}}^{c} = \overline{\mathrm{span}}\{\gamma_A(A)(1 \otimes \widehat{\mathcal{G}})\}.$$

In this case, we denote $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}} := A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}^{\mathrm{op}}}^{c}$ and $\widehat{\gamma}_A := \widehat{\gamma}_A^{\mathrm{c}}$. Note that $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}$ has now canonical homomorphisms $j_A^{\mathrm{r}} : A \to \mathcal{M}(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}})$ and $j_{\widehat{\mathcal{G}}}^{\mathrm{r}} : \widehat{\mathcal{G}} \to \mathcal{M}(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}})$ and a dual coaction of $\widehat{\mathcal{G}}$ which acts trivially on A and by the comultiplication $\hat{\Delta}$ on $\widehat{\mathcal{G}}$. If $A = \mathbb{C}$ with trivial coaction γ_{tr} of $\mathcal{G}^{\mathrm{op}}$ then $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}} \cong \widehat{\mathcal{G}}$ and $\widehat{\gamma}_{tr}$ is identified with the comultiplication $\hat{\Delta}$ of $\widehat{\mathcal{G}}$.

Remark 2.7.1. (1) Let us describe what is happening in the group case $\mathcal{G} = \mathcal{C}_0(G)$. In this case, continuous coactions of $\mathcal{G}$ correspond to (strongly) continuous actions of G. Suppose that γ_A is a continuous coaction of $\mathcal{G}$ on a C^*-algebra A. Then the corresponding action of G on A, which we denote by α, is given by the formula $\alpha_t(a) := \gamma_A(a)(t)$, where we identify $\tilde{\mathcal{M}}(A \otimes \mathcal{C}_0(G)) \cong \mathcal{C}_b(G, A)$. The dual of $\mathcal{G}$ is $\widehat{\mathcal{G}} = C_{\mathrm{r}}^*(G)$, the reduced C^*-algebra of G. According to our definition, we have

$$A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big(j_A^{\mathrm{r}}(A)(1_A \otimes \hat{J} C_{\mathrm{r}}^*(G)\hat{J})\big) \subseteq \mathcal{L}\big(A \otimes L^2(G)\big),$$

We identify $A \otimes L^2(G) \cong L^2(G, A)$ in the usual way.[6] Under this identification, j_A^{r} is given by the formula $\big(j_A^{\mathrm{r}}(a)\xi\big)(t) = \alpha_t(a)\xi(t)$ for all $\xi \in \mathcal{C}_c(G, A)$ and $t \in G$. The modular conjugation $\hat{J}$ of $\widehat{\mathcal{G}}$ is given by $\hat{J}f(t) := \delta_G(t)^{-\frac{1}{2}}\overline{f(t^{-1})}$ for all $f \in L^2(G)$ and $t \in G$. Now take any $f \in \mathcal{C}_c(G)$. Then, for all $\xi \in \mathcal{C}_c(G, A)$, we have

$$\begin{aligned}\big(\gamma_A(a)(1_A \otimes \hat{J}\lambda(f)\hat{J})\big)\xi|_t &= \alpha_t(a)\int_G \overline{f(t^{-1}s)}\delta_G(t^{-1}s)^{\frac{1}{2}}\xi(s)\,\mathrm{d}s \\ &= \int_G \alpha_t\big(K(t^{-1}s)\big)\delta_G(t^{-1}s)^{\frac{1}{2}}\xi(s)\,\mathrm{d}s = \tilde{\rho}_K(\xi)|_t,\end{aligned}$$

where $K(r) := \overline{f(r)}a$ for all $r \in G$, and

$$\tilde{\rho} : \mathcal{C}_c(G, A) \to \mathcal{L}\big(L^2(G, A)\big), \quad \tilde{\rho}_K(\xi)|_t := \int_G \alpha_t\big(K(t^{-1}s)\big)\delta_G(t^{-1}s)^{\frac{1}{2}}\xi(s)\,\mathrm{d}s.$$

It follows that

$$A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big\{\tilde{\rho}_K : K \in \mathcal{C}_c(G, A)\big\} = \overline{\tilde{\rho}\big(\mathcal{C}_c(G, A)\big)}.$$

Straightforward calculations show that $\tilde{\rho}$ is a $*$-homomorphism if we equip $\mathcal{C}_c(G, A)$ with the following $*$-algebra structure:

$$(K * L)(t) := \int_G K(s)\alpha_s\big(L(s^{-1}t)\big)\,\mathrm{d}s, \quad K^*(t) := \delta_G(t)^{-1}\alpha_t\big(K(t^{-1})\big)^*.$$

[6] Recall that $L^2(G, A)$ is the completion of the pre-Hilbert A-module $\mathcal{C}_c(G, A)$ with respect to the A-inner product $\langle \xi \,|\, \eta \rangle_A := \int_G \xi(t)^*\eta(t)\,\mathrm{d}t$ and the canonical right A-action.

One can also leave out the modular function in the above formulas. More precisely, define

$$\rho : \mathcal{C}_c(G, A) \to \mathcal{L}\big(L^2(G, A)\big), \quad \rho_K(\xi)|_t := \int_G \alpha_t\big(K(t^{-1}s)\big)\xi(s)\,\mathrm{d}s.$$

Then ρ is a $*$-homomorphism if we equip $\mathcal{C}_c(G, A)$ with the same product as above and the modified involution given by the formula $\tilde{K}(t) := \alpha_t\big(K(t^{-1})\big)^*$. These two $*$-algebra structures are isomorphic. The map

$$\mu : \mathcal{C}_c(G, A) \to \mathcal{C}_c(G, A), \quad \mu(K)(t) := \delta_G(t)^{1/2}K(t)$$

is a $*$-isomorphism and we have $\tilde{\rho} = \rho \circ \mu$. Therefore, we also have

$$A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\operatorname{span}}\{\rho_K : K \in \mathcal{C}_c(G, A)\} = \overline{\rho\big(\mathcal{C}_c(G, A)\big)}.$$

This is exactly the representation of $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ used in [48] where the notation $C^*_{\mathrm{r}}(G, A)$ is used instead. We shall also use this notation. But one should keep in mind that this is not the usual definition of $C^*_{\mathrm{r}}(G, A)$. For example, if $A = \mathbb{C}$ with the trivial action of G, then $C^*_{\mathrm{r}}(G, \mathbb{C}) = \mathbb{C} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = C^*_{\mathrm{r}}(G)^c$, the C^*-commutant of the quantum group $C^*_{\mathrm{r}}(G)$. To compare our definition of $C^*_{\mathrm{r}}(G, A)$ with the usual one, we define a unitary $U \in \mathcal{L}\big(L^2(G, A)\big)$ by the formula $U\xi|_t := \delta_G(t)^{-\frac{1}{2}}\xi(t^{-1})$. It is easy to see that

$$(U \circ \tilde{\rho}_K \circ U^*)\xi|_t = \int_G \alpha_{t^{-1}}(K(s))\xi(s^{-1}t)\,\mathrm{d}s = \int_G \alpha_{t^{-1}}(K(ts^{-1}))\delta_G(s)^{-1}\xi(s)\,\mathrm{d}s. \quad (*)$$

Therefore $C^*_{\mathrm{r}}(G, A)$ is isomorphic to the C^*-subalgebra of $\mathcal{L}\big(L^2(G, A)\big)$ generated by the operators of the form $(*)$. This is the representation of $C^*_{\mathrm{r}}(G, A)$ used most frequently (see, for example, [58, 7.7.1]). Note that if $A = \mathbb{C}$ (with the trivial action of G), then $\mathrm{Ad}_U \circ \tilde{\rho} = \lambda$ and therefore the C^*-subalgebra of $\mathcal{L}\big(L^2(G)\big)$ generated by the operators $(*)$ is exactly $C^*_{\mathrm{r}}(G)$.

(2) If A is a $\widehat{G}$-C^*-algebra, that is, a C^*-algebra with a continuous coaction γ_A of $\mathcal{G} = C^*_{\mathrm{r}}(G)$ (see Example 2.6.3(3)), then the crossed product will also be denoted by $A \rtimes_{\mathrm{r}} G$. In this case, we have $\widehat{\mathcal{G}}^{\mathrm{c}} = \widehat{\mathcal{G}} = M\big(\mathcal{C}_0(G)\big)$, where $M : \mathcal{C}_0(G) \to \mathcal{L}\big(L^2(G)\big)$ denotes the multiplication representation. Thus the crossed product is given by

$$A \rtimes_{\mathrm{r}} G = \overline{\operatorname{span}}\Big(\gamma_A(A)\big(1 \otimes M\big(\mathcal{C}_0(G)\big)\big)\Big) \subseteq \mathcal{L}\big(A \otimes L^2(G)\big).$$

The dual coaction of $\mathcal{C}_0(G)$ corresponds to the (continuous) action β of G on $A \rtimes_{\mathrm{r}} G$ given by

$$\beta_t\big(\gamma_A(a)(1 \otimes M_f)\big) = \gamma_A(a)(1 \otimes M_{f_t}),$$

where $f_t(s) := f(st)$ for all $s, t \in G$.

(3) The assignment $A \mapsto A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is functorial. Given a (possibly degenerate) $\mathcal{G}$-equivariant $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ (see Remark 2.6.5), there is a unique $*$-homomorphism $\pi \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} : A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \to \mathcal{M}(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})$ satisfying

$$(\pi \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big(\gamma_A(a)(1 \otimes x)\big) = \gamma_B\big(\pi(a)\big)(1 \otimes x) \quad \text{for all } a \in A,\ x \in \widehat{\mathcal{G}}^{\mathrm{c}}.$$

Consider the $*$-homomorphism $\pi \otimes \mathrm{id}_{\mathcal{K}} : \tilde{\mathcal{M}}(A \otimes \mathcal{K}) \to \mathcal{M}(B \otimes \mathcal{K})$, where $\mathcal{K} = \mathcal{K}(H)$. Since $\tilde{\mathcal{M}}(A \otimes \mathcal{G}) \subseteq \tilde{\mathcal{M}}(A \otimes \mathcal{K})$, it follows that $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \subseteq \tilde{\mathcal{M}}(A \otimes \mathcal{K})$. The map $\pi \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ coincides with the restriction of $\pi \otimes \mathrm{id}_{\mathcal{K}}$ to the reduced crossed products, that is, we have

$$(\pi \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(c) = (\pi \otimes \mathrm{id}_{\mathcal{K}})(c) \quad \text{for all } c \in A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}. \tag{2.22}$$

If π is nondegenerate, then so is $\pi \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.

2.7.2 Full crossed products

Let (A, γ_A) be a (continuous) coaction of a locally compact quantum group $\mathcal{G}$. A *covariant homomorphism* from (A, γ_A) to a C^*-algebra B is a pair (π, u) consisting of a $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ and a unitary (right) corepresentation $u \in \mathcal{M}(B \otimes \mathcal{G})$ such that

$$(\pi \otimes \mathrm{id}_{\mathcal{G}})\gamma_A(a) = u(\pi(a) \otimes 1_{\mathcal{G}})u^* \quad \text{for all } a \in A,$$

that is, $(\pi \otimes \mathrm{id}_{\mathcal{G}}) \circ \gamma_A = \mathrm{Ad}_u \circ (\pi \otimes 1_{\mathcal{G}})$. Note that we allow π to be degenerate. The map $\pi \otimes \mathrm{id}_{\mathcal{G}}$ is thus defined from $\tilde{\mathcal{M}}(A \otimes \mathcal{G})$ to $\mathcal{M}(B \otimes \mathcal{G})$. Since $\gamma_A(A)$ is contained in $\tilde{\mathcal{M}}(A \otimes \mathcal{G})$, the covariance condition makes sense. We say that (π, u) is nondegenerate if π is nondegenerate. A *full crossed product* of (A, γ_A) is a triple $(C, j_A, \mathcal{U}_A)$ where C is a C^*-algebra and $(j_A, \mathcal{U}_A)$ is a nondegenerate covariant homomorphism from (A, γ_A) to C such that for any nondegenerate covariant homomorphism (π, u) from (A, γ_A) to B, there is a unique nondegenerate $*$-homomorphism $\pi \rtimes u : C \to \mathcal{M}(B)$ satisfying $(\pi \rtimes u) \circ j_A = \pi$ and $\big((\pi \rtimes u) \otimes \mathrm{id}_{\mathcal{G}}\big)(\mathcal{U}_A) = u$. A full crossed product exists and is unique up to isomorphism ([26, Theorem 1.10]), and it will be denoted by $A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$.

As already mentioned, there is a universal corepresentation $\hat{\mathcal{V}} \in \mathcal{M}(\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \otimes \mathcal{G})$ of $\mathcal{G}$ such that the formula $(\mu \otimes \mathrm{id}_{\mathcal{G}})(\hat{\mathcal{V}}) = u$ gives a bijective correspondence between (right) corepresentations $u \in \mathcal{M}(B \otimes \mathcal{G})$ of $\mathcal{G}$ and nondegenerate $*$-homomorphisms $\mu : \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to \mathcal{M}(B)$ (see comments after Definition 2.5.6).

Using the relationship above, we can now describe a covariant homomorphism from (A, γ_A) to B as a pair (π, μ) where π is as before and μ is now a nondegenerate $*$-homomorphism $\mu : \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to \mathcal{M}(B)$ satisfying

$$(\pi \otimes \mathrm{id}_{\mathcal{G}})\gamma_A(a) = (\mu \otimes \mathrm{id}_{\mathcal{G}})(\hat{\mathcal{V}})(\pi(a) \otimes 1_{\mathcal{G}})(\mu \otimes \mathrm{id}_{\mathcal{G}})(\hat{\mathcal{V}})^* \quad \text{for all } a \in A.$$

It follows that the full crossed product of (A, γ_A) can be described alternatively by a triple $(C, j_A, j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}})$, where C is a C^*-algebra and $(j_A, j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}})$ is a nondegenerate covariant homomorphism from (A, γ_A) to C in the sense just defined above, and it has the universal property that for any other nondegenerate covariant homomorphism (π, μ) from (A, γ_A) to B there is a unique nondegenerate $*$-homomorphism $\pi \rtimes \mu : C \to \mathcal{M}(B)$ satisfying $\pi \rtimes \mu \circ j_A = \pi$ and $\pi \rtimes \mu \circ j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}} = \mu$.

The restriction to nondegenerate covariant homomorphisms is not necessary. It is there only to simplify the definition. If (π, μ) is a (possibly degenerate) covariant homomorphism of (A, γ_A) then there is a unique (possibly degenerate) $*$-homomorphism (also denoted by) $\pi \rtimes \mu$ satisfying $(\pi \rtimes \mu)\big(j_A(a) j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}(x)\big) = \pi(a)\mu(x)$ for all $a \in A$ and $x \in \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$.

As for reduced crossed products, one can also define a dual coaction of $\widehat{\mathcal{G}}^{\mathrm{c}}$ on the full crossed product $A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$. In fact, define $\pi(a) := j_A(a) \otimes 1$ and $\mu(x) := (j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}} \otimes \lambda^{\mathrm{op}})\hat{\Delta}_{\mathrm{u}}^{\mathrm{c}}(x)$. Then the pair (π, μ) is a nondegenerate covariant homomorphism from (A, γ_A) to $A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \otimes \widehat{\mathcal{G}}^{\mathrm{c}}$. Thus there is a unique nondegenerate $*$-homomorphism $\widehat{\gamma}_A^{\mathrm{c,u}} := \pi \rtimes \mu : A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to \mathcal{M}(A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \otimes \widehat{\mathcal{G}}^{\mathrm{c}})$ satisfying $\widehat{\gamma}_A^{\mathrm{c,u}}\big(j_A(a)\big) = j_A(a) \otimes 1$ and $\widehat{\gamma}_A^{\mathrm{c,u}}\big(j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}(x)\big) = (j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}} \otimes \lambda^{\mathrm{op}})\hat{\Delta}_{\mathrm{u}}^{\mathrm{c}}(x)$. One checks that $\widehat{\gamma}_A^{\mathrm{c,u}}$ is, in fact, a continuous coaction of $\widehat{\mathcal{G}}^{\mathrm{c}}$ on $A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$, and it is called the *dual coaction* of $\widehat{\mathcal{G}}^{\mathrm{c}}$ on the full crossed product $A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$.

Given a covariant homomorphism (π, μ) from (A, γ_A) to a C^*-algebra B, one has that

$$C^*(\pi, \mu) := \overline{\mathrm{span}}\big(\pi(A)\mu(\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}})\big) = \overline{\mathrm{span}}\big(\mu(\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}})\pi(A)\big)$$

is a C^*-subalgebra of $\mathcal{M}(B)$. Moreover, for the pair $(j_A, j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}})$ we have $C^*(j_A, j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}) = A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$. The reduced crossed product of (A, γ_A) can also be described in this way. Define $j_A^{\mathrm{r}}(a) := \gamma_A(a) \in \mathcal{M}(A \otimes \mathcal{G}) \subseteq \mathcal{M}\big(A \otimes \mathcal{K}(H)\big)$ and $j_{\widehat{\mathcal{G}}^{\mathrm{c}}}^{\mathrm{r}}(x) := 1_A \otimes x \in \mathcal{M}\big(A \otimes \mathcal{K}(H)\big)$, $x \in \widehat{\mathcal{G}}^{\mathrm{c}}$ and define also $j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}^{\mathrm{r}} := j_{\widehat{\mathcal{G}}^{\mathrm{c}}}^{\mathrm{r}} \circ \lambda^{\mathrm{op}}$. We claim that the pair $(j_A^{\mathrm{r}}, j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}^{\mathrm{r}})$ is a covariant homomorphism of (A, γ_A). In fact, we already know that $(\lambda^{\mathrm{op}} \otimes \mathrm{id}_{\mathcal{G}})(\hat{\mathcal{V}}) = V$. Now using that $\Delta(y) = V(y \otimes 1)V^*$, the desired covariance condition follows. Finally, note that

$$C^*(j_A^{\mathrm{r}}, j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}^{\mathrm{r}}) = \overline{\mathrm{span}}\big(j_A^{\mathrm{r}}(A) j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}^{\mathrm{r}}(\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}})\big) = \overline{\mathrm{span}}\big(\gamma_A(A)(1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big) = A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.$$

Thus $\lambda_A^{\mathrm{op}} := j_A^{\mathrm{r}} \rtimes j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}^{\mathrm{r}} : A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is a surjective $*$-homomorphism. It is easy to see that λ_A^{op} is equivariant with respect to the dual coactions. Note that if $A = \mathbb{C}$ with trivial coaction of $\mathcal{G}$, then $A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \cong \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$ and the dual coaction is identified with the canonical coaction of $\widehat{\mathcal{G}}^{\mathrm{c}}$ on $\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$ given by the map $x \mapsto (\mathrm{id} \otimes \lambda^{\mathrm{op}})\hat{\Delta}_{\mathrm{u}}^{\mathrm{c}}(x)$. In this case we also have $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \widehat{\mathcal{G}}^{\mathrm{c}}$ and λ_A^{op} is identified with λ^{op}.

As for reduced crossed products, the assignment $A \mapsto A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$ is functorial. Given a $\mathcal{G}$-equivariant $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ between $\mathcal{G}$-C^*-algebras A and B, there is a $*$-homomorphism $\pi \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} : A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to \mathcal{M}(B \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}})$ satisfying $(\pi \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}})\big(j_A(a) j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}(x)\big) = j_B\big(\pi(a)\big) j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}(x)$ for all $a \in A$ and $x \in \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$. In fact, it is easy to see that the pair $(j_B \circ \pi, j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}})$ is a covariant homomorphism from (A, γ_A) to $B \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$. Thus, $\pi \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$ is equal to $(j_B \circ \pi) \rtimes j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}$. If π is nondegenerate, then so is $\pi \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$.

If $\mathcal{G} = \mathcal{C}_0(G)$ for some locally compact group G and A is a G-C^*-algebra, then we shall also use the notation $C^*(G, A)$ for the full crossed product $A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$.

Finally, we mention that for $\mathcal{G} = C_{\mathrm{r}}^*(G)$, there is no difference between full and reduced crossed products, that is, $A \rtimes G \cong A \rtimes_{\mathrm{r}} G$, for any $\widehat{G}$-C^*-algebra A, where $A \rtimes G$ denotes the full crossed product. The reason is that $\mathcal{G}$ is amenable as a locally compact quantum group (see [8]) which means in this case that $\widehat{\mathcal{G}}_{\mathrm{u}} \cong \widehat{\mathcal{G}} \cong \mathcal{C}_0(G)$.

2.7.3 Reduction of coactions

Recall that a $\mathcal{G}$-C^*-algebra (A, γ_A) is reduced if γ_A is injective. Also recall that $(j_A^{\mathrm{r}}, j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}^{\mathrm{r}})$ is a covariant homomorphism from (A, γ_A) to $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, whose integrated form is $\lambda_A^{\mathrm{op}} : A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to$

$A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. Remember that $j_A^{\mathrm{r}}(c) = \gamma_A(c)$ for all $c \in A$. Thus $\ker(j_A^{\mathrm{r}}) = \ker(\gamma_A)$. Moreover, since $\lambda_A^{\mathrm{op}} \circ j_A = j_A^{\mathrm{r}}$, we also have $\ker(j_A) \subseteq \ker(j_A^{\mathrm{r}}) = \ker(\gamma_A)$. And because $(j_A, \mathcal{U}_A)$ is a covariant homomorphism of (A, γ_A) we have $(j_A \otimes \mathrm{id}) \circ \gamma_A = \mathrm{Ad}_{\mathcal{U}_A} \circ (j_A \otimes 1)$ which implies that $\ker(\gamma_A) \subseteq \ker(j_A)$. Therefore $\ker(j_A) = \ker(j_A^{\mathrm{r}}) = \ker(\gamma_A)$. In particular, (A, γ_A) is reduced if and only if j_A or j_A^{r} is injective. Thus, if (A, γ_A) is reduced then there exists a covariant homomorphism (π, μ) with π faithful. Since any covariant homomorphism factors through j_A, the converse also holds.

We can always reduce a coaction as follows. Let (A, γ_A) be a $\mathcal{G}$-C^*-algebra and define $I := \ker(\gamma_A)$ and $A_{\mathrm{r}} := A/I$. Let $q : A \to A_{\mathrm{r}}$ be the quotient map. Then there is a unique (continuous) coaction γ_A^{r} of $\mathcal{G}$ on A satisfying $\gamma_A^{\mathrm{r}}\big(q(a)\big) = (q \otimes \mathrm{id}_{\mathcal{G}})\big(\gamma_A(a)\big)$ for all $a \in A$ (see [51, Lemma 2.17]). Moreover, γ_A^{r} is injective. We call $(A_{\mathrm{r}}, \gamma_A^{\mathrm{r}})$ the *canonical reduction* of (A, γ_A).

Let (A, γ_A) be a $\mathcal{G}$-C^*-algebra, and let $(A_{\mathrm{r}}, \gamma_A^{\mathrm{r}})$ be its canonical reduction. Then the full crossed products coincide. More precisely, if $q : A \to A_{\mathrm{r}}$ denotes the quotient map, then the induced $*$-homomorphism $q \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} : A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to A_{\mathrm{r}} \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$ is a $*$-isomorphism (see [51, Proposition 2.18]).

In general, a *reduction* of a $\mathcal{G}$-C^*-algebra (A, γ_A) is a reduced $\mathcal{G}$-C^*-algebra (B, γ_B) together with a *reduction map* which is an equivariant surjection $\vartheta : A \to B$ such that the induced map $\vartheta \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} : A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to B \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$ is an isomorphism. This is very similar to the "normalization" of full coactions (see [26] for details). In fact, the following universal property for reductions is analogous to the universal property for normalizations proved in [26, Lemma 4.2]. Moreover, the same proof also works for reductions. For convenience we provide the proof here.

Lemma 2.7.2. *Let $\vartheta : A \to B$ be a reduction of a continuous coaction (A, γ_A) and suppose that $\kappa : A \to \mathcal{M}(C)$ is a $\mathcal{G}$-equivariant homomorphism where C is a reduced $\mathcal{G}$-C^*-algebra. Then κ factors uniquely through B, that is, there is a unique $\mathcal{G}$-equivariant homomorphism $\varrho : B \to \mathcal{M}(C)$ such that $\kappa = \varrho \circ \vartheta$.*

Proof. Consider the homomorphism $\tilde{\varrho} := (\kappa \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}) \circ (\vartheta \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}})^{-1} \circ j_B : B \to \mathcal{M}(C \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}})$. Note that

$$(\tilde{\varrho} \circ \vartheta)(a) = (\kappa \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}) \circ (\vartheta \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}})^{-1} \circ (\vartheta \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}) \circ j_A(a) = (j_C \circ \kappa)(a).$$

Since j_C is injective this equation implies the existence of a $\mathcal{G}$-equivariant homomorphism $\varrho : B \to \mathcal{M}(C)$ satisfying $\kappa = \varrho \circ \vartheta$. Since ϑ is surjective, ϱ is necessarily unique. $\square$

Remark 2.7.3. **(1)** Let A and C be $\mathcal{G}$-C^*-algebras. Lemma 2.7.2 implies that given reductions $\vartheta : A \to B$ and $\nu : C \to D$, and given a $\mathcal{G}$-equivariant homomorphism $\pi : A \to$

C, there is a unique homomorphism $\rho : B \to D$ completing the following diagram:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "\vartheta"'] & C \arrow[d, "\nu"] \\ B \arrow[r, "\rho"'] & D \end{tikzcd}$$

Note that if π is an isomorphism, then so is ρ, because one can apply the same to π^{-1} to get the inverse of ρ. In particular, a reduction is uniquely determined up to isomorphism compatible with the reduction maps.

(2) Let (A, γ_A) be a $\mathcal{G}$-C^*-algebra. Then $\lambda_A^{\mathrm{op}} : A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is a reduction of the dual coaction $(A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}, \widehat{\gamma}_A^{\mathrm{c,u}})$. In fact, one can follow the same proof of [26, Lemma 4.11]. Let $\vartheta : A \to B$ be a reduction of (A, γ_A). It is easy to see that the following diagram commutes:

$$\begin{tikzcd} A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \arrow[r, "\vartheta \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}"] \arrow[d, "\lambda_A^{\mathrm{op}}"'] & B \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \arrow[d, "\lambda_B^{\mathrm{op}}"] \\ A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \arrow[r, "\vartheta \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}"'] & B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \end{tikzcd}$$

Since $\vartheta : A \to B$ is a reduction of (A, γ_A), the map $\vartheta \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$ is an isomorphism. Combining this with (1) above, we get that $\vartheta \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is an isomorphism as well.

2.7.4 Regularity and semi-regularity of quantum groups

Consider a locally compact quantum group $\mathcal{G}$ and let $\mathcal{G}$ coact on itself by the comultiplication Δ. Then, by definition, the reduced crossed product $\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is given by

$$\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big(\Delta(\mathcal{G})(1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big) \subseteq \mathcal{L}(\mathcal{G} \otimes H) \subseteq \mathcal{L}(H \otimes H).$$

Recall that the comultiplication satisfies $\Delta(x) = W^*(1 \otimes x)W$ for all $x \in \mathcal{G}$, where $W \in \mathcal{M}(\mathcal{G} \otimes \widehat{\mathcal{G}})$ is the left regular corepresentation of $\mathcal{G}$. Thus, for all $\hat{x} \in \widehat{\mathcal{G}}^{\mathrm{c}}$, we have

$$\Delta(x)(1 \otimes \hat{x}) = W^*(1 \otimes x)W(1 \otimes \hat{x}) = W^*(1 \otimes x\hat{x})W.$$

Hence the map

$$\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}}) \ni T \mapsto W^*(1 \otimes T)W \in \mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, \tag{2.23}$$

defines an isomorphism $\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}}) \cong \mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. In particular, $\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})$ is a C^*-subalgebra of $\mathcal{L}(H)$, that is, $\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}}) = \overline{\mathrm{span}}(\widehat{\mathcal{G}}^{\mathrm{c}}\mathcal{G})$.

The following concept was introduced by Baaj and Skandalis in [4, 6].

Definition 2.7.4. A locally compact quantum group $(\mathcal{G}, \Delta)$ is called *semi-regular* if $\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})$ contains $\mathcal{K}(H)$ and it is called *regular* if $\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})$ equals $\mathcal{K}(H)$.

Therefore, under the canonical identification (2.23), $\mathcal{G}$ is semi-regular if and only if $\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ contains $\mathcal{K}(H)$ and it is regular if and only if $\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is equal to $\mathcal{K}(H)$.

Example 2.7.5. Compact and discrete quantum groups are regular. More generally, all the locally compact quantum groups arising from *algebraic quantum groups* in the sense of [40] are regular (see [75, Remark 2.12]).

All the *Kac algebras* are regular quantum groups (see [6, 16]). In particular, commutative and cocommutative quantum groups are regular.

The so-called $E_\mu(2)$ quantum group is an example of a semi-regular quantum group which is not regular (see [4, 32]).

Quite surprisingly, the existence of non-semi-regular locally compact quantum groups was established recently by Baaj, Skandalis and Vaes in [7].

Remark 2.7.6. **(1)** Our definition of (semi-)regularity is not the original definition appearing in [4, 6], but one of its characterizations (see [7, Proposition 2.6]). The definitions in [4, 6] apply not only to locally compact quantum but to any multiplicative unitary: if $V \in \mathcal{L}(K \otimes K)$ is a multiplicative unitary, where K is some Hilbert space, then one defines $\mathcal{C}(V) := \{(\mathrm{id} \otimes \omega)(\Sigma V) : \omega \in \mathcal{L}(K)_*\}$. This space is a subalgebra of $\mathcal{L}(K)$, so that its closure is a Banach algebra. But, in general, it is not a C^*-algebra, that is, it is not invariant under involution. The multiplicative unitary V is called semi-regular if the closure of $\mathcal{C}(V)$ contains the compact operators and it is regular if the equality holds. With this new terminology one can now say that a locally compact quantum group $\mathcal{G}$ is regular if its right (or, equivalently, left) regular corepresentation is regular. Proposition 2.6 in [7] says that the closure of $\mathcal{C}(V)$ is isomorphic to $\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, where now $V \in \mathcal{L}(H \otimes H)$ is the right regular corepresentation of $\mathcal{G}$. In fact, from the proof of this proposition, we have

$$\overline{\mathcal{C}(V)} = UCU^*, \quad \text{where } U := J\hat{J} \quad \text{and } C := \overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}}).$$

(2) Let $\mathcal{G}_{\mathrm{u}}$ be the universal companion of $\mathcal{G}$, and consider the canonical coaction of $\mathcal{G}$ on $\mathcal{G}_{\mathrm{u}}$ given by the map $(\mathrm{id} \otimes \hat{\lambda}) \circ \Delta_{\mathrm{u}} : \mathcal{G}_{\mathrm{u}} \to \mathcal{M}(\mathcal{G}_{\mathrm{u}} \otimes \mathcal{G})$. Then the reduced crossed product is by definition

$$\mathcal{G}_{\mathrm{u}} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big((\mathrm{id} \otimes \hat{\lambda})\Delta_{\mathrm{u}}(\mathcal{G}_{\mathrm{u}})(1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big)$$

Recall that $(\mathrm{id} \otimes \hat{\lambda})\Delta_{\mathrm{u}}(x) = \mathcal{W}^*(1 \otimes \hat{\lambda}(x))\mathcal{W}$ for all $x \in \mathcal{G}_{\mathrm{u}}$, where $\mathcal{W}$ is the left regular corepresentation of $\mathcal{G}_{\mathrm{u}}$. Since $\mathcal{W} \in \mathcal{M}(\mathcal{G}_{\mathrm{u}} \otimes \widehat{\mathcal{G}})$, we have

$$(\mathrm{id} \otimes \hat{\lambda})\big(\Delta_{\mathrm{u}}(x)\big)(1 \otimes y) = \mathcal{W}^*\big(1 \otimes \hat{\lambda}(x)y\big)\mathcal{W}, \quad \text{for all } y \in \widehat{\mathcal{G}}^{\mathrm{c}}.$$

Thus $\mathcal{G}_{\mathrm{u}} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big(\mathcal{W}^*(1 \otimes \mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})\mathcal{W}\big) \cong \overline{\mathrm{span}}\,\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}} \cong \mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, and therefore $\mathcal{G}$ is regular if and only if $\mathcal{G}_{\mathrm{u}} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \cong \mathcal{K}(H)$. The quantum group $\mathcal{G}$ is called *strongly regular* if the full crossed product $\mathcal{G}_{\mathrm{u}} \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$ is isomorphic to $\mathcal{K}(H)$. More precisely, there is a canonical map $\Omega : \mathcal{G}_{\mathrm{u}} \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to \mathcal{L}(H)$ given by $j_{\mathcal{G}_{\mathrm{u}}}(x)j_{\widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}}(y) \mapsto \hat{\lambda}(x)\lambda^{\mathrm{op}}(y)$ for all $x \in \mathcal{G}_{\mathrm{u}}$ and $y \in \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$.

In fact, Ω is the composition of the canonical surjection $\mathcal{G}_{\mathrm{u}} \ltimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to \mathcal{G}_{\mathrm{u}} \ltimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ with the canonical isomorphism $\mathcal{G}_{\mathrm{u}} \ltimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \cong \mathcal{G} \ltimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \cong \overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})$ described above. In particular, the image of Ω is equal to $\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})$. By definition, $\mathcal{G}$ is strongly regular if and only if Ω is an isomorphism onto $\mathcal{K}(H)$ (this is the precise definition of strong regularity). Thus $\mathcal{G}$ is strongly regular if and only if $\mathcal{G}$ is regular and the canonical surjection $\mathcal{G}_{\mathrm{u}} \ltimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to \mathcal{G} \ltimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is injective (and therefore an isomorphism). It is not known whether there exist examples of regular quantum groups which are not strongly regular (see [26, 75] for further discussion).

Proposition 2.7.7. *Let $\mathcal{G}$ be a locally compact quantum group. The following assertions are equivalent*:

(i) *$\mathcal{G}$ is regular, that is, $\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}}) = \mathcal{K}(H)$,*

(ii) *$\mathcal{G}^{\mathrm{op}}$ is regular, that is, $\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}) = \mathcal{K}(H)$,*

(iii) *$\mathcal{G}^{\mathrm{c}}$ is regular, that is, $\overline{\mathrm{span}}(\mathcal{G}^{\mathrm{c}}\widehat{\mathcal{G}}^{\mathrm{c}}) = \mathcal{K}(H)$,*

(iv) *$\widehat{\mathcal{G}}$ is regular, that is, $\overline{\mathrm{span}}(\widehat{\mathcal{G}}\mathcal{G}^{\mathrm{c}}) = \mathcal{K}(H)$.*

The same statements also hold for semi-regularity if one replaces the equalities above by the inclusion $\supseteq$.

Proof. This follows from the relations $J\mathcal{G}J = \mathcal{G}^{\mathrm{c}}$, $J\widehat{\mathcal{G}}J = \widehat{\mathcal{G}}$, $\hat{J}\mathcal{G}\hat{J} = \mathcal{G}$, $\hat{J}\widehat{\mathcal{G}}\hat{J} = \widehat{\mathcal{G}}^{\mathrm{c}}$ and $J\mathcal{K}(H)J = \hat{J}\mathcal{K}(H)\hat{J} = \mathcal{K}(H)$ (this last one follows from the equalities $J|\xi\rangle\langle\eta|J = |J\xi\rangle\langle J\eta|$ and $\hat{J}|\xi\rangle\langle\eta|\hat{J} = |\hat{J}\xi\rangle\langle\hat{J}\eta|$ for all $\xi, \eta \in H$). For example, if $\mathcal{G}$ is regular then $\mathcal{K}(H) = \hat{J}\mathcal{K}(H)\hat{J} = \hat{J}\,\overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})\hat{J} = \overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}})$, that is, $\mathcal{G}^{\mathrm{op}}$ is regular. ☐

Lemma 2.7.8. *Let $\mathcal{G}$ be a locally compact quantum group. Then $C := \overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})$ is an irreducible C^*-subalgebra of $\mathcal{L}(H)$.*

Proof. Note that the commutant of C is $C' = \mathcal{G}' \cap (\widehat{\mathcal{G}}^{\mathrm{c}})' = M' \cap \hat{M}$, where $M = \mathcal{G}''$ and $\hat{M} = \widehat{\mathcal{G}}''$ are the von Neumann algebraic quantum groups of $\mathcal{G}$ and $\widehat{\mathcal{G}}$, respectively. Take any $x \in M' \cap \hat{M}$. Since $\hat{W} \in \hat{M} \otimes M$ (here $\otimes$ denotes the von Neumann algebraic tensor product), we have

$$\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W} = 1 \otimes x.$$

It follows from [73, Proposition 1.5.5] that $x \in \mathbb{C} \cdot 1$. Therefore $C' = \mathbb{C} \cdot 1$, that is, C acts irreducibly on H. ☐

Note that irreducibility of a C^*-subalgebra $C \subseteq \mathcal{L}(H)$ is equivalent to $C'' = \mathcal{L}(H)$. The reader should compare the following result with [7, Proposition 5.6].

Proposition 2.7.9. *Let $\mathcal{G}$ be a locally compact quantum group, and define $C = \overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})$. Then $\mathcal{G}$ is semi-regular if and only if $C \cap \mathcal{K}(H) \neq \{0\}$.*

Proof. The non-trivial direction follows from the fact that any irreducible C^*-sub-algebra of $\mathcal{L}(H)$ whose intersection with $\mathcal{K}(H)$ is non-trivial must contain $\mathcal{K}(H)$ (see [50, Theorem 2.4.9] or [24, Lemma 3.11.2]). ☐

Thus, under the isomorphism (2.23), we can say that either $\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ contains all the compact operators $\mathcal{K}(H)$ (in that case $\mathcal{G}$ is semi-regular), or it contains only the zero compact operator (in that case $\mathcal{G}$ is not semi-regular).

In what follows we give some more characterizations of (semi-)regularity in terms of continuity of coactions. The results are very similar to those appearing in [7].

Remark 2.7.10. **(1)** Let $\mathcal{G}$ be a locally compact quantum group and define C to be the closed linear span of $\widehat{\mathcal{G}}\mathcal{G} \subseteq \mathcal{L}(H)$. We already know that C is a C^*-algebra which is isomorphic to the reduced crossed product $B := \widehat{\mathcal{G}}^{\mathrm{op}} \rtimes_{\mathrm{r}} \mathcal{G} = \overline{\mathrm{span}}\big(\hat{\Delta}^{\mathrm{op}}(\widehat{\mathcal{G}}^{\mathrm{op}})(1 \otimes \mathcal{G})\big) \subseteq \mathcal{L}(H \otimes H)$, where we let $\widehat{\mathcal{G}}^{\mathrm{op}}$ coact on itself by the comultiplication $\hat{\Delta}^{\mathrm{op}}$. The isomorphism $\pi : C \to B$ is given by $\pi(xy) = \hat{\Delta}^{\mathrm{op}}(x)(1 \otimes y) = \tilde{W}^*(1 \otimes xy)\tilde{W}$ for all $x \in \widehat{\mathcal{G}}^{\mathrm{op}}$ and $y \in \mathcal{G}$, where $\tilde{W} := \hat{W}^{\mathrm{op}}$ is the left regular corepresentation of $\widehat{\mathcal{G}}^{\mathrm{op}}$. This is analogous to the isomorphism (2.23). Let us consider on B the dual coaction of $\mathcal{G}$, which is given by $\gamma_B\big(\hat{\Delta}^{\mathrm{op}}(x)(1 \otimes y)\big) = (\hat{\Delta}^{\mathrm{op}}(x) \otimes 1)(1 \otimes \Delta(y)) = V_{23}\big(\hat{\Delta}^{\mathrm{op}}(x)(1 \otimes y) \otimes 1\big)V_{23}^*$ for all $x \in \widehat{\mathcal{G}}^{\mathrm{op}}$ and $y \in \mathcal{G}$, where V is the right regular corepresentation of $\mathcal{G}$. It is easy to see that, under the isomorphism π, the dual coaction γ_B on B corresponds to the coaction $\tilde{\gamma}_C$ on C defined by $\tilde{\gamma}_C(xy) = (x \otimes 1)\Delta(y) = V(xy \otimes 1)V^*$ for all $x \in \widehat{\mathcal{G}}^{\mathrm{op}}$ and $y \in \mathcal{G}$. In particular, the coaction $\tilde{\gamma}_C$ is continuous, that is, C is a $\mathcal{G}$-C^*-algebra and π is an isomorphism of $\mathcal{G}$-C^*-algebras.

(2) Example 2.6.18 yields two canonical equivalent coactions γ_H and $\tilde{\gamma}_H$ of $\mathcal{G}$ on the Hilbert space $H = L^2(\mathcal{G})$. They are given by $\gamma_H(\xi) = \hat{W}^*(\xi \otimes 1)$ and $\tilde{\gamma}_H(\xi) = V(\xi \otimes 1)$. The corresponding coactions on $\mathcal{K} = \mathcal{K}(H)$ are given by $\gamma_{\mathcal{K}}(x) = \hat{W}^*(x \otimes 1)\hat{W}$ and $\tilde{\gamma}_{\mathcal{K}}(x) = V(x \otimes 1)V^*$, respectively. A natural question is: when are these coactions continuous? Note that $\mathcal{G}$ is regular if and only if $C = \mathcal{K}$, and in this case the coaction $\tilde{\gamma}_C$ defined in (1) is the coaction $\tilde{\gamma}_{\mathcal{K}}$. In particular, the coaction $\tilde{\gamma}_{\mathcal{K}}$ (and so also $\gamma_{\mathcal{K}}$) is continuous for regular quantum groups. The following result says that this is the only case where this happens.

Proposition 2.7.11. *Let $\mathcal{G}$ be a locally compact quantum group. Then the following statements are equivalent:*

(i) *$\mathcal{G}$ is regular,*

(ii) *the coaction γ_H (or, equivalently, $\tilde{\gamma}_H$) of $\mathcal{G}$ on H is continuous,*

(iii) *the corepresentation $\hat{W}^*$ (or, equivalently, V) of $\mathcal{G}$ on H is continuous,*

(iv) *the coaction $\gamma_{\mathcal{K}}$ (or, equivalently, $\tilde{\gamma}_{\mathcal{K}}$) of $\mathcal{G}$ on $\mathcal{K} := \mathcal{K}(H)$ is continuous,*

(v) $\overline{\mathrm{span}}\big((1 \otimes \mathcal{G})V(\mathcal{K} \otimes 1)\big) = \mathcal{K} \otimes \mathcal{G}$,

(vi) $\overline{\mathrm{span}}\big((1 \otimes \mathcal{K})V(\mathcal{K} \otimes 1)\big) = \mathcal{K} \otimes \mathcal{K}$,

(vii) $\overline{\mathrm{span}}\big((\mathcal{K} \otimes 1)V(1 \otimes \mathcal{K})\big) = \mathcal{K} \otimes \mathcal{K}$.

Proof. By Example 2.6.18, the coactions γ_H and $\tilde{\gamma}_H$ are equivalent. Thus continuity of γ_H is equivalent to that of $\tilde{\gamma}_H$. The same holds for the corepresentations $\hat{W}^*$ and V and also for the coactions $\gamma_{\mathcal{K}}$ and $\tilde{\gamma}_{\mathcal{K}}$ on $\mathcal{K}$. The coaction $\tilde{\gamma}_H$ is given by $\tilde{\gamma}_H(\xi) = V(\xi \otimes 1)$ for all $\xi \in H$. Thus continuity of $\tilde{\gamma}_H$ is equivalent to the condition $\overline{\mathrm{span}}\big((1\otimes\mathcal{G})V(H\otimes 1)\big) = H\otimes\mathcal{G}$ or, equivalently, $\overline{\mathrm{span}}\big((1\otimes\mathcal{G})V(\mathcal{K}\otimes 1)\big) = \mathcal{K}\otimes\mathcal{G}$. This is, by definition, equivalent to the continuity of V (see Remark 2.6.17(2)), and it is also equivalent to the continuity of $\tilde{\gamma}_{\mathcal{K}}$, that is, $\overline{\mathrm{span}}\big((1\otimes\mathcal{G})V(\mathcal{K}\otimes 1)V^*\big) = \mathcal{K}\otimes\mathcal{G}$. We conclude that (ii), (iii) and (iv) are equivalent to (v): $\overline{\mathrm{span}}\big((1\otimes\mathcal{G})V(\mathcal{K}\otimes 1)\big) = \mathcal{K}\otimes\mathcal{G}$. Multiplying this equation from the left by $1\otimes\mathcal{K}$ and using that $\mathcal{K}\cdot\mathcal{G} = \mathcal{K}$ we see that (v) also implies (vi). The equivalence between (vi) and (vii) follows by taking adjoints and using the relation $V = (J\otimes\hat{J})V^*(J\otimes\hat{J})$ (this relation can be derived from Equations (2.15) and (2.16) and the equality $\hat{J}J = \nu^{\frac{i}{4}}J\hat{J}$; see [73, Corollary 1.13.15]). And the equivalence between (vii) and (i) is the content of [6, Proposition 3.2(ii)]. Therefore (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (i). Finally, if (i) is true, then so is (iii) by Remark 2.7.10(2). $\square$

Corollary 2.7.12. *Let $\mathcal{G}$ be a regular locally compact quantum group and suppose that $\gamma_{\mathcal{E}}$ is a continuous coaction of $\mathcal{G}$ on a Hilbert B-module $\mathcal{E}$. Then the coactions $\gamma_{\mathcal{E}\otimes H}$ and $\tilde{\gamma}_{\mathcal{E}\otimes H}$ on $\mathcal{E}\otimes H$ defined in Example* 2.6.18 *are continuous.*

Proof. This follows from Example 2.6.18(2) and Proposition 2.7.11. $\square$

Proposition 2.7.11 says that continuity of coactions is somewhat complicated for non-regular quantum groups. This has been observed already in [7]. Another natural condition on a coaction is weak continuity. However, we are going to see that even this weaker condition turns out to be too strong for the coaction on $\mathcal{K}$. First we need a preliminary result which is very similar to [7, Proposition 5.6].

Proposition 2.7.13. *Let $\mathcal{G}$ be a locally compact quantum group and define*

$$D := \overline{\mathrm{span}}\Big(\big(\mathrm{id}\otimes\mathcal{L}(H)_*\big)\big(V(\mathcal{K}\otimes 1)V^*\big)\Big) \subseteq \mathcal{L}(H),$$

where V is the right regular corepresentation of $\mathcal{G}$ and $\mathcal{K} := \mathcal{K}(H)$. Then $\mathcal{G}$ is semi-regular if and only if $D\cap\mathcal{K} \neq \{0\}$. In this case $\mathcal{K}\subseteq D$ and $\mathcal{G}$ is regular if and only if the equality holds. Moreover, we have $D = \overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}})$.

Proof. Define $\tilde{V} := \Sigma V^*\Sigma$, where Σ denotes the flip operator (we remark that $\tilde{V} = W^{\mathrm{op}}$ is the left regular corepresentation of the opposite quantum group $\mathcal{G}^{\mathrm{op}}$; see [73, Proposition 1.14.10]). Note that $\mathcal{C}(\tilde{V}) = \mathcal{C}(V)^*$ (see Remark 2.7.6(1) for the definition of $\mathcal{C}(V)$). Using the relation $V = (J\otimes\hat{J})V^*(J\otimes\hat{J})$ (see the proof of Proposition 2.7.11), we get

$$\begin{aligned}
D &= \overline{\mathrm{span}}\Big(\big(\mathrm{id}\otimes\mathcal{L}(H)_*\big)\big(\Sigma\tilde{V}^*(1\otimes\mathcal{K})\tilde{V}\Sigma\big)\Big)\\
&= \overline{\mathrm{span}}\Big(\big(\mathrm{id}\otimes\mathcal{L}(H)_*\big)\big((J\otimes\hat{J})\Sigma\tilde{V}(1\otimes\mathcal{K})\tilde{V}^*\Sigma(J\otimes\hat{J})\big)\Big)\\
&= J\,\overline{\mathrm{span}}\Big(\big(\mathrm{id}\otimes\mathcal{L}(H)_*\big)\big(\Sigma\tilde{V}(1\otimes\mathcal{K})\tilde{V}^*\Sigma\big)\Big)J\\
&= J\,\overline{\mathrm{span}}\big(\mathcal{C}(\tilde{V})\mathcal{C}(\tilde{V})^*\big)J = J\overline{\mathcal{C}(V)}J.
\end{aligned}$$

From Remark 2.7.6(1) we have $J\overline{\mathcal{C}(V)}J = \hat{J}C\hat{J} = \overline{\operatorname{span}}(\mathcal{G}\widehat{\mathcal{G}})$, where $C := \overline{\operatorname{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})$ (here we are using the equalities $\hat{J}\mathcal{G}\hat{J} = \mathcal{G}$ and $\hat{J}\widehat{\mathcal{G}}^{\mathrm{c}}\hat{J} = \widehat{\mathcal{G}}$). The result now follows from Proposition 2.7.9. $\square$

As a consequence, we get the following result.

Proposition 2.7.14. *Let $\mathcal{G}$ be a locally compact quantum group and consider on H the coaction $\tilde{\gamma}_H$ (or, equivalently, γ_H) of $\mathcal{G}$ and on $\mathcal{K} := \mathcal{K}(H)$ the coaction $\tilde{\gamma}_{\mathcal{K}}$ (or, equivalently, $\gamma_{\mathcal{K}}$) defined in Example* 2.6.18(3). *Then the following statements are equivalent:*

(i) *$\mathcal{G}$ is regular,*

(ii) *$\tilde{\gamma}_H$ is admissible, that is, $\gamma_H(H) \subseteq \tilde{\mathcal{M}}(H \otimes \mathcal{G})$,*

(iii) *$\tilde{\gamma}_{\mathcal{K}}$ is admissible, that is, $\tilde{\gamma}_{\mathcal{K}}(\mathcal{K}) \subseteq \tilde{\mathcal{M}}(\mathcal{K} \otimes \mathcal{G})$*

(iv) *$\tilde{\gamma}_{\mathcal{K}}$ is weakly admissible, that is, $\omega * \xi \in \mathcal{K}$ for all $\omega \in L^1(\mathcal{G})$ and $\xi \in \mathcal{K}$,*

(v) *$\tilde{\gamma}_{\mathcal{K}}$ is weakly continuous, that is, $\overline{\operatorname{span}}\{\omega * \xi : \omega \in L^1(\mathcal{G}), \xi \in \mathcal{K}\} = \mathcal{K}$.*

Proof. If $\mathcal{G}$ is regular, then we already know from Proposition 2.7.11 that γ_H is continuous, and in particular it is admissible. Thus (i) $\Rightarrow$ (ii). It is also clear that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). We recall here that $\omega * \xi = (\mathrm{id} \otimes \omega)\tilde{\gamma}_{\mathcal{K}}(\xi)$ and $\tilde{\gamma}_{\mathcal{K}}(\xi) = V(\xi \otimes 1)V^*$ for all $\omega \in L^1(\mathcal{G})$ and $\xi \in \mathcal{K}$. Since $L^1(\mathcal{G}) = \{\omega|_{\mathcal{G}} : \omega \in \mathcal{L}(H)_*\}$, the closed linear span of $\{\omega * \xi : \omega \in L^1(\mathcal{G}), \xi \in \mathcal{K}\}$ coincides with the space D defined in Proposition 2.7.13. Thus the condition (iv) means that $D \subseteq \mathcal{K}$ which by Proposition 2.7.13 is equivalent to $D = \mathcal{K}$, that is, to the weak continuity of $\tilde{\gamma}_{\mathcal{K}}$. Again by Proposition 2.7.13 the equality $D = \mathcal{K}$ is also equivalent to the regularity of $\mathcal{G}$. $\square$

We conclude from the result above that, unless $\mathcal{G}$ is regular, the coaction $\tilde{\gamma}_{\mathcal{K}}$ (or, equivalently, $\gamma_{\mathcal{K}}$) of $\mathcal{G}$ on $\mathcal{K}$ is badly behaved with respect to any kind of continuity.

Remark 2.7.15. (1) Note that we can also characterize the semi-regularity of $\mathcal{G}$ in terms of the coaction $\tilde{\gamma}_{\mathcal{K}}$ (or, equivalently, $\gamma_{\mathcal{K}}$). In fact, we can rephrase Proposition 2.7.13 by saying that $\mathcal{G}$ is semi-regular if and only if $\omega * \xi \in \mathcal{K}$, for some $\omega \in L^1(\mathcal{G})$ and $\xi \in \mathcal{K}$ with $\omega * \xi \neq 0$.

(2) It might seem that the conditions in Proposition 2.7.14 are also equivalent to the weak continuity of $\tilde{\gamma}_H$ (or, equivalently, of γ_H). But this is not true because $\tilde{\gamma}_H$ (and so also γ_H) is always weakly continuous, for any locally compact quantum group. In fact, note that this follows from Proposition 2.6.10. This can also be proved directly by using that the closure of $\{(\mathrm{id} \otimes \omega)(V) : \omega \in \mathcal{L}(H)_*\} \subseteq \mathcal{L}(H)$ is $\widehat{\mathcal{G}}^{\mathrm{c}}$. In fact, since $\widehat{\mathcal{G}}^{\mathrm{c}}$ is a nondegenerate C^*-subalgebra of $\mathcal{L}(H)$, we get

$$\overline{\operatorname{span}}\big(L^1(\mathcal{G}) * H\big) = \overline{\operatorname{span}}\big\{(\mathrm{id} \otimes \omega)(V)\xi : \xi \in H, \omega \in \mathcal{L}(H)_*\big\} = \overline{\operatorname{span}}(\widehat{\mathcal{G}}^{\mathrm{c}} H) = H.$$

In particular, we see that, in general, weak continuity of a coaction of $\mathcal{G}$ on a Hilbert module $\mathcal{E}$ does not imply weak continuity of the corresponding coaction on the compact operators $\mathcal{K}(\mathcal{E})$. However, if $\mathcal{G}$ is regular, then this is true, because in this case continuity and weak continuity are equivalent concepts ([7, Propostion 5.8]).

(3) Let $\mathcal{G}$ be a locally compact quantum group and define $C := \overline{\mathrm{span}}(\widehat{\mathcal{G}}\mathcal{G}) \cong \widehat{\mathcal{G}}^{\mathrm{op}} \rtimes_{\mathrm{r}} \mathcal{G}$ with the dual coaction $\tilde{\gamma}_C(c) = V(c \otimes 1)V^*$ as in Remark 2.7.10(1). Suppose that $\mathcal{G}$ is semi-regular, that is, $C \subseteq \mathcal{K}$. In this case we can consider the C^*-algebra $A \subseteq \mathcal{L}(H \oplus \mathbb{C})$ defined by

$$A := \begin{pmatrix} C & H \\ H^* & \mathbb{C} \end{pmatrix}.$$

Note that A is the linking algebra of the Hilbert $C, \mathbb{C}$-bimodule ${}_C H_{\mathbb{C}}$ (with left C-inner product ${}_C\langle \xi \,|\, \eta \rangle := |\xi\rangle\langle\eta|$). We can also define a coaction on A which combines the coaction $\tilde{\gamma}_C$ on C and the coaction $\tilde{\gamma}_H$ on H:

$$\tilde{\gamma}_A \begin{pmatrix} c & \xi \\ \eta^* & z \end{pmatrix} := \begin{pmatrix} \tilde{\gamma}_C(c) & \tilde{\gamma}_H(\xi) \\ \tilde{\gamma}_H(\eta)^* & z \otimes 1 \end{pmatrix} = \begin{pmatrix} V(c \otimes 1)V^* & V(\xi \otimes 1) \\ (\eta^* \otimes 1)V^* & z \otimes 1 \end{pmatrix}.$$

Since $\tilde{\gamma}_C$ is continuous and $\tilde{\gamma}_H$ is weakly continuous, it follows that $\tilde{\gamma}_A$ is weakly continuous. But if $\mathcal{G}$ is not regular, then we already know that $\tilde{\gamma}_H$ is not admissible, which implies that $\tilde{\gamma}_A$ is also not admissible (and in particular not continuous). Thus, if $\mathcal{G}$ is semi-regular, but not regular, then $\tilde{\gamma}_A$ is an example of a weakly continuous coaction which is not admissible. This example is an adaptation of the example appearing in the proof of Proposition 5.8 in [7] (where left coactions are used instead; see Remark 2.6.2(3)).

2.7.5 Crossed product duality

Let A be a C^*-algebra with a continuous coaction γ_A of $\mathcal{G}$. Then we can form the crossed product $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, which carries the (continuous) dual coaction $\widehat{\gamma}_A^{\mathrm{c}}$ of $\widehat{\mathcal{G}}^{\mathrm{c}}$. Repeating this process we get the double crossed product, which we denote by $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}^{\mathrm{op,c}}$. The reason for this notation is that the C^*-commutant of the dual of $\widehat{\mathcal{G}}^{\mathrm{c}}$ is equal to $\mathcal{G}^{\mathrm{op,c}}$. Note that $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}^{\mathrm{op,c}}$ has a bidual coaction of $\mathcal{G}^{\mathrm{op,c}}$, which we denote by $\hat{\hat{\gamma}}_A^{\mathrm{op,c}}$. As already mentioned, the unitary $U := \hat{J}J$ implements an isomorphism $\mathrm{Ad}_U : \mathcal{G}^{\mathrm{op,c}} \to \mathcal{G}$ of locally compact quantum groups (see discussion preceding Example 2.5.2). Thus $\hat{\hat{\gamma}}_A := \mathrm{Ad}_{(1\otimes U)} \circ \hat{\hat{\gamma}}_A^{\mathrm{op,c}}$ is a coaction of $\mathcal{G}$ on $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}^{\mathrm{op,c}}$. We shall write $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}$ for $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}^{\mathrm{op,c}}$ equipped with the coaction $\hat{\hat{\gamma}}_A$. In this way, we have started with a coaction (A, γ_A) of $\mathcal{G}$ and obtained a new coaction $(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}, \hat{\hat{\gamma}}_A)$ of $\mathcal{G}$. So it is natural to ask what is the relationship between these two coactions. The following result, due to Baaj and Skandalis in [6], gives the answer. For convenience we provide the proof.

Proposition 2.7.16. *Let $\mathcal{G}$ be a regular locally compact quantum group $\mathcal{G}$. Let (A, γ_A) be a reduced $\mathcal{G}$-C^*-algebra. Then the bidual coaction $(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}, \hat{\hat{\gamma}}_A, \mathcal{G})$ is isomorphic to $(A \otimes \mathcal{K}, \gamma_{A\otimes\mathcal{K}}, \mathcal{G})$, where $\mathcal{K} := \mathcal{K}(H)$ and $\gamma_{A\otimes\mathcal{K}}$ is the coaction of $\mathcal{G}$ on $A \otimes \mathcal{K}$ defined as in Example* 2.6.18 *by*

$$\gamma_{A\otimes\mathcal{K}}(T) := \hat{W}_{23}^* \Sigma_{23} (\gamma_A \otimes \mathrm{id})(T) \Sigma_{23}^* \hat{W}_{23},$$

where $\hat{W}$ is the left regular corepresentation of the dual $\widehat{\mathcal{G}}$, and $\Sigma : \mathcal{G} \otimes H \to H \otimes \mathcal{G}$ is the flip operator.

Proof. Note that by definition we have

$$A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}^{\mathrm{op,c}} = \overline{\mathrm{span}}\big((\gamma_A(A)\otimes 1)(1\otimes \hat{\Delta}^{\mathrm{c}}(\widehat{\mathcal{G}}^{\mathrm{c}}))(1\otimes 1\otimes \mathcal{G}^{\mathrm{op,c}})\big) \subseteq \mathcal{L}(A\otimes H\otimes H).$$

Since $\Delta(x) = V(x\otimes 1)V^* = \mathrm{Ad}_V(x\otimes 1)$ for all $x\in\mathcal{G}$, we have

$$\mathrm{Ad}_{(1\otimes V)}(\gamma_A(a)\otimes 1) = \gamma_A^{(2)}(a) := (\mathrm{id}\otimes\Delta)\gamma_A(a) = (\gamma_A\otimes\mathrm{id})\gamma_A(a)$$

for all $a\in A$. The relations $\hat{W} = (\hat{J}\otimes\hat{J})V(\hat{J}\otimes\hat{J})$ and $\hat{\Delta}(y) = \hat{W}^*(1\otimes y)\hat{W}$, $y\in\hat{\mathcal{G}}$, imply that $\hat{\Delta}^{\mathrm{c}}(\hat{x}) = V^*(1\otimes\hat{x})V$, $\hat{x}\in\widehat{\mathcal{G}}^{\mathrm{c}}$, and hence

$$\mathrm{Ad}_{(1\otimes V)}\big(1\otimes\hat{\Delta}^{\mathrm{c}}(\hat{x})\big) = 1\otimes 1\otimes\hat{x}, \quad \text{for all } \hat{x}\in\widehat{\mathcal{G}}^{\mathrm{c}}.$$

And because $V\in\mathcal{M}(\widehat{\mathcal{G}}^{\mathrm{c}}\otimes\mathcal{G})$ we have for all $x\in\mathcal{G}^{\mathrm{op,c}}\subseteq\mathcal{G}'$ that

$$\mathrm{Ad}_{(1\otimes V)}(1\otimes 1\otimes x) = (1\otimes 1\otimes x).$$

Thus $\mathrm{Ad}_{(1\otimes V)}$ defines an isomorphism

$$\begin{aligned} A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}^{\mathrm{op,c}} &\cong \overline{\mathrm{span}}\big((\gamma_A^{(2)}(A)(1\otimes 1\otimes\widehat{\mathcal{G}}^{\mathrm{c}})(1\otimes 1\otimes\mathcal{G}^{\mathrm{op,c}})\big)\\ &= \overline{\mathrm{span}}\big((\gamma_A\otimes\mathrm{id})(\gamma_A(A)(1\otimes\widehat{\mathcal{G}}^{\mathrm{c}}\mathcal{G}^{\mathrm{op,c}})\big). \end{aligned}$$

Since γ_A is injective the map $(\gamma_A\otimes\mathrm{id})\big(\gamma_A(a)(1\otimes x\hat{x})\big)\mapsto\gamma_A(a)(1\otimes x\hat{x})$ defines an isomorphism

$$\overline{\mathrm{span}}\Big((\gamma_A\otimes\mathrm{id})\big(\gamma_A(A)(1\otimes\widehat{\mathcal{G}}^{\mathrm{c}}\mathcal{G}^{\mathrm{op,c}}\big)\Big) \cong \overline{\mathrm{span}}\big((\gamma_A(A)(1\otimes\widehat{\mathcal{G}}^{\mathrm{c}}\mathcal{G}^{\mathrm{op,c}})\big).$$

Let us denote the isomorphism above by ρ. Since $\mathcal{G}$ is regular we have (by Proposition 2.7.7) $\overline{\mathrm{span}}(\widehat{\mathcal{G}}^{\mathrm{c}}\mathcal{G}^{\mathrm{op,c}}) = \overline{\mathrm{span}}(\widehat{\mathcal{G}}^{\mathrm{c}}\mathcal{G}^{\mathrm{c}}) = \mathcal{K}$ and therefore, using that γ_A is continuous, we finally get an isomorphism

$$\begin{aligned} A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G} = A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}^{\mathrm{op,c}} &\cong \overline{\mathrm{span}}\big((\gamma_A(A)(1\otimes\widehat{\mathcal{G}}^{\mathrm{c}}\mathcal{G}^{\mathrm{op,c}})\big)\\ &= \overline{\mathrm{span}}\big((\gamma_A(A)(1\otimes\mathcal{K})\big) = \overline{\mathrm{span}}\big((\gamma_A(A)(1\otimes\mathcal{G}\mathcal{K})\big)\\ &= \overline{\mathrm{span}}\big((A\otimes\mathcal{G})(1\otimes\mathcal{K})\big) = A\otimes\mathcal{K}. \end{aligned}$$

This isomorphism, which we denote by $\pi = \rho\circ\mathrm{Ad}_{(1\otimes V)}$, is given by the formula

$$\pi\big((\gamma_A(a)\otimes 1)(1\otimes\hat{\Delta}^{\mathrm{c}}(\hat{x}))(1\otimes 1\otimes x)\big) = \gamma_A(a)(1\otimes\hat{x}x), \quad a\in A,\ \hat{x}\in\widehat{\mathcal{G}}^{\mathrm{c}},\ x\in\mathcal{G}^{\mathrm{op,c}}.$$

We prove now that π is equivariant with respect to the coactions $\hat{\hat{\gamma}}_A$ and $\gamma_{A\otimes\mathcal{K}}$, that is, we prove that $(\pi\otimes\mathrm{id})\circ\hat{\hat{\gamma}}_A = \gamma_{A\otimes\mathcal{K}}\circ\pi$. For all $a\in A$, $\hat{x}\in\widehat{\mathcal{G}}^{\mathrm{c}}$ and $x\in\mathcal{G}^{\mathrm{op,c}}$ we have

$$\begin{aligned} &(\pi\otimes\mathrm{id})\hat{\hat{\gamma}}_A\big((\gamma_A(a)\otimes 1)(1\otimes\hat{\Delta}^{\mathrm{c}}(\hat{x}))(1\otimes 1\otimes x)\big)\\ &\quad = (\pi\otimes\mathrm{id})\Big(\big(\gamma_A(a)\otimes 1\otimes 1\big)\big(1\otimes\hat{\Delta}^{\mathrm{c}}(\hat{x})\otimes 1\big)\big(1\otimes 1\otimes\mathrm{Ad}_{(1\otimes U)}(\Delta^{\mathrm{op,c}}(x))\big)\Big)\\ &\quad = \big(\gamma_A(a)\otimes 1\big)(1\otimes\hat{x}\otimes 1)(\pi\otimes\mathrm{id})\big(1\otimes 1\otimes\mathrm{Ad}_{(1\otimes U)}(\Delta^{\mathrm{op,c}}(x))\big), \end{aligned}$$

and

$$\gamma_{A\otimes\mathcal{K}}\Big(\pi\big((\gamma_A(a)\otimes 1)(1\otimes\hat{\Delta}^{\mathrm{c}}(\hat{x}))(1\otimes 1\otimes x)\big)\Big)$$
$$= \gamma_{A\otimes\mathcal{K}}(\gamma_A(a)(1\otimes\hat{x}x)) = \gamma_{A\otimes\mathcal{K}}(\gamma_A(a))\gamma_{A\otimes\mathcal{K}}(1\otimes\hat{x})\gamma_{A\otimes\mathcal{K}}(1\otimes x).$$

Thus all we have to prove are the following equalities

$$\gamma_{A\otimes\mathcal{K}}\big(\gamma_A(a)\big) = \gamma_A(a)\otimes 1, \tag{2.24}$$

$$\gamma_{A\otimes\mathcal{K}}(1\otimes\hat{x}) = 1\otimes\hat{x}\otimes 1 \tag{2.25}$$

and

$$\gamma_{A\otimes\mathcal{K}}(1\otimes x) = (\pi\otimes\mathrm{id})\big(1\otimes 1\otimes\mathrm{Ad}_{(1\otimes U)}(\Delta^{\mathrm{op,c}}(x))\big).$$

For the first equality we use $\Delta(y) = W^*(1\otimes y)W$ and $\hat{W}^*\Sigma = \Sigma W$ and calculate

$$\begin{aligned}\gamma_{A\otimes\mathcal{K}}(\gamma_A(a)) &= \hat{W}_{23}^*\Sigma_{23}\gamma_A^{(2)}(a)\Sigma_{23}^*\hat{W}_{23}\\ &= \Sigma_{23}W_{23}(\mathrm{id}\otimes\Delta)(\gamma_A(a))W_{23}^*\Sigma_{23}^*\\ &= \Sigma_{23}\gamma_A(a)_{13}\Sigma_{23}^* = \gamma_A(a)\otimes 1.\end{aligned}$$

For the second equality we use that $\hat{W}\in\mathcal{M}(\widehat{\mathcal{G}}\otimes\mathcal{G})$ and $\hat{x}\in\widehat{\mathcal{G}}^{\mathrm{c}}\subseteq\widehat{\mathcal{G}}'$ to get

$$\begin{aligned}\gamma_{A\otimes\mathcal{K}}(1\otimes\hat{x}) &= \hat{W}_{23}^*\Sigma_{23}(1\otimes 1\otimes\hat{x})\Sigma_{23}^*\hat{W}_{23}\\ &= \hat{W}_{23}^*(1\otimes\hat{x}\otimes 1)\hat{W}_{23} = 1\otimes\hat{x}\otimes 1.\end{aligned}$$

Finally, for the last equality, note that because Ad_U is an isomorphism between $\mathcal{G}^{\mathrm{op,c}}$ and $\mathcal{G}$ we have $\mathrm{Ad}_{(U\otimes U)}\big(\Delta^{\mathrm{op,c}}(x)\big) = \Delta\big(\mathrm{Ad}_U(x)\big)$. Using this and the relations $\Delta(y) = V(y\otimes 1)V^*$ and $V(U\otimes 1) = (U\otimes 1)\hat{W}^*$ (this last one follows from Equations (2.15) and (2.16)), we get that $\mathrm{Ad}_{(1\otimes U)}(\Delta^{\mathrm{op,c}}(x)) = \hat{W}^*(x\otimes 1)\hat{W}\in\mathrm{Ad}_{(1\otimes U)}\mathcal{M}(\mathcal{G}^{\mathrm{op,c}}\otimes\mathcal{G}^{\mathrm{op,c}}) = \mathcal{M}(\mathcal{G}^{\mathrm{c}}\otimes\mathcal{G})$. Thus

$$\mathrm{Ad}_{(V\otimes 1)}\big(1\otimes\mathrm{Ad}_{(1\otimes U)}\Delta^{\mathrm{op,c}}(x)\big) = 1\otimes\hat{W}^*(x\otimes 1)\hat{W},$$

because $V\in\mathcal{M}(\widehat{\mathcal{G}}^{\mathrm{c}}\otimes\mathcal{G})$. We conclude that

$$\begin{aligned}&(\pi\otimes\mathrm{id})\big(1\otimes 1\otimes\mathrm{Ad}_{(1\otimes U)}(\Delta^{\mathrm{op,c}}(x))\big)\\ &\qquad = (\rho\otimes\mathrm{id})(\mathrm{Ad}_{(1\otimes V\otimes 1)})\big(1\otimes 1\otimes\mathrm{Ad}_{(1\otimes U)}(\Delta^{\mathrm{op,c}}(x))\big)\\ &\qquad = (\rho\otimes\mathrm{id})(1\otimes 1\otimes\hat{W}^*(x\otimes 1)\hat{W})\\ &\qquad = 1\otimes\hat{W}^*(x\otimes 1)\hat{W} = \gamma_{A\otimes\mathcal{K}}(1\otimes x).\end{aligned}$$ □

Given a C^*-algebra A with a coaction γ_A of $\mathcal{G}$ we can define a coaction of $\mathcal{G}$ on the Hilbert A-module $A\otimes H$ by the following formula (see Example 2.6.18(3)):

$$\gamma_{A\otimes H}: A\otimes H\to\mathcal{M}(A\otimes H), \quad \gamma_{A\otimes H}(\zeta) := \hat{W}_{23}^*\Sigma_{23}(\gamma_A\otimes\mathrm{id})(\zeta), \tag{2.26}$$

for all $\zeta\in A\otimes H$, where $\Sigma:\mathcal{G}\otimes H\to H\otimes\mathcal{G}$ is the flip operator. Note that the corresponding coaction of $\mathcal{G}$ on $\mathcal{K}(A\otimes H)\cong A\otimes\mathcal{K}$ coincides with the coaction defined in Proposition 2.7.16. Therefore we immediately get the following result.

Corollary 2.7.17. *Let* (A, γ_A) *be a reduced* $\mathcal{G}$-C^**-algebra, where* $\mathcal{G}$ *is a regular locally compact quantum group. Then* $(A \otimes H, \gamma_{A\otimes H})$ *is a* $\mathcal{G}$*-equivariant Morita equivalence between* $(A \otimes \mathcal{K}, \gamma_{A\otimes\mathcal{K}}) \cong (A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}, \hat{\hat{\gamma}}_A)$ *and* (A, γ_A).

Note that Equations (2.24) and (2.25) imply that elements of $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \subseteq \mathcal{M}(A \otimes \mathcal{K})$ are fixed by the coaction $\gamma_{A\otimes\mathcal{K}}$, or what is equivalent (see Proposition 2.6.13), that all the operators in $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \subseteq \mathcal{L}(A \otimes H)$ are $\mathcal{G}$-equivariant with respect to $\gamma_{A\otimes H}$, that is,

$$A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \subseteq \mathcal{L}^{\mathcal{G}}(A \otimes H). \tag{2.27}$$

It is important to note here that we do not need regularity of the quantum group $\mathcal{G}$ or injectivity of the coaction of A. In other words, the relation above holds for any locally compact quantum group $\mathcal{G}$ and any $\mathcal{G}$-C^*-algebra A.

Remark 2.7.18. Let $\mathcal{G}$ be a regular locally compact quantum group. If (A, γ_A) is a $\mathcal{G}$-C^*-algebra, then there is a canonical surjective $*$-homomorphism Ω_A from the double full crossed product $A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \rtimes \mathcal{G}_{\mathrm{u}}$ onto $A \otimes \mathcal{K}(H)$.

The coaction (A, γ_A) is called *maximal* if the canonical surjection Ω_A is an isomorphism. Thus maximal coactions are exactly those where full crossed product duality holds. A *maximalization* of (A, γ_A) is a maximal coaction (A^m, γ_A^m) together with a $\mathcal{G}$-equivariant surjection $\nu : A^m \to A$ such that $\nu \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} : A^m \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}} \to A \rtimes \widehat{\mathcal{G}}_{\mathrm{u}}^{\mathrm{c}}$ is an isomorphism. If $\mathcal{G}$ is strongly regular, then every continuous coaction of $\mathcal{G}$ admits a maximalization which is unique up to isomorphism compatible with the maximalization maps. Conversely, if the trivial coaction admits a maximalization, then $\mathcal{G}$ is strongly regular. See [26] for further details.

Chapter 3

Integrable coactions on C^*-algebras

3.1 Motivation: the group case

Recall that an action of a locally compact group G on a locally compact (Hausdorff) space X is called proper if the map $G \times X \to X \times X$, $(t, p) \mapsto (t \cdot p, p)$ is proper in the sense that inverse images of compact subsets are compact. This is equivalent to say that for all (relatively) compact subsets $K, L \subseteq X$

$$[[K, L]] := \{t \in G : t \cdot K \cap L \neq \emptyset\}$$

is a (relatively) compact subset of G. Proper actions have many nice properties. One of the most important ones is that the quotient space $G\backslash X$ is again a locally compact Hausdorff space.

Given a locally compact G-space X, we can associate to it a commutative G-C^*-algebra $A := \mathcal{C}_0(X)$, where the action α is defined by $\alpha_t(f)(p) := f(t^{-1} \cdot p)$. One interesting question is: Can we characterize the properness of the action on X by the action α on A?

Suppose that the action on X is proper. Then for any $f \in \mathcal{C}_c(X)$, the function $E_1(f) : X \to \mathbb{C}$ defined by

$$E_1(f)(p) := \int_G \alpha_t(f)(p)\,\mathrm{d}t = \int_G f(t^{-1} \cdot p)\,\mathrm{d}t$$

is continuous and bounded on X. That is, it defines an element $E_1(f) \in \mathcal{C}_b(X) = \mathcal{M}(A)$, the multiplier algebra of $A = \mathcal{C}_0(X)$. It is natural to denote the element $E_1(f)$ by

$$\int \alpha_t(f)\,\mathrm{d}t.$$

But it should be noted that, unless G is compact or $f = 0$, the integral above does not converge in Bochner's sense because $\|\alpha_t(f)\| = \|f\|$ for all $t \in G$. But one can prove that the integral above converges in the sense that for any $\theta \in A^*$, the continuous complex valued function $t \mapsto \theta\big(\alpha_t(f)\big)$ is integrable (in the ordinary sense) and the integral is equal to $\theta\big(E_1(f)\big)$. Thus, if X is a proper G-space, then the algebra A contains a dense subspace of integrable elements in the sense above. In fact, it was proved by Rieffel in [66] that this property characterizes the proper G-spaces.

To be more precise, given any C^*-algebra A (not necessarily commutative) with a (strongly) continuous action α of G, let us say that an element $a \in A^+$ is *integrable* if there is an element $b \in \mathcal{M}(A)$ such that for all $\theta \in A^*$, the function $t \mapsto \theta(\alpha_t(a))$ is integrable (in the ordinary sense) and

$$\int_G \theta(\alpha_t(a))\,\mathrm{d}t = \theta(b).$$

We denote by A_{i}^+ the set of positive integrable elements, and say that A is *integrable* if A_{i}^+ is dense in A^+ (also called *proper* in [66]).

A commutative G-C^*-algebra $A = \mathcal{C}_0(X)$ is integrable if and only if X is a proper G-space (see [66, Theorem 4.7]). In this sense integrable G-C^*-algebras are generalizations of proper G-spaces. As already mentioned above, such algebras were also called proper by Rieffel in [66]. But, as we will discuss later in more details, integrability is in general not enough to construct a "generalized fixed-point algebra" which corresponds to the algebra $\mathcal{C}_0(G\backslash X)$ in the commutative case. In any way, as we are going to see, integrable algebras share many properties with proper actions on spaces.

The construction above for a locally compact group G leads to the following question: if $\mathcal{G}$ is a locally compact quantum group, is it possible to extend the notion of integrability for coactions of $\mathcal{G}$? The answer is yes, and this is what we are going to see in the next section. For coactions of von Neumann algebraic quantum groups on von Neumann algebras this was defined by Vaes in [74]. There are crucial differences between von Neumann and C^*-algebraic settings and we want to emphasize here the C^*-algebraic case.

3.2 Definition of integrable coactions

Let $(\mathcal{G}, \Delta)$ be a locally compact quantum group and let φ be the left Haar weight of $(\mathcal{G}, \Delta)$. We also fix a GNS-construction for φ of the form (H, ι, Λ), where we assume $\mathcal{G} \subseteq \mathcal{L}(H)$ and write ι for the inclusion map $\mathcal{G} \hookrightarrow \mathcal{L}(H)$.

In what follows we are going to use notations and definitions from Section 2.4.1.

Definition 3.2.1. Let A be a C^*-algebra and let γ_A be a coaction of $\mathcal{G}$ on A.

(i) We say that an element $a \in A^+$ is *integrable* if $\gamma_A(a) \in \bar{\mathcal{M}}^+_{\mathrm{id}_A \otimes \varphi}$. We denote by A_{i}^+ the set of integrable elements of A^+. We say that A (or the coaction γ_A) is *integrable* if A_{i}^+ is dense in A^+.

(ii) We say that an element $a \in A$ is *square-integrable* if aa^* is integrable. We denote by A_{si} the set of square-integrable elements of A. We say that A (or the coaction γ_A) is *square-integrable* if A_{si} is dense in A.

Note that by definition $a \in A_{\mathrm{si}}$ if and only if $\gamma_A(a)^* \in \bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi}$. We have chosen this convention (and not $\gamma_A(a) \in \bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi}$) because this fits better into the setting of square-integrable coactions on Hilbert modules as we will see later.

We always have the following structure:

Proposition 3.2.2. *Let γ_A be a coaction of $\mathcal{G}$ on A and write $A_{\mathrm{i}} := \operatorname{span} A_{\mathrm{i}}^+$. Then $A_{\mathrm{i}}^+ = A^+ \cap A_{\mathrm{i}}$ is a hereditary cone, A_{i} is a hereditary $*$-subalgebra of A, A_{si} is a right ideal in A and*

$$A_{\mathrm{i}} = A_{\mathrm{si}}A_{\mathrm{si}}^* := \operatorname{span}\{xy^* : x, y \in A_{\mathrm{si}}\}.$$

Proof. Since $\bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$ is a hereditary cone of $\mathcal{M}(A\otimes\mathcal{G})^+$ and γ_A is a $*$-homomorphism, it follows that A_{i}^+ is a hereditary cone of A^+. The assertions now follow from [69, Lemma VII.1.2]. □

Proposition 3.2.3. *Let γ_A be a coaction of $\mathcal{G}$ on A. The following statements are equivalent:*

(i) *A_{i} is dense in A.*

(ii) *γ_A is square-integrable.*

(iii) *γ_A is integrable.*

(iv) *$A_{\mathrm{si}}^+ := A^+ \cap A_{\mathrm{si}}$ is dense in A^+.*

Proof. Since A_{si} is a right ideal, we have $A_{\mathrm{i}} = A_{\mathrm{si}}A_{\mathrm{si}}^* \subseteq A_{\mathrm{si}}$, so that (i) implies (ii). Suppose (ii) and let $a \in A^+$. There is a sequence (b_n) in A_{si} such that $b_n \to a^{\frac{1}{2}}$. Then $a_n := b_n b_n^* \in A_{\mathrm{i}}^+$ and $a_n \to a$. So (ii) implies (iii). Suppose (iii). Let again $a \in A^+$. Then there is $b_n \in A_{\mathrm{i}}^+$ such that $b_n \to a^2$ and hence $a_n = b_n^{\frac{1}{2}} \in A_{\mathrm{si}}^+$ and $a_n \to a$. Therefore (iii) implies (iv). Finally, if (iv) is true, then so is (ii). But we have just proved that (ii) implies (iii). And obviously (iii) implies (i). Therefore (iv) implies (i). □

Example 3.2.4. A trivial example is given when the quantum group $\mathcal{G}$ is compact, that is, if the Haar weight φ is bounded. In this case we have $\bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi} = \mathcal{M}(A\otimes\mathcal{G})$ for every C^*-algebra A. Hence every element and hence every coaction of $\mathcal{G}$ is integrable.

The following result shows that if we restrict to unital C^*-algebras, then the only possible way to obtain integrable coactions is to have compact quantum groups.

Proposition 3.2.5. *Let A be a $\mathcal{G}$-C^*-algebra, and suppose that A has a unit $1_A \neq 0$. Then A is integrable if and only if $\mathcal{G}$ is compact.*

Proof. Suppose that A is integrable. Then by definition A_{si} is a dense right ideal of A. Since A is unital, it follows that $A_{\mathrm{si}} = A$. In particular, $1_A \in A_{\mathrm{si}}$, that is, $1_A \otimes 1_{\mathcal{G}} = \gamma_A(1_A) \in \bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$. Thus, if θ is a state of A we get that $1_{\mathcal{G}} = (\theta\otimes\mathrm{id}_{\mathcal{G}})(1_A\otimes 1_{\mathcal{G}})$ belongs to $\bar{\mathcal{N}}_\varphi$. Therefore φ is bounded, that is, $\mathcal{G}$ is compact. □

Thinking of integrable coactions as generalizations to quantum groups of proper actions of groups on topological spaces, the proposition above is the quantum version of the fact that only compact groups can act properly on compact spaces.

Notation 3.2.6. For a C^*-algebra A with a coaction γ_A of $\mathcal{G}$, we introduce the following sets

$$\begin{aligned}\mathcal{M}(A)_{\mathrm{i}}^{+} &:= \{a \in \mathcal{M}(A)^{+} : \gamma_A(a) \in \bar{\mathcal{M}}^{+}_{\mathrm{id}_A\otimes\varphi}\},\\ \mathcal{M}(A)_{\mathrm{si}} &:= \{a \in \mathcal{M}(A) : aa^* \in \mathcal{M}(A)_{\mathrm{i}}^{+}\},\\ \mathcal{M}(A)_{\mathrm{i}} &:= \operatorname{span}\mathcal{M}(A)_{\mathrm{i}}.\end{aligned}$$

As in Proposition 3.2.2, one proves that $\mathcal{M}(A)_{\mathrm{i}}^{+}$ is a hereditary cone of $\mathcal{M}(A)^{+}$, $\mathcal{M}(A)_{\mathrm{si}}$ is a right ideal of $\mathcal{M}(A)$ and $\mathcal{M}(A)_{\mathrm{i}}$ is a hereditary $*$-subalgebra of $\mathcal{M}(A)$ with

$$\mathcal{M}(A)_{\mathrm{i}} \cap \mathcal{M}(A)^{+} = \mathcal{M}(A)_{\mathrm{i}}^{+} \quad \text{and} \quad \mathcal{M}(A)_{\mathrm{i}} = \operatorname{span}\mathcal{M}(A)_{\mathrm{si}}\mathcal{M}(A)_{\mathrm{si}}^{*}.$$

Note also that $A_{\mathrm{i}}^{+} = \mathcal{M}(A)_{\mathrm{i}}^{+} \cap A$ and $A_{\mathrm{si}} = \mathcal{M}(A)_{\mathrm{si}} \cap A$.

Proposition 3.2.7. *Let $\mathcal{G}$ be a locally compact quantum group, let A be a C^*-algebra with a coaction of $\mathcal{G}$ and suppose that $a \in \mathcal{M}(A)^{+}$. Then $a \in \mathcal{M}(A)_{i}^{+}$ if and only if there is $b \in \mathcal{M}(A)^{+}$ such that the following property holds:*

$$\text{for all } \theta \in A_{+}^{*} \text{ we have } (\theta\otimes\mathrm{id}_{\mathcal{G}})\big(\gamma_A(a)\big) \in \bar{\mathcal{M}}_{\varphi}^{+} \text{ and } \varphi\big((\theta\otimes\mathrm{id}_{\mathcal{G}})\gamma_A(a)\big)= \theta(b).$$

In this case we have $(\mathrm{id}_A\otimes\varphi)\big(\gamma_A(a)\big) = b$.

Proof. This follows directly from Proposition 2.4.5. □

We can also characterize (square-)integrability using (square-)integrable elements in $\mathcal{M}(A)$ in the following way.

Proposition 3.2.8. *Let γ_A be a coaction of $\mathcal{G}$ on A. The following assertions are equivalent.*

(i) $\mathcal{M}(A)_{\mathrm{i}}^{+}$ *is strictly dense in* $\mathcal{M}(A)^{+}$*;*

(ii) $\mathcal{M}(A)_{\mathrm{i}}$ *is strictly dense in* $\mathcal{M}(A)$*;*

(iii) $\mathcal{M}(A)_{\mathrm{si}}$ *is strictly dense in* $\mathcal{M}(A)$*;*

(iv) γ_A *is square-integrable.*

Proof. It is clear that (i) implies (ii). Since $\mathcal{M}(A)_{\mathrm{i}} = \operatorname{span}\mathcal{M}(A)_{\mathrm{si}}\mathcal{M}(A)_{\mathrm{si}}^{*} \subseteq \mathcal{M}(A)_{\mathrm{si}}$ it is also clear that (ii) implies (iii). Suppose (iii) is true. Thus, if $a \in A$, there is a net $\{a_i\}$ in $\mathcal{M}(A)_{\mathrm{si}}$ such that $a_i \to a$ strictly and hence $a_ib \to ab$ for all $b \in A$. Now note that $\mathcal{M}(A)_{\mathrm{si}}A \subseteq \mathcal{M}(A)_{\mathrm{si}} \cap A = A_{\mathrm{si}}$. Therefore (iii) implies (iv). Finally, suppose that (iv) is true. Then by 3.2.3, A_{i}^{+} is norm dense and hence also strictly dense in A^{+}. Thus the strict closure of $\mathcal{M}(A)_{\mathrm{i}}^{+}$ contains the strict closure of A^{+}. So it suffices to show that any $a \in \mathcal{M}(A)^{+}$ is a strict limit of a net in A^{+}. But taking an approximate unit of A one can find a bounded net $\{b_i\}$ in A such that $b_i \to a^{\frac{1}{2}}$ strictly, and therefore also $b_i^*b_i \to a$ strictly. □

Definition 3.2.9. Let (A, γ_A) be a coaction of $\mathcal{G}$. For each $a \in \mathcal{M}(A)_{\mathrm{i}}$, we define the element

$$E_1(a) := (\mathrm{id}_A \otimes \varphi)\big(\gamma_A(a)\big) \in \mathcal{M}(A),$$

and for $\xi \in \mathcal{M}(A)_{\mathrm{si}}$ we define the operators (see Proposition 2.4.6)

$$\langle\!\langle \xi| := (\mathrm{id}_A \otimes \Lambda)\big(\gamma_A(\xi)^*\big) \in \mathcal{L}(A, A \otimes H) = \mathcal{M}(A \otimes H)$$

and

$$|\xi\rangle\!\rangle := \langle\!\langle \xi|^* = (\mathrm{id}_A \otimes \Lambda)\big(\gamma_A(\xi)^*\big)^* \in \mathcal{L}(A \otimes H, A) = \mathcal{M}(A \otimes H)^*.$$

Proposition 3.2.10. *Let $(A, \gamma_A, \mathcal{G})$ be a coaction, and let $\xi, \eta \in \mathcal{M}(A)_{\mathrm{si}}$. Then*

$$|\xi\rangle\!\rangle\big(a \otimes \Lambda(x)\big) = (\mathrm{id}_A \otimes \varphi)\big(\gamma_A(\xi)(a \otimes x)\big), \quad \textit{for all } a \in A,\ x \in \bar{\mathcal{N}}_\varphi,$$

and

$$|\xi\rangle\!\rangle\langle\!\langle \eta| = E_1(\xi\eta^*).$$

Proof. This follows directly from the definitions and Proposition 2.4.6. □

We are now going to prove that $\mathcal{G}$ itself is an integrable $\mathcal{G}$-C^*-algebra. First we need a preparation.

Lemma 3.2.11. *Let $\mathcal{G}$ be a locally compact quantum group.*

(i) *If $x \in \bar{\mathcal{M}}_\varphi$ then $\Delta(x) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{G}} \otimes \varphi}$ and $(\mathrm{id}_{\mathcal{G}} \otimes \varphi)\Delta(x) = \varphi(x)1_{\mathcal{G}}$.*

(ii) *If $x \in \mathcal{M}(\mathcal{G})^+$ is such that $(\omega_{v,v} \otimes \mathrm{id})\Delta(x) \in \bar{\mathcal{M}}_\varphi^+$ for all $v \in H$, then $x \in \bar{\mathcal{M}}_\varphi^+$.*

Proof. The first statement is a consequence of the left invariance of φ and is exactly [41, Result 2.4]. And the second statement follows from [41, Proposition 5.15]. □

From the definition of the left regular corepresentation W of $\mathcal{G}$ (see Equation (2.14)), it follows that (see [41, 2.10])

$$(\omega \otimes \mathrm{id})(W^*)\Lambda(b) = \Lambda\big((\omega \otimes \mathrm{id})\Delta(b)\big) \quad \text{for all } b \in \bar{\mathcal{N}}_\varphi,\ \omega \in \mathcal{L}(H)_*. \tag{3.1}$$

This is equivalent to the following equation

$$W^*\big(1 \otimes \Lambda(b)\big) = (\mathrm{id} \otimes \Lambda)\big(\Delta(b)\big) \quad \text{for all } b \in \bar{\mathcal{N}}_\varphi. \tag{3.2}$$

Note that, by Lemma 3.2.11(i) above, we have $\Delta(b) \in \bar{\mathcal{N}}_{\mathrm{id} \otimes \varphi}$ for all $b \in \bar{\mathcal{N}}_\varphi$, so that both equations above make sense. Remember that W belongs to $\mathcal{M}(\mathcal{G} \otimes \widehat{\mathcal{G}}) \subseteq \mathcal{L}(\mathcal{G} \otimes H)$.

Recall that for $v \in H$, v^* denotes the element in $H^* = \mathcal{K}(H, \mathbb{C})$ given by $v^*(\zeta) = \langle v|\zeta\rangle$ for all $\zeta \in H$. For $u, v \in H$, $|u\rangle\langle v| \in \mathcal{K}(H)$ denotes the operator $|u\rangle\langle v|w = u\langle v|w\rangle$ for all $w \in H$.

Proposition 3.2.12. *Let $(\mathcal{G},\Delta)$ be a locally compact quantum group. Let $\mathcal{G}$ coact on itself by the comultiplication Δ. Then $\mathcal{G}$ is integrable. Moreover, we have*

$$\mathcal{G}_{\mathrm{i}}^+ = \mathcal{M}_\varphi^+, \quad \mathcal{G}_{\mathrm{i}} = \mathcal{M}_\varphi, \quad \mathcal{G}_{\mathrm{si}} = \mathcal{N}_\varphi^*,$$

$$\mathcal{M}(\mathcal{G})_{\mathrm{i}}^+ = \bar{\mathcal{M}}_\varphi^+, \quad \mathcal{M}(\mathcal{G})_{\mathrm{i}} = \bar{\mathcal{M}}_\varphi, \quad \mathcal{M}(\mathcal{G})_{\mathrm{si}} = \bar{\mathcal{N}}_\varphi^*,$$

and

$$E_1(b) = \varphi(b)1_{\mathcal{G}} \quad \text{for all } b \in \bar{\mathcal{M}}_\varphi^+.$$

Let $W \in \mathcal{M}(\mathcal{G}\otimes\widehat{\mathcal{G}}) \subseteq \mathcal{L}(\mathcal{G}\otimes H)$ be the left regular corepresentation of $\mathcal{G}$. Then

$$\langle\!\langle\xi| = W^*\big(1\otimes\Lambda(\xi^*)\big), \quad |\xi\rangle\!\rangle = \big(1\otimes\Lambda(\xi^*)^*\big)W \quad \text{and}$$

$$\langle\!\langle\xi\,|\,\eta\rangle\!\rangle = W^*\big(1\otimes|\Lambda(\xi^*)\rangle\langle\Lambda(\eta^*)|\big)W, \quad \text{for all } \xi,\eta\in\bar{\mathcal{N}}_\varphi^*.$$

Proof. Let $b \in \bar{\mathcal{M}}_\varphi^+$. Lemma 3.2.11(i) says that $\Delta(b) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{G}}\otimes\varphi}^+$, that is, $b \in \mathcal{M}(\mathcal{G})_{\mathrm{i}}^+$ and

$$E_1(b) = (\mathrm{id}\otimes\varphi)\Delta(b) = \varphi(b)1_{\mathcal{G}}.$$

Since φ is densely defined we conclude that $\mathcal{G}$ is integrable. Suppose now that $b \in \mathcal{M}(\mathcal{G})_{\mathrm{i}}^+$, that is, $\Delta(b) \in \bar{\mathcal{M}}_{\mathrm{id}\otimes\varphi}^+$. Then for all $\theta \in \mathcal{G}_+^*$ we have $(\theta\otimes\mathrm{id})\Delta(b) \in \bar{\mathcal{M}}_\varphi^+$. In particular, $(\omega_{v,v}\otimes\mathrm{id})\Delta(b) \in \bar{\mathcal{M}}_\varphi^+$ for all $v \in H$. Lemma 3.2.11(ii) implies that $b \in \bar{\mathcal{M}}_\varphi^+$. Therefore $\mathcal{M}(\mathcal{G})_{\mathrm{i}}^+ = \bar{\mathcal{M}}_\varphi^+$. From this the other equalities $\mathcal{G}_{\mathrm{i}}^+ = \mathcal{M}_\varphi^+$, $\mathcal{G}_{\mathrm{i}} = \mathcal{M}_\varphi$, $\mathcal{G}_{\mathrm{si}} = \mathcal{N}_\varphi^*$, $\mathcal{M}(\mathcal{G})_{\mathrm{i}} = \bar{\mathcal{M}}_\varphi$ and $\mathcal{M}(\mathcal{G})_{\mathrm{si}} = \bar{\mathcal{N}}_\varphi^*$ follow. Now, for $\xi \in \bar{\mathcal{N}}_\varphi^*$, Equation (3.2) yields

$$\langle\!\langle\xi| = (\mathrm{id}_{\mathcal{G}}\otimes\Lambda)\big(\Delta(\xi^*)\big) = W^*\big(1\otimes\Lambda(\xi^*)\big).$$

The other equalities follow. $\square$

There is an analogue of Proposition 3.2.12 for the universal locally compact quantum group $\mathcal{G}_{\mathrm{u}}$ of $\mathcal{G}$. To prove it we need first an analogue of Lemma 3.2.11. Recall that φ_{u} denotes the left Haar weight of $\mathcal{G}_{\mathrm{u}}$ and $\hat{\lambda} : \mathcal{G}_{\mathrm{u}} \to \mathcal{G}$ denotes the canonical surjection. Note that $\hat{\lambda}$ induces a canonical inclusion $\mathcal{G}^* \hookrightarrow \mathcal{G}_{\mathrm{u}}^*$, $\omega \mapsto \omega\circ\hat{\lambda}$. This is, in fact, an inclusion of Banach algebras (see comments before Proposition 5.3 in [39]). We shall identify $\mathcal{G}^* \subseteq \mathcal{G}_{\mathrm{u}}^*$ in this way, that is, we identify an element $\omega \in \mathcal{G}^*$ with the element $\omega\circ\hat{\lambda} \in \mathcal{G}_{\mathrm{u}}^*$.

Lemma 3.2.13. *Let $\mathcal{G}_{\mathrm{u}}$ be the universal locally compact quantum group of $\mathcal{G}$.*

(i) *If $x \in \bar{\mathcal{M}}_{\varphi_{\mathrm{u}}}$ then $\Delta_{\mathrm{u}}(x) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{G}_{\mathrm{u}}}\otimes\varphi_{\mathrm{u}}}$ and $(\mathrm{id}_{\mathcal{G}_{\mathrm{u}}}\otimes\varphi_{\mathrm{u}})\Delta_{\mathrm{u}}(x) = \varphi_{\mathrm{u}}(x)1_{\mathcal{G}_{\mathrm{u}}}$.*

(ii) *If $x \in \mathcal{M}(\mathcal{G}_{\mathrm{u}})^+$ is such that $(\omega_{v,v}\otimes\mathrm{id})\Delta_{\mathrm{u}}(x) \in \bar{\mathcal{M}}_{\varphi_{\mathrm{u}}}^+$ for all $v \in H$, then $x \in \bar{\mathcal{M}}_{\varphi_{\mathrm{u}}}^+$.*

Proof. The first statement is [41, Result 2.4] applied to the left invariant weight φ_{u}. The second statement follows from Lemma 3.2.11(ii) and the relations $\varphi_{\mathrm{u}} = \varphi\circ\hat{\lambda}$ and $(\hat{\lambda}\otimes\hat{\lambda})\circ\Delta_{\mathrm{u}} = \Delta\circ\hat{\lambda}$ (see also [39, Result 5.4]). $\square$

Equation (3.2) also has its analogue in the universal setting. More precisely, we have

$$\mathcal{W}^*\big(1\otimes\Lambda_{\mathrm{u}}(b)\big)=(\mathrm{id}\otimes\Lambda_{\mathrm{u}})\big(\Delta_{\mathrm{u}}(b)\big)\quad\text{for all } b\in\bar{\mathcal{N}}_{\varphi_{\mathrm{u}}},\tag{3.3}$$

where $\Lambda_{\mathrm{u}}=\Lambda\circ\hat{\lambda}$ is the canonical GNS-map for φ_{u} and $\mathcal{W}\in\mathcal{M}(\mathcal{G}_{\mathrm{u}}\otimes\widehat{\mathcal{G}})$ is the left regular corepresentation of $\mathcal{G}_{\mathrm{u}}$. In fact, formula (3.3) follows from [39, Corollary 5.8(1)], which is the analogue of formula (3.1) for the universal setting.

Proposition 3.2.14. *Let $\mathcal{G}_{\mathrm{u}}$ be the universal locally compact quantum group of $\mathcal{G}$ and let $\mathcal{G}$ coact on $\mathcal{G}_{\mathrm{u}}$ by the canonical coaction $(\mathrm{id}\otimes\hat{\lambda})\circ\Delta_{\mathrm{u}}:\mathcal{G}_{\mathrm{u}}\mapsto\mathcal{M}(\mathcal{G}_{\mathrm{u}}\otimes\mathcal{G})$. Then $\mathcal{G}_{\mathrm{u}}$ is integrable. Moreover, we have*

$$(\mathcal{G}_{\mathrm{u}})_{\mathrm{i}}^{+}=\mathcal{M}_{\varphi_{\mathrm{u}}}^{+},\quad(\mathcal{G}_{\mathrm{u}})_{\mathrm{i}}=\mathcal{M}_{\varphi_{\mathrm{u}}},\quad(\mathcal{G}_{\mathrm{u}})_{\mathrm{si}}=\mathcal{N}_{\varphi_{\mathrm{u}}}^{*},$$

$$\mathcal{M}(\mathcal{G}_{\mathrm{u}})_{\mathrm{i}}^{+}=\bar{\mathcal{M}}_{\varphi_{\mathrm{u}}}^{+},\quad\mathcal{M}(\mathcal{G}_{\mathrm{u}})_{\mathrm{i}}=\bar{\mathcal{M}}_{\varphi_{\mathrm{u}}},\quad\mathcal{M}(\mathcal{G}_{\mathrm{u}})_{\mathrm{si}}=\bar{\mathcal{N}}_{\varphi_{\mathrm{u}}}^{*},$$

and

$$E_1(b)=\varphi_{\mathrm{u}}(b)1_{\mathcal{G}_{\mathrm{u}}}\quad\textit{for all } b\in\bar{\mathcal{M}}_{\varphi_{\mathrm{u}}}^{+}.$$

Let $\mathcal{W}\in\mathcal{M}(\mathcal{G}_{\mathrm{u}}\otimes\widehat{\mathcal{G}})\subseteq\mathcal{L}(\mathcal{G}_{\mathrm{u}}\otimes H)$ be the left regular corepresentation of $\mathcal{G}_{\mathrm{u}}$. Then

$$\langle\!\langle\xi|=\mathcal{W}^*\big(1\otimes\Lambda_{\mathrm{u}}(\xi^*)\big),\quad|\xi\rangle\!\rangle=\big(1\otimes\Lambda_{\mathrm{u}}(\xi^*)^*\big)\mathcal{W}\quad\textit{and}$$

$$\langle\!\langle\xi\,|\,\eta\rangle\!\rangle=\mathcal{W}^*\big(1\otimes|\Lambda_{\mathrm{u}}(\xi^*)\rangle\langle\Lambda_{\mathrm{u}}(\eta^*)|\big)\mathcal{W},\quad\textit{for all } \xi,\eta\in\bar{\mathcal{N}}_{\varphi_{\mathrm{u}}}^{*}.$$

Proof. This is analogous to the proof of Proposition 3.2.12, using Lemma 3.2.13 and Equation (3.3) instead of Lemma 3.2.11 and Equation (3.2). □

We are going to prove now that in the case of a locally compact group, that is, in the case of a commutative locally compact quantum group, our definition of an integrable coaction coincides with the definition of an integrable action of the underlying group in the sense of Exel [18, 19] or a proper action in the sense of Rieffel [66].

Let G be a locally compact group and consider the associated quantum group $\mathcal{G}=\mathcal{C}_0(G)$ with comultiplication Δ defined by $\Delta(f)(s,t)=f(st)$ for $f\in\mathcal{G}$ and left Haar weight given by

$$\varphi(f):=\int_G f(s)\,\mathrm{d}s\quad\text{for all } f\in\mathcal{C}_0(G)^+,$$

where $\mathrm{d}s$ is a fixed left Haar measure on G.

Proposition 3.2.15. *Let A be a C^*-algebra and consider $\mathcal{G}=\mathcal{C}_0(G)$ as above. Under the identification $\mathcal{M}(A\otimes\mathcal{G})\cong\mathcal{C}_b\big(G,\mathcal{M}^{\mathrm{s}}(A)\big)$, where $\mathcal{M}^{\mathrm{s}}(A)$ is $\mathcal{M}(A)$ considered with the strict topology, we have $x\in\bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}^{+}$ if and only if there is $a\in\mathcal{M}(A)^+$ such that for every $\theta\in A_+^*$ the function $G\ni s\mapsto\theta\big(x(s)\big)\in\mathbb{R}^+$ is integrable in the ordinary sense and*

$$\int_G\theta\big(x(s)\big)\,\mathrm{d}s=\theta(a).$$

In this case we have $(\mathrm{id}_A\otimes\varphi)(x)=a$.

Proof. Although this follows from Proposition 2.4.5, we give here a direct proof. Suppose first that there is $a \in \mathcal{M}(A)^+$ with

$$\int_G \theta\big(x(s)\big)\,\mathrm{d}s = \theta(a) \quad \text{for every } \theta \in A^*_+.$$

Note that for $\omega \in \mathcal{G}^*_+ = \mathcal{M}(G)_+$ (bounded positive measures on G) we have

$$(\mathrm{id}_A \otimes \omega)(x) = \int_G^{\mathrm{s}} x(s)\,\mathrm{d}\omega(s),$$

where the superscript "s" in the integral above stands for strict integral. Now, if $\omega \in \mathcal{F}_\varphi$,[1] then, by definition, $\omega \leq \varphi$ and hence

$$0 \leq \int_G \theta\big(x(s)\big)\,\mathrm{d}\omega(s) \leq \int_G \theta\big(x(s)\big)\,\mathrm{d}s = \theta(a).$$

Since φ is lower semi-continuous (Fatou's Lemma), it follows from Equation (2.4) that

$$\lim_{\omega\in\mathcal{G}_\varphi} \theta\big((\mathrm{id}_A \otimes \omega)(x)\big) = \lim_{\omega\in\mathcal{G}_\varphi} \int_G \theta\big(x(s)\big)\,\mathrm{d}\omega(s) = \int_G \theta\big(x(s)\big)\,\mathrm{d}s = \theta(a).$$

Lemma 2.4.3 implies that the net $\big((\mathrm{id}_A \otimes \omega)(x)\big)_{\omega\in\mathcal{G}_\varphi}$ converges strictly to a, that is, $x \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$ and $(\mathrm{id}_A\otimes\varphi)(x) = a$. Conversely, suppose $x \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$ and let $a := (\mathrm{id}_A\otimes\varphi)(x)$. Take any $\theta \in A^*_+$. We have $(\mathrm{id}_A \otimes \omega)(x) \leq a$, so that

$$\int_G \theta\big(x(s)\big)\,\mathrm{d}\omega(s) = \theta\big((\mathrm{id}_A \otimes \omega)(s)\big) \leq \theta(a).$$

Equation (2.3) yields

$$0 \leq \int_G \theta\big(x(s)\big)\,\mathrm{d}s = \sup_{\omega\in\mathcal{F}_\varphi} \int_G \theta\big(x(s)\big)\,\mathrm{d}\omega(s) \leq \theta(a) < \infty.$$

So $s \mapsto \theta\big(x(s)\big)$ is integrable. Finally, Equation (2.4) gives

$$\int_G \theta\big(x(s)\big)\,\mathrm{d}s = \lim_{\omega\in\mathcal{G}_\varphi} \int_G \theta\big(x(s)\big)\,\mathrm{d}\omega(s) = \lim_{\omega\in\mathcal{G}_\varphi} \theta\big((\mathrm{id} \otimes \omega)(x)\big) = \theta(a). \qquad \square$$

Definition 3.2.16 ([18])**.** Let A be a C^*-algebra. A function $f : G \to A$ is *locally integrable* if it is Bochner integrable over every measurable relatively compact subset $K \subseteq G$, that is, $f|_K \in L^1(K, A)$. We say that f is *unconditionally integrable* if it is locally integrable and the net $\big(\int_K f(t)\,\mathrm{d}t\big)_{K\in\mathcal{C}}$ of Bochner integrals converges in the norm topology of A, where $\mathcal{C}$ is the direct set of all measurable relatively compact subsets of G ordered by inclusion. The limit of this net is called the *unconditional integral* and is denoted by $\int_G^{\mathrm{u}} f(t)\,\mathrm{d}t$.

[1]See Section 2.4 for the definitions of $\mathcal{F}_\varphi$ and $\mathcal{G}_\varphi$ used in this proof.

A function $f : G \to \mathcal{M}(A)$ is *strictly-unconditionally integrable* if the functions $t \mapsto af(t)$ and $t \mapsto f(t)a$ are unconditionally integrable for all $a \in A$. In this case, the *strict unconditional integral* $\int_G^{\mathrm{su}} f(t)\,\mathrm{d}t$ is the multiplier of A given by

$$a\left(\int_G^{\mathrm{su}} f(t)\,\mathrm{d}t\right) := \int_G^{\mathrm{u}} af(t)\,\mathrm{d}t \quad\text{and}\quad \left(\int_G^{\mathrm{su}} f(t)\,\mathrm{d}t\right)a := \int_G^{\mathrm{u}} f(t)a\,\mathrm{d}t.$$

Thus, by definition, a function $f : G \to \mathcal{M}(A)$ is strictly-unconditionally integrable if and only if it is *locally strictly integrable* (meaning that the strict integrals $\int_K^{\mathrm{s}} f(t)\,\mathrm{d}t$ exist for any measurable relatively compact subset $K \subseteq G$) and the net $\left(\int_K^{\mathrm{s}} f(t)\,\mathrm{d}t\right)_{K\in\mathcal{C}}$ converges in the strict topology of $\mathcal{M}(A)$ (to $\int_G^{\mathrm{su}} f(t)\,\mathrm{d}t$).

For positive-valued functions one has the following characterization.

Proposition 3.2.17. *Let $f : G \to \mathcal{M}(A)$ be a locally strictly integrable function such that $f(t) \geq 0$ for all $t \in G$. Then the following assertions are equivalent:*

(i) *f is strictly-unconditionally integrable;*

(ii) *$t \mapsto a^*f(t)a$ is unconditionally integrable for all $a \in A$;*

(iii) *there is $b \in \mathcal{M}(A)^+$ such that for all $\theta \in A_+^*$, the function $t \mapsto \theta(f(t))$ is integrable in the ordinary sense, and $\int_G \theta(f(t))\,\mathrm{d}t = \theta(b)$. In this case*

$$\int_G^{\mathrm{su}} f(t)\,\mathrm{d}t = b.$$

Proof. Obviously (i) implies (ii) and (iii). Since $f(t) \geq 0$, the net $\left(\int_K^{\mathrm{s}} f(t)\,\mathrm{d}t\right)_{K\in\mathcal{C}}$ is increasing. The implication (ii)⇒(i) now follows from Lemma 2.4.4, and the implication (iii)⇒(i) follows from Lemma 2.4.3. □

Corollary 3.2.18. *Let $\mathcal{G} = \mathcal{C}_0(G)$ and $x \in \mathcal{M}(A\otimes\mathcal{G})^+ = \mathcal{C}_b(G, \mathcal{M}^{\mathrm{s}}(A))^+$. Then $x \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$ if and only if $s \mapsto x(s)$ is strictly-unconditionally integrable. In this case, we have*

$$(\mathrm{id}_A \otimes \varphi)(x) = \int_G^{\mathrm{su}} x(s)\,\mathrm{d}s.$$

Proof. Follows from Propositions 3.2.15 and 3.2.17. □

Corollary 3.2.19. *For $\mathcal{G} = \mathcal{C}_0(G)$ the definitions of integrable coactions of $\mathcal{G}$ (Definition 3.2.1) and integrable (or proper) actions of G as defined in [19, 18, 66] coincide. If A is a C^*-algebra with a coaction of $\mathcal{G}$ corresponding to an action α of G, then $a \in A^+$ is integrable if and only if $t \mapsto \alpha_t(a)$ is strictly-unconditionally integrable. In this case*

$$E_1(a) = \int_G^{\mathrm{su}} \alpha_t(a)\,\mathrm{d}t.$$

Proof. This follows from Proposition 3.2.15 and [66, Theorem 4.3, Proposition 4.4]. □

3.3 Functoriality and further examples

If we have two locally compact spaces X and Y with continuous actions α and β of G, and if $\theta : X \to Y$ is a G-equivariant continuous function, then the properness of β implies that of α. A generalization of this result to non-commutative dynamical systems was proved by Rieffel in [66]. We now prove that this continues to hold in our setting.

Recall that a nondegenerate $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ between C^*-algebras A and B with coactions γ_A and γ_B of $\mathcal{G}$ is $\mathcal{G}$-equivariant if $\gamma_B(\pi(a)) = (\pi \otimes \mathrm{id}_{\mathcal{G}})(\gamma_A(a))$ for all $a \in A$.

Proposition 3.3.1. *Let A and B be C^*-algebras with coactions γ_A and γ_B of $(\mathcal{G}, \Delta)$, respectively. Let $\pi : A \to \mathcal{M}(B)$ be a $\mathcal{G}$-equivariant nondegenerate $*$-homomorphism. Then $\pi(A_{\mathrm{si}})B \subseteq B_{\mathrm{si}}$. In particular, if γ_A is integrable, then so is γ_B. Moreover, we have*

$$\pi\big(\mathcal{M}(A)_{\mathrm{i}}^+\big) \subseteq \mathcal{M}(B)_{\mathrm{i}}^+, \quad \pi\big(\mathcal{M}(A)_{\mathrm{i}}\big) \subseteq \mathcal{M}(B)_{\mathrm{i}} \quad \textit{and} \quad \pi\big(\mathcal{M}(A)_{\mathrm{si}}\big) \subseteq \mathcal{M}(B)_{\mathrm{si}}.$$

The following properties hold:

(i) *If $a \in \mathcal{M}(A)_{\mathrm{i}}$ then $E_1\big(\pi(a)\big) = \pi\big(E_1(a)\big)$.*

(ii) *If $\xi \in \mathcal{M}(A)_{\mathrm{si}}$ then $\langle\!\langle \pi(\xi)| = (\pi \otimes \mathrm{id}_H)(\langle\!\langle \xi|)$ and $|\pi(\xi)\rangle\!\rangle = (\pi \otimes \mathrm{id}_{H^*})(|\xi\rangle\!\rangle)$.*

Proof. We have $\gamma_B\big(\pi(a)\big) = (\pi \otimes \mathrm{id}_{\mathcal{G}})\big(\gamma_A(a)\big)$ because π is $\mathcal{G}$-equivariant. Thus, if $a \in \mathcal{M}(A)_{\mathrm{si}}$, then by definition, $\gamma_A(a) \in \bar{\mathcal{N}}^*_{\mathrm{id}_A \otimes \varphi}$ and hence Lemma 2.4.8 says that $\gamma_B\big(\pi(a)\big) \in \bar{\mathcal{N}}^*_{\mathrm{id}_B \otimes \varphi}$, that is, we have $\pi(a) \in \mathcal{M}(B)_{\mathrm{si}}$. Since $\mathcal{M}(B)_{\mathrm{si}}$ is a right ideal in $\mathcal{M}(B)$ we get that $\pi(a)b \in \mathcal{M}(B)_{\mathrm{si}} \cdot B \subseteq \mathcal{M}(B)_{\mathrm{si}} \cap B = B_{\mathrm{si}}$ for all $b \in B$. Therefore if γ_A is integrable, then so is γ_B. From $\pi(\mathcal{M}(A)_{\mathrm{si}}) \subseteq \mathcal{M}(B)_{\mathrm{si}}$ it is clear that $\pi(\mathcal{M}(A)_{\mathrm{i}}^+) \subseteq \mathcal{M}(B)_{\mathrm{i}}^+$ and $\pi(\mathcal{M}(A)_{\mathrm{i}}) \subseteq \mathcal{M}(B)_{\mathrm{i}}$. Lemma 2.4.8 and the equivariance of π yield, for every $a \in \mathcal{M}(A)_{\mathrm{i}}$,

$$\begin{aligned} E_1\big(\pi(a)\big) &= (\mathrm{id}_B \otimes \varphi)\big(\gamma_B(\pi(a)\big) \\ &= (\mathrm{id}_B \otimes \varphi)\big((\pi \otimes \mathrm{id}_{\mathcal{G}})\gamma_A(a)\big) \\ &= \pi\Big((\mathrm{id}_A \otimes \varphi)\big(\gamma_A(a)\big)\Big) \\ &= \pi\big(E_1(a)\big). \end{aligned}$$

And also for every $\xi \in \mathcal{M}(A)_{\mathrm{si}}$,

$$\begin{aligned} \langle\!\langle \pi(\xi)| &= (\mathrm{id}_B \otimes \Lambda)\big(\gamma_B(\pi(\xi))^*\big) \\ &= (\mathrm{id}_B \otimes \Lambda)\big((\pi \otimes \mathrm{id}_{\mathcal{G}})\gamma_A(\xi)^*\big) \\ &= (\pi \otimes \mathrm{id}_H)\Big((\mathrm{id}_A \otimes \Lambda)\big(\gamma_A(\xi)^*\big)\Big) \\ &= (\pi \otimes \mathrm{id}_H)(\langle\!\langle \xi|). \end{aligned}$$

Since $(\pi \otimes \mathrm{id}_H)(x)^* = (\pi \otimes \mathrm{id}_{H^*})(x^*)$, it follows also from this equation that

$$|\pi(\xi)\rangle\!\rangle = (\pi \otimes \mathrm{id}_{H^*})(|\xi\rangle\!\rangle). \qquad \square$$

Note that in the conditions of the proposition above, if $a \in A$ is such that $\pi(a) \in B$, then $\pi(a) \in B_{\mathrm{i}}^{+}$ whenever $a \in A_{\mathrm{i}}^{+}$ and $\pi(a) \in B_{\mathrm{si}}$ whenever $a \in A_{\mathrm{si}}$. In particular, we get the following result.

Corollary 3.3.2. *Let $(A, \gamma_A, \mathcal{G})$ and $(B, \gamma_B, \mathcal{G})$ be coactions. Let $\pi : A \to B$ be a $\mathcal{G}$-equivariant nondegenerate $*$-homomorphism. Then*

$$\pi(A_{\mathrm{i}}^{+}) \subseteq B_{\mathrm{i}}^{+}, \quad \pi(A_{\mathrm{i}}) \subseteq B_{\mathrm{i}} \quad \text{and} \quad \pi(A_{\mathrm{si}}) \subseteq B_{\mathrm{si}}.$$

In particular, if γ_A is integrable, then so is γ_B.

Corollary 3.3.3. *Let A and B be $\mathcal{G}$-C^*-algebras. Suppose that $\pi : A \to \mathcal{M}(B)$ is an equivariant nondegenerate $*$-homomorphism. Let $\xi, \eta \in \mathcal{M}(A)_{\mathrm{si}}$ such that $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in \mathcal{M}(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})$. Then*

$$(\pi \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(\langle\!\langle \xi \,|\, \eta \rangle\!\rangle) = \langle\!\langle \pi(\xi) \,|\, \pi(\eta) \rangle\!\rangle.$$

In particular, if $\pi : A \to B$ and $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ then $\langle\!\langle \pi(\xi) \,|\, \pi(\eta) \rangle\!\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.

Proof. By Proposition 3.3.1 and Equation (2.22) we have

$$\langle\!\langle \pi(\xi) \,|\, \pi(\eta) \rangle\!\rangle = (\pi \otimes \mathrm{id}_H)(\langle\!\langle \xi |)(\pi \otimes \mathrm{id}_{H^*})(| \eta \rangle\!\rangle) = (\pi \otimes \mathrm{id}_{\mathcal{K}})(\langle\!\langle \xi \,|\, \eta \rangle\!\rangle) = (\pi \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(\langle\!\langle \xi \,|\, \eta \rangle\!\rangle).$$

□

Proposition 3.3.1 is a very useful tool providing an indirect method of proving that a coaction is integrable. We give immediately some applications. First recall that if X and Y are topological spaces with continuous actions α and β of a locally compact group G and if we consider the diagonal action γ of G on $Z = X \times Y$, then γ is proper if α or β is proper. Indeed, this follows by observing that the canonical projections $X \times Y \to X$ and $X \times Y \to Y$ are G-equivariant. This still holds if we have C^*-algebras A and B with actions α and β of G, respectively, and consider the diagonal action γ of G on $C = A \otimes B$, then γ is integrable if α or β is integrable. Again, this follows from Proposition 3.3.1 by observing that the canonical $*$-homomorphisms $A \to \mathcal{M}(A \otimes B)$, $a \mapsto a \otimes 1$ and $B \to \mathcal{M}(A \otimes B)$, $b \mapsto 1 \otimes b$, are G-equivariant.

Unfortunately, the concept of diagonal coaction does not make sense in the general setting of locally compact quantum groups. But it does, for example, if one of the coactions is trivial.

Corollary 3.3.4. *Let B be a C^*-algebra with a coaction γ_B of $\mathcal{G}$. Let A be any other C^*-algebra and consider on the tensor product $A \otimes B$ the coaction $\gamma_{A \otimes B} : A \otimes B \to \mathcal{M}(A \otimes B \otimes \mathcal{G})$ given by $\gamma_{A \otimes B}(a \otimes b) = a \otimes \gamma_B(b)$, $a \in A$, $b \in B$. If γ_B is integrable, then so is $\gamma_{A \otimes B}$.*

Proof. This follows from Proposition 3.3.1 because the canonical $*$-homomorphism $B \ni b \mapsto 1 \otimes b \in \mathcal{M}(A \otimes B)$ is $\mathcal{G}$-equivariant. □

Thus although trivial coactions are almost never integrable (only if $\mathcal{G}$ is compact or the algebra is zero), when we tensor with any integrable coaction, the resulting coaction, as above defined, is integrable.

The second important application of Proposition 3.3.1 is to prove that any dual coaction is integrable.

Corollary 3.3.5. *Suppose $(\mathcal{G},\Delta)$ is a locally compact quantum group. Let $(A,\gamma_A,\mathcal{G})$ be a continuous coaction of $\mathcal{G}$. Then the dual coactions $(A\rtimes\widehat{\mathcal{G}}^{\mathrm{c}}_{\mathrm{u}},\widehat{\gamma}_A^{\mathrm{c,u}},\widehat{\mathcal{G}}^{\mathrm{c}})$ and $(A\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}},\widehat{\gamma}_A^{\mathrm{c}},\widehat{\mathcal{G}}^{\mathrm{c}})$ of $\widehat{\mathcal{G}}^{\mathrm{c}}$ are integrable.*

Proof. Consider $\widehat{\mathcal{G}}^{\mathrm{c}}$ and $\widehat{\mathcal{G}}^{\mathrm{c}}_{\mathrm{u}}$ with the canonical coactions of $\widehat{\mathcal{G}}^{\mathrm{c}}$. By definition of $\widehat{\gamma}_A^{\mathrm{c,u}}$ and $\widehat{\gamma}_A^{\mathrm{c}}$, the canonical $*$-homomorphisms $j_{\widehat{\mathcal{G}}^{\mathrm{c}}_{\mathrm{u}}}:\widehat{\mathcal{G}}^{\mathrm{c}}_{\mathrm{u}}\to\mathcal{M}(A\rtimes\widehat{\mathcal{G}}^{\mathrm{c}}_{\mathrm{u}})$ and $j_{\widehat{\mathcal{G}}^{\mathrm{c}}}:\widehat{\mathcal{G}}^{\mathrm{c}}\to\mathcal{M}(A\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})$ are $\widehat{\mathcal{G}}^{\mathrm{c}}$-equivariant. Since these homomorphisms are nondegenerate, the result now follows from Propositions 3.2.12, 3.2.14 and 3.3.1. □

In particular, if $\mathcal{G}$ is regular, we get that $\mathcal{K}:=\mathcal{K}(H)\cong\widehat{\mathcal{G}}^{\mathrm{c}}\rtimes_{\mathrm{r}}\mathcal{G}$ is an integrable $\mathcal{G}$-C^*-algebra. Recall that the coaction $\gamma_{\mathcal{K}}$ on $\mathcal{K}$ is given by $\gamma_{\mathcal{K}}(x)=\hat{W}^*(x\otimes 1)\hat{W}$ for all $x\in\mathcal{K}$, where $\hat{W}$ is the left regular corepresentation of $\widehat{\mathcal{G}}$ (see Example 2.6.18(3)). More generally, we have:

Corollary 3.3.6. *Suppose that $\mathcal{G}$ is regular. Then for any coaction (A,γ_A) of $\mathcal{G}$, the coaction $(A\otimes\mathcal{K},\gamma_{A\otimes\mathcal{K}})$ is integrable, where $\mathcal{K}:=\mathcal{K}(H)$ and*

$$\gamma_{A\otimes\mathcal{K}}(T)=\hat{W}_{23}^*\Sigma_{23}(\gamma_A\otimes\mathrm{id})(T)\Sigma_{23}^*\hat{W}_{23},\quad T\in A\otimes\mathcal{K}$$

Proof. This follows from Proposition 3.3.1 by observing that the map $x\mapsto 1_A\otimes x$ from $\mathcal{K}$ into $\mathcal{M}(A\otimes\mathcal{K})$ is a nondegenerate $\mathcal{G}$-equivariant $*$-homomorphism (see Example 2.6.18(1)). Note that if γ_A is injective, then this also follows from Corollary 2.4.4 and Proposition 2.7.16. □

We are going to see later that $\mathcal{K}$ is integrable even if $\mathcal{G}$ is not regular. Thus the same argument above shows that $A\otimes\mathcal{K}$ is always integrable for any $\mathcal{G}$-coaction (A,γ_A), where $\mathcal{G}$ is an arbitrary locally compact quantum group. In particular, any $\mathcal{G}$-coaction is Morita equivalent to an integrable $\mathcal{G}$-coaction. Thus, unless $\mathcal{G}$ is compact, integrability is not invariant under Morita equivalence.

Our last application of Proposition 3.3.1 is the following result.

Corollary 3.3.7. *Let A be a $\mathcal{G}$-C^*-algebra and suppose that I is a $\mathcal{G}$-invariant ideal of A. If A is integrable, then so are I and the quotient A/I.*

Proof. This follows from the facts that the restriction map $r:A\to\mathcal{M}(I)$ and quotient map $q:A\to A/I$ are nondegenerate $\mathcal{G}$-equivariant $*$-homomorphisms. □

In the case of groups, the result above was proved by Rieffel in [66, Corollary 5.4, Proposition 5.5].

Chapter 4

Square-integrable coactions on Hilbert modules

4.1 Definition of square-integrable coactions

Throughout this section we fix a locally compact quantum group $\mathcal{G}$ and denote its left Haar weight by φ. As in the previous chapter, we also fix a GNS-construction for φ of the form (H, ι, Λ), where ι denotes the inclusion map $\mathcal{G} \hookrightarrow \mathcal{L}(H)$. In what follows, we are going to use some notations and definitions from Section 2.4.1.

Definition 4.1.1. Let $\mathcal{E}$ be a Hilbert B-module with a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$. We say that $\xi \in \mathcal{E}$ is *square-integrable* if $\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}$, for all $\eta \in \mathcal{E}$. We denote by $\mathcal{E}_{\mathrm{si}}$ the set of all square-integrable elements of $\mathcal{E}$. We say that $\mathcal{E}$ (or the coaction $\gamma_{\mathcal{E}}$) is *square-integrable* if $\mathcal{E}_{\mathrm{si}}$ is dense in $\mathcal{E}$.

For $\eta \in \mathcal{E}$ we have used the element $\eta \otimes 1 \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ in the definition above. It is defined in the natural way: $(\eta \otimes 1)(b \otimes x) = \eta \cdot b \otimes x$, for $b \in B$, $x \in \mathcal{G}$. Recall that $\mathcal{M}(\mathcal{E} \otimes \mathcal{G}) = \mathcal{L}(B \otimes \mathcal{G}, \mathcal{E} \otimes \mathcal{G})$ is a Hilbert $\mathcal{M}(B \otimes \mathcal{G})$-module, where the $\mathcal{M}(B \otimes \mathcal{G})$-inner product is given by $\langle x \,|\, y \rangle_{\mathcal{M}(B \otimes \mathcal{G})} = x^* y$ for all $x, y \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$. Thus we can also write $\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1) = \langle \gamma_{\mathcal{E}}(\xi) \,|\, \eta \otimes 1 \rangle_{\mathcal{M}(B \otimes \mathcal{G})}$ for all $\xi, \eta \in \mathcal{E}$.

We mainly use square-integrable elements of $\mathcal{E}$, but note that because the coaction $\gamma_{\mathcal{E}}$ extends to the multiplier space $\mathcal{M}(\mathcal{E})$ of $\mathcal{E}$, the same definition above can be used to define square-integrable elements in $\mathcal{M}(\mathcal{E})$. We shall denote the space of square-integrable elements in $\mathcal{M}(\mathcal{E})$ by $\mathcal{M}(\mathcal{E})_{\mathrm{si}}$. Thus $\xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ if and only if $\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}$ for all $\eta \in \mathcal{E}$. Note that $\mathcal{E}_{\mathrm{si}} = \mathcal{M}(\mathcal{E})_{\mathrm{si}} \cap \mathcal{E}$.

If A is a C^*-algebra with a coaction γ_A of $\mathcal{G}$, we can consider A as a Hilbert A-module in the usual way. In this case, we have two definitions for square-integrable elements and coactions and we have to prove that they coincide. This will follow from the next two results.

Proposition 4.1.2. *Let $\gamma_{\mathcal{E}}$ be a coaction of $\mathcal{G}$ on a Hilbert B-module $\mathcal{E}$ and consider on $\mathcal{K}(\mathcal{E})$ the induced coaction $\gamma_{\mathcal{K}(\mathcal{E})}$ of $\mathcal{G}$. Suppose that $\xi \in \mathcal{E}$. Then $\xi \in \mathcal{E}_{\mathrm{si}}$ if and only if $|\xi\rangle\langle\xi| \in \mathcal{K}(\mathcal{E})_{\mathrm{i}}$ (as defined in Definition 3.2.1).*

Proof. By definition, $\xi \in \mathcal{E}_{\mathrm{si}}$ if and only if $\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1) \in \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}$ for all $\eta \in \mathcal{E}$. This is equivalent to

$$(\eta \otimes 1)^* \gamma_{\mathcal{K}(\mathcal{E})}(|\xi\rangle\langle\xi|)(\eta \otimes 1) = (\eta \otimes 1)^* \gamma_{\mathcal{E}}(\xi)\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1) \in \bar{\mathcal{M}}_{\mathrm{id}_B \otimes \varphi} \text{ for all } \eta \in \mathcal{E}.$$

On the other hand, setting $x := \gamma_{\mathcal{K}(\mathcal{E})}(|\xi\rangle\langle\xi|)$, we have $|\xi\rangle\langle\xi| \in \mathcal{K}(\mathcal{E})_{\mathrm{i}}$ if and only if $x \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi}$. Given $\omega \in \mathcal{G}^*$, it follows from Proposition 2.4.14(iii),(iv) that

$$(\mathrm{id}_B \otimes \omega)\big((\eta \otimes 1)^* x (\eta \otimes 1)\big) = \langle \eta \,|\, (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \omega)(x)\eta\rangle \text{ for all } \eta \in \mathcal{E}.$$

So we get that $\xi \in \mathcal{E}_{\mathrm{si}}$ if and only if the net $\big(\langle \eta \,|\, (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \omega)(x)\eta\rangle\big)_{\omega \in \mathcal{G}_\varphi}$ converges strictly in $\mathcal{M}(B)$ for all $\eta \in \mathcal{E}$. Since $\mathcal{E} \cdot B = \mathcal{E}$, the strict convergence is equivalent to convergence in the norm of B. The assertion now follows from Lemma 2.4.4. $\square$

Corollary 4.1.3. *Let $\mathcal{E}$ be a Hilbert B-module with coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$, and let $\gamma_{\mathcal{K}(\mathcal{E})}$ be the induced coaction of $\mathcal{G}$ on $\mathcal{K}(\mathcal{E})$. Then $\mathcal{E}$ is square-integrable if and only if $\mathcal{K}(\mathcal{E})$ is integrable* (*see Definition* 3.2.1).

Proof. Using Proposition 4.1.2 the same proof of [47, Proposition 8.3] works. For convenience, we reproduce the proof here. Proposition 4.1.2 and polarization imply that $|\xi\rangle\langle\eta| \in \mathcal{K}(\mathcal{E})_{\mathrm{i}}$ for all $\xi, \eta \in \mathcal{E}_{\mathrm{si}}$. Thus, if $\mathcal{E}$ is square-integrable, then so is $\mathcal{K}(\mathcal{E})$. Conversely, assume that $\mathcal{K}(\mathcal{E})$ is square-integrable. If $T \in \mathcal{K}(\mathcal{E})_{\mathrm{si}}$, then $|T\xi\rangle\langle T\xi| = T|\xi\rangle\langle\xi|T^* \leq \|\xi\|^2 TT^*$, which implies that $|T\xi\rangle\langle T\xi| \in \mathcal{K}(\mathcal{E})_{\mathrm{i}}^+$ because $TT^* \in \mathcal{K}(\mathcal{E})_{\mathrm{i}}^+$ and $\mathcal{K}(\mathcal{E})_{\mathrm{i}}^+$ is a hereditary cone. Thus $T\xi \in \mathcal{E}_{\mathrm{si}}$ by Proposition 4.1.2. Since $\mathcal{K}(\mathcal{E})$ is square-integrable, $\mathcal{K}(\mathcal{E})_{\mathrm{si}}$ is dense in $\mathcal{K}(\mathcal{E})$ and therefore the set of elements of the form $T\xi$ with $T \in \mathcal{K}(\mathcal{E})_{\mathrm{si}}$ and $\xi \in \mathcal{E}$ is dense in $\mathcal{E}$. This shows that $\mathcal{E}$ is square-integrable. $\square$

If a Hilbert B-module $\mathcal{E}$ is square-integrable, then we can construct many adjointable operators $\mathcal{E} \to B \otimes H$. We are going to explain how we can construct such operators in what follows. We shall need the KSGNS-map $\mathrm{id}_B \otimes \Lambda$ as well as the generalized KSGNS-map $\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda$ defined in Sections 2.4.2 and 2.4.4.

Proposition 4.1.4. *Let $\mathcal{E}$ be a Hilbert B-module with a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$ and suppose that $\xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$. Then the equation*

$$\langle\!\langle \xi | \eta := (\mathrm{id}_B \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1)\big)$$

defines an adjointable operator $\langle\!\langle \xi| : \mathcal{E} \to B \otimes H$. For all $b \in B$ and $s \in \mathcal{N}_\varphi$, we have $\gamma_{\mathcal{E}}(\xi)(b \otimes s) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}} \otimes \varphi}$ and the adjoint operator $|\xi\rangle\!\rangle := \langle\!\langle \xi|^*$ is given by the formula

$$|\xi\rangle\!\rangle\big(b \otimes \Lambda(s)\big) = (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)(b \otimes s)\big)$$

for all $b \in B$ and $s \in \mathcal{N}_\varphi$. Moreover, $\xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ if and only if $\gamma_{\mathcal{E}}(\xi)^* \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$, and in this case

$$\langle\!\langle \xi| = (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*\big).$$

Proof. By Proposition 2.4.20(iii), $\xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ if and only if $\gamma_{\mathcal{E}}(\xi)^* \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$, and in this case the operator $(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*\big) \in \mathcal{L}(\mathcal{E}, B \otimes H)$ is given by

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*\big)\eta = (\mathrm{id}_B \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1)\big) = \langle\!\langle \xi|\eta$$

for all $\eta \in \mathcal{E}$. Thus $\langle\!\langle \xi|$ is an adjointable operator and it is equal to $(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*\big)$. Proposition 2.4.20(iv) implies that $\gamma_{\mathcal{E}}(\xi)(b \otimes s) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$ whenever $\xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$, $b \in B$ and $s \in \mathcal{N}_{\varphi}$, and the adjoint of $\langle\!\langle \xi|$ is given by

$$|\xi\rangle\!\rangle\big(b \otimes \Lambda(s)\big) = (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)(b \otimes s)\big). \qquad \square$$

Remark 4.1.5. Let us keep the notations of Proposition 4.1.4. Note that the element $(\mathrm{id}_B\otimes\Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*(\eta\otimes1)\big)$ is, *a priori*, only in $\mathcal{M}(B\otimes H) = \mathcal{L}(B, B\otimes H)$, but Proposition 4.1.4 says that it is, in fact, in $B \otimes H \subseteq \mathcal{M}(B \otimes H)$. This can also be seen directly by using that $\mathcal{E} = \mathcal{E} \cdot B$. In fact, we can factor $\eta = \zeta \cdot c$ for some $\zeta \in \mathcal{E}$ and $c \in B$, so that (by Proposition 2.4.6(iv))

$$(\mathrm{id}_B \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1)\big) = (\mathrm{id}_B \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*(\zeta \otimes 1)\big)c \in B \otimes H.$$

Note also that, *a priori*, we only have $(\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)(b \otimes s)\big) \in \mathcal{M}(\mathcal{E})$. But we see that it is, in fact, an element of $\mathcal{E} \subseteq \mathcal{M}(\mathcal{E})$ because

$$(\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)(b \otimes s)\big) = (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)(1_B \otimes s)\big)b.$$

We have used above that $\gamma_{\mathcal{E}}(\xi)(1_B \otimes s) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$. In fact, since $\gamma_{\mathcal{E}}(\xi)^* \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ and $1_B \otimes s \in \mathcal{N}_{\mathrm{id}_B\otimes\varphi}$, this follows from Proposition 2.4.20(iv).

Corollary 4.1.6. *The map $\langle\!\langle \cdot| : \mathcal{M}(\mathcal{E})_{\mathrm{si}} \to \mathcal{L}(\mathcal{E}, B \otimes H)$, $\xi \mapsto \langle\!\langle \xi|$, is a closed anti-linear map when we consider on $\mathcal{M}(\mathcal{E})$ the strict topology and on $\mathcal{L}(\mathcal{E}, B \otimes H)$ the $\mathcal{K}$-strong topology (see Definition 2.1.13).*

Proof. This follows directly from Propositions 4.1.4 and 2.4.20(ix). $\square$

Example 4.1.7. Suppose that $\mathcal{G}$ is a compact quantum group, that is, suppose that the Haar weight φ is bounded. Then every Hilbert B-module $\mathcal{E}$ with a coaction of $\mathcal{G}$ is square-integrable. In fact, note that $\bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} = \mathcal{M}(B \otimes \mathcal{G})$, so that $\mathcal{M}(\mathcal{E})_{\mathrm{si}} = \mathcal{M}(\mathcal{E})$ and, in particular, $\mathcal{E}_{\mathrm{si}} = \mathcal{E}$. Therefore, given any $\xi \in \mathcal{M}(\mathcal{E})$ we have an adjointable operator $\langle\!\langle \xi| \in \mathcal{L}(\mathcal{E}, B \otimes H)$. This operator can be described in the following way. First note that the GNS-map Λ is given by

$$\Lambda(x) = \Lambda(x \cdot 1) = x\Lambda(1) = x(\delta_1),$$

where 1 is the unit of $\mathcal{G}$ and $\delta_1 := \Lambda(1)$. More generally, the KSGNS-map $\mathrm{id}_B \otimes \Lambda$ is given by

$$(\mathrm{id}_B \otimes \Lambda)(x) = x(\mathrm{id}_B \otimes \Lambda)(1_B \otimes 1) = x(1_B \otimes \delta_1)$$

for all $x \in \mathcal{M}(B \otimes \mathcal{G})$, where we have identified $\mathcal{M}(B \otimes \mathcal{G}) \subseteq \mathcal{L}(B \otimes H)$. Even more generally, the KSGNS-map $\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda$ can also be written in the form

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x) = x(1_{\mathcal{E}} \otimes \delta_1)$$

for all $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi} = \mathcal{L}(\mathcal{E}\otimes\mathcal{G}, B\otimes\mathcal{G})$, where $1_{\mathcal{E}}$ denotes the identity operator on $\mathcal{E}$. Thus $1_{\mathcal{E}}\otimes\delta_1$ is an element of $\mathcal{L}(\mathcal{E})\otimes H \subseteq \mathcal{L}(\mathcal{E}, \mathcal{E}\otimes H)$. Here we are identifying $\mathcal{L}(B\otimes\mathcal{G}, \mathcal{E}\otimes H) = \mathcal{M}(\mathcal{E}\otimes\mathcal{G}) \subseteq \mathcal{M}(\mathcal{E}\otimes\mathcal{K}(H)) \cong \mathcal{L}(B\otimes H, \mathcal{E}\otimes H)$ (this last identification follows from Proposition 2.1.11 and Remark 2.1.12(2)) and therefore x is considered as an element of $\mathcal{L}(\mathcal{E}\otimes\mathcal{G}, B\otimes\mathcal{G}) \subseteq \mathcal{L}(\mathcal{E}\otimes H, B\otimes H)$. In particular, we get

$$\langle\!\langle\xi| = (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*\big) = \gamma_{\mathcal{E}}(\xi)^*(1_{\mathcal{E}}\otimes\delta_1)$$

for all $\xi \in \mathcal{M}(\mathcal{E})$. The adjoint operator $|\xi\rangle\!\rangle \in \mathcal{L}(B\otimes H, \mathcal{E})$ is therefore given by

$$|\xi\rangle\!\rangle = (1_{\mathcal{E}}\otimes\delta_1^*)\gamma_{\mathcal{E}}(\xi),$$

where δ_1^* denotes the element of $\mathcal{L}(H, \mathbb{C})$ given by $\delta_1^*(v) = \langle\delta_1\,|\,v\rangle$ for all $v \in H$.

Example 4.1.8. We analyze the case of a locally compact group G, that is, we consider the quantum group $\mathcal{G} = \mathcal{C}_0(G)$. Let us fix a left Haar measure $\mathrm{d}t$ on G and consider the corresponding left Haar weight on $\mathcal{G}$: $\varphi(f) = \int_G f(t)\,\mathrm{d}t$ for all $f \in \mathcal{C}_0(G)^+$. In this case we have a canonical GNS-construction $(L^2(G), M, \Lambda)$ for φ, where M is the multiplication representation of $\mathcal{C}_0(G)$ on $L^2(G)$ and Λ is the inclusion of $\mathcal{N}_\varphi = \mathcal{C}_0(G)\cap L^2(G)$ into $L^2(G)$.

Let B be a C^*-algebra. We identify $\mathcal{M}(B\otimes\mathcal{G})$ with $\mathcal{C}_b\big(G, \mathcal{M}^{\mathrm{s}}(B)\big)$ (the algebra of bounded strictly continuous functions $G \to \mathcal{M}(B)$). Under this identification, the KSGNS-map $\mathrm{id}_B\otimes\Lambda$ can be described as a canonical inclusion of the space $\bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi} \subseteq \mathcal{M}(B\otimes\mathcal{G}) \cong \mathcal{C}_b(G, \mathcal{M}^{\mathrm{s}}(B))$ into $\mathcal{M}(B\otimes L^2(G)) \cong \mathcal{L}\big(B, L^2(G, B)\big)$. We explain how this is done in what follows.

Following [47], we say that a function $f \in \mathcal{C}_b(G, B)$ is *square-integrable* if the net $(\chi_j\cdot f) \subseteq \mathcal{C}_c(G, B)$ is Cauchy (and hence converges) in $L^2(G, B)$, where (χ_j) is a net of compactly supported continuous functions $G \to [0, 1]$ such that $\chi_j(t) \to 1$ uniformly on compact subsets of G. This definition does not depend on the choice of the net $\{\chi_j\}$. Moreover, Proposition 12 in [9] shows that $f \in \mathcal{C}_b(G, B)$ is square-integrable if and only if the function $t \mapsto f(t)^*f(t)$ is unconditionally integrable (see Definition 3.2.16), and in this case we have

$$\int_G^{\mathrm{u}} f(t)^*f(t)\,\mathrm{d}t = \lim_j \int_G \chi_j(t)^2 f(t)^*f(t)\,\mathrm{d}t = \langle\lim_j \chi_j\cdot f\,|\,\lim_j \chi_j\cdot f\rangle_B.$$

We denote the set of square-integrable functions in $\mathcal{C}_b(G, B)$ by $\mathcal{C}_b^{\mathrm{si}}(G, B)$. Thus, if we identify each $f \in \mathcal{C}_b^{\mathrm{si}}(G, B)$ with the limit of $(\chi_j\cdot f)$ in $L^2(G, B)$ we get a canonical inclusion of $\mathcal{C}_b^{\mathrm{si}}(G, B)$ into $L^2(G, B)$. We shall identify $\mathcal{C}_b^{\mathrm{si}}(G, B) \subseteq L^2(G, B)$ via this inclusion.

This can be generalized to functions in $\mathcal{C}_b\big(G, \mathcal{M}^{\mathrm{s}}(B)\big)$ if we replace norm by strict topology. We say that $f \in \mathcal{C}_b\big(G, \mathcal{M}^{\mathrm{s}}(B)\big)$ is *strictly square-integrable* if the function $f\cdot b := [t \mapsto f(t)b]$ is square-integrable for every $b \in B$. We denote the set of strictly square-integrable functions in $\mathcal{C}_b(G, \mathcal{M}^{\mathrm{s}}(B))$ by $\mathcal{C}_b^{\mathrm{si}}\big(G, \mathcal{M}^{\mathrm{s}}(B)\big)$. It follows from Proposition 3.2.17 that $f \in \mathcal{C}_b^{\mathrm{si}}\big(G, \mathcal{M}^{\mathrm{s}}(B)\big)$ if and only if $t \mapsto f(t)^*f(t)$ is strictly-unconditionally integrable. By Corollary 3.2.18, this means that $f \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$. Hence $\mathcal{C}_b^{\mathrm{si}}\big(G, \mathcal{M}^{\mathrm{s}}(B)\big) = \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ and, for every $f \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$, we have

$$(\mathrm{id}_B\otimes\varphi)(f^*f) = \int_G^{\mathrm{su}} f(t)^*f(t)\,\mathrm{d}t \in \mathcal{M}(B).$$

Given $f \in \mathcal{C}_b^{\mathrm{si}}\big(G, \mathcal{M}^{\mathrm{s}}(B)\big)$ we have, by definition, that $f \cdot b \in \mathcal{C}_b^{\mathrm{si}}(G, B) \subseteq L^2(G, B)$. Thus f induces a map $F : B \to L^2(G, B)$, $F(b) := f \cdot b = \lim_j \chi_j(f \cdot b)$. We claim that $F \in \mathcal{M}\big(L^2(G, B)\big) = \mathcal{L}\big(B, L^2(G, B)\big)$. First note that

$$\begin{aligned}\langle F(b) \,|\, F(b)\rangle_B &= \lim_j \langle \chi_j(f \cdot b) \,|\, \chi_j(f \cdot b)\rangle_B \\ &= \int_G^{\mathrm{u}} b^* f(t)^* f(t) b \,\mathrm{d}t \leq \left\| \int_G^{\mathrm{su}} f(t)^* f(t) \,\mathrm{d}t \right\| b^* b.\end{aligned}$$

This says that F is a bounded operator with $\|F\| \leq \| \int_G^{\mathrm{su}} f(t)^* f(t) \,\mathrm{d}t \|$. Now define $E : \mathcal{C}_c(G, B) \subseteq L^2(G, B) \to B$ by

$$E(g) := \int_G f(t)^* g(t) \,\mathrm{d}t, \qquad \text{for all } g \in \mathcal{C}_c(G, B).$$

Since $f \in \mathcal{C}_b(G, \mathcal{M}^{\mathrm{s}}(B))$, the function $[t \mapsto f(t)^* g(t)]$ belongs to $\mathcal{C}_c(G, B)$ for all $g \in \mathcal{C}_c(G, B)$, so that the Bochner integral above is well-defined. Moreover, for every $b \in B$ and $g \in \mathcal{C}_c(G, B)$, we have

$$\begin{aligned}\langle F(b) \,|\, g\rangle_B &= \lim_j \langle \chi_j(f \cdot b) \,|\, g\rangle_B = \lim_j \int_G \chi_j(t) b^* f(t) g(t) \,\mathrm{d}t \\ &= \int_G^{\mathrm{u}} b^* f(t)^* g(t) \,\mathrm{d}t = \int_G b^* f(t)^* g(t) \,\mathrm{d}t \\ &= b^* \int_G f(t)^* g(t) \,\mathrm{d}t = \langle b \,|\, E(g)\rangle_B.\end{aligned}$$

It follows that E extends to an adjointable operator $L^2(G, B) \to B$ and $F = E^*$. Therefore, $F \in \mathcal{L}\big(B, L^2(G, B)\big) = \mathcal{M}\big(L^2(G, B)\big)$ and hence we get a canonical inclusion

$$\bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi} = \mathcal{C}_b^{\mathrm{si}}\big(G, \mathcal{M}^{\mathrm{s}}(B)\big) \ni f \mapsto F \in \mathcal{M}\big(L^2(G, B)\big).$$

Finally, we prove that the KSGNS-map $\mathrm{id}_B \otimes \Lambda$ is given by this inclusion. This is now easy because, by Proposition 2.4.6, we have, for every $f \in \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi} = \mathcal{C}_b^{\mathrm{si}}\big(G, \mathcal{M}^{\mathrm{s}}(B)\big)$ and $g \in \mathcal{C}_c(G) \odot B \subseteq \mathcal{N}_\varphi \odot B$, that

$$(\mathrm{id}_B \otimes \Lambda)(f)^* g = (\mathrm{id}_B \otimes \varphi)(f^* g) = \int_G^{\mathrm{su}} f(t)^* g(t) \,\mathrm{d}t = \int_G f(t)^* g(t) \,\mathrm{d}t = F^*(g).$$

Now suppose that B has an action of B, let $\mathcal{E}$ be a Hilbert B, G-module with an action γ of G, and consider the corresponding coaction of $\mathcal{G}$ on $\mathcal{E}$ given by

$$\gamma_{\mathcal{E}}(\xi) := [t \mapsto \gamma_t(\xi)] \in \mathcal{C}_b(G, \mathcal{E}),$$

where we identify $\mathcal{C}_b(G, \mathcal{E}) \subseteq \mathcal{M}(\mathcal{E} \otimes \mathcal{G})$ in the canonical way. Given $\xi, \eta \in \mathcal{E}$, note that $\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1) \in \mathcal{M}(B \otimes \mathcal{G})$ corresponds to the function $t \mapsto \langle \gamma_t(\xi) \mid \eta\rangle_B \in \mathcal{C}_b(G, B)$. According to our definition, $\xi \in \mathcal{E}$ is square-integrable if and only if this function belongs to $\bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi} = \mathcal{C}_b^{\mathrm{si}}(G, B)$ for all $\eta \in \mathcal{E}$. And by what we have seen above, we have

$$\langle\!\langle \xi | \eta = (\mathrm{id}_B \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1)\big) = [t \mapsto \langle \gamma_t(\xi) \,|\, \eta\rangle_B] \in \mathcal{C}_b^{\mathrm{si}}(G, B) \subseteq L^2(G, B)$$

for all $\xi \in \mathcal{E}_{\mathrm{si}}$ and $\eta \in \mathcal{E}$. The adjoint of $\langle\!\langle\xi|$ is given by

$$|\xi\rangle\!\rangle f = (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)f\big) = \int_G \gamma_t(\xi)f(t)\,\mathrm{d}t \in \mathcal{E}, \tag{4.1}$$

for all $\xi \in \mathcal{E}_{\mathrm{si}}$ and $f \in \mathcal{C}_c(G,B)$. Note that $\mathcal{C}_c(G,B) \subseteq \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$. In fact, this follows from Corollary 3.2.18 because any function in $\mathcal{C}_c(G,B)$ is Bochner integrable and therefore also (strictly) unconditionally integrable. By Proposition 2.4.20(iv), the element $\gamma_{\mathcal{E}}(\xi)f$ belongs to $\bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$, so that it makes sense to write $(\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)f\big)$. On the other hand, the element $\gamma_{\mathcal{E}}(\xi)f$ corresponds to the function $t \mapsto \gamma_t(\xi)f(t)$ which belongs to $\mathcal{C}_c(G,\mathcal{E})$ and therefore is Bochner integrable. Moreover, its integral $\int_G \gamma_t(\xi)f(t)\,\mathrm{d}t$ coincides with $(\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)f\big)$. In fact, if $\eta \in \mathcal{E}$, then we have

$$\begin{aligned}\big\langle\eta\big|(\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)f\big)\big\rangle_B &= (\mathrm{id}_B \otimes \varphi)\big((\eta^* \otimes 1)\gamma_{\mathcal{E}}(\xi)f\big)\\ &= \int_G \langle\eta\,|\,\gamma_t(\xi)f(t)\rangle_B\,\mathrm{d}t\\ &= \left\langle\eta\,\middle|\,\int_G \gamma_t(\xi)f(t)\,\mathrm{d}t\right\rangle_B.\end{aligned}$$

This justifies Equation (4.1). We conclude that in the group case our definition of square-integrability coincides with the definition appearing in [47, 48].

Remark 4.1.9. We have seen in Example 4.1.8 that for the case $\mathcal{G} = \mathcal{C}_0(G)$, where G is some locally compact group, a function $f \in \mathcal{C}_b\big(G,\mathcal{M}^{\mathrm{s}}(B)\big) = \mathcal{M}(B\otimes\mathcal{G})$ belongs to $\bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ if and only if the net $(\chi_j \cdot f) \subseteq \mathcal{C}_c\big(G,\mathcal{M}^{\mathrm{s}}(B)\big)$ converges strongly in $\mathcal{M}\big(L^2(G,B)\big)$ and in this case $(\mathrm{id}_B\otimes\Lambda)(f)$ coincides with the strong limit of $(\chi_j\cdot f)$ in $\mathcal{M}\big(L^2(G,B)\big)$, where (χ_j) is some net of compactly supported continuous functions $G \to [0,1]$ converging uniformly to 1 on compact subsets. Here we are using the canonical embedding $\mathcal{C}_c(G,\mathcal{M}^{\mathrm{s}}(B)\big) \hookrightarrow \mathcal{M}\big(L^2(G,B)\big)$ which takes any function g in $\mathcal{C}_c\big(G,\mathcal{M}^{\mathrm{s}}(B)\big)$ and associates the operator $B \ni b \mapsto g \cdot b \in \mathcal{C}_c(G,B) \subseteq L^2(G,B)$ in $\mathcal{M}\big(L^2(G,B)\big)$, where $(g\cdot b)(t) := g(t)b$ for all $t \in G$ (notation as in Example 4.1.8).

This can be generalized to arbitrary locally compact quantum groups in the following way. Take any net (e_j) in $\mathcal{G}$ consisting of analytic elements $e_j \in \mathcal{N}_\varphi$ with respect to the modular group σ for φ such that $\sigma_z(e_j)$ converges strictly to 1 in $\mathcal{M}(\mathcal{G})$ for all $z \in \mathbb{C}$ (see Lemma 2.4.10). The elements e_j will play the role of χ_j above.[1] Let $x \in \mathcal{M}(B \otimes \mathcal{G})$. We claim that $x \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$ if and only if the net $\big(x(1_B \otimes \Lambda(e_j))\big)$ converges strongly in $\mathcal{M}(B \otimes H)$ and in this case $(\mathrm{id}_B \otimes \Lambda)(x)$ coincides with the strong limit of this net. Here we consider the embedding $\mathcal{M}(B \otimes \mathcal{G}) \subseteq \mathcal{L}(B \otimes H)$, so that $x(1_B \otimes \Lambda(e_j))$ can be considered as an element of $\mathcal{M}(B \otimes H)$. First suppose that $x \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$. Then, by Propositions 2.4.6(ii),(iii) and 2.4.13(i), we have

$$x\big(1_B \otimes \Lambda(e_j)\big) = (\mathrm{id}_B \otimes \Lambda)\big(x(1 \otimes e_j)\big) = \big(1 \otimes J\sigma_{\frac{\mathrm{i}}{2}}(e_j)^*J\big)(\mathrm{id}_B \otimes \Lambda)(x)$$

which converges strongly to $(\mathrm{id}_B \otimes \Lambda)(x)$ in $\mathcal{M}(B \otimes H)$ because $\sigma_{\frac{\mathrm{i}}{2}}(e_j)$ converges strictly to 1. Conversely, if $x(1_B\otimes\Lambda(e_i))$ converges strongly to some element $y \in \mathcal{M}(B\otimes H)$, then

[1] Note that in the group case the modular group is trivial, so that any element is analytic.

because $x(1_B \otimes e_j)$ converges strictly to x in $\mathcal{M}(B \otimes \mathcal{G})$, it follows from Proposition 2.4.6(v) that $x \in \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}$ and $(\mathrm{id}_B \otimes \Lambda)(x) = y$.

The next result gives some basic properties of the bra-ket operators $\langle\langle \xi|$ and $|\xi\rangle\rangle$. Recall that given a C^*-algebra A with a coaction γ_A of $\mathcal{G}$ and an element $a \in \mathcal{M}(A)_{\mathrm{i}}$, the symbol $E_1(a)$ denotes the element $(\mathrm{id}_A \otimes \varphi)\big(\gamma_A(a)\big) \in \mathcal{M}(A)$ (see Definition 3.2.9).

Proposition 4.1.10. *Let $\mathcal{E}$ be a Hilbert B-module with a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$.*

(i) *If $\xi, \eta \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$, then $\xi \circ \eta^* \in \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)_{\mathrm{i}}$ and $|\xi\rangle\rangle\langle\langle\eta| = E_1(\xi \circ \eta^*)$.*
In particular, if $\xi, \eta \in \mathcal{E}_{\mathrm{si}}$, then $|\xi\rangle\langle\eta| \in \mathcal{K}(\mathcal{E})_{\mathrm{i}}$ and $|\xi\rangle\rangle\langle\langle\eta| = E_1(|\xi\rangle\langle\eta|)$.

(ii) *If $\xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ and $b \in \mathcal{M}(B)$, then $\xi \cdot b \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ and $|\xi \cdot b\rangle\rangle = |\xi\rangle\rangle \circ \gamma_B(b)$, where we have identified $\gamma_B(b) \in \mathcal{M}(B \otimes \mathcal{G}) \subseteq \mathcal{L}(B \otimes H)$.*

In particular, if $\xi \in \mathcal{E}_{\mathrm{si}}$ (or even in $\mathcal{M}(\mathcal{E})_{\mathrm{si}}$) and $b \in B$, then $\xi \cdot b \in \mathcal{E}_{\mathrm{si}}$ and $|\xi \cdot b\rangle\rangle = |\xi\rangle\rangle \circ \gamma_B(b)$.

(iii) *Let $\mathcal{F}$ be another Hilbert B-module with a coaction of $\mathcal{G}$. If $\xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ and $T \in \mathcal{L}^{\mathcal{G}}(\mathcal{E}, \mathcal{F})$, then $T \circ \xi \in \mathcal{M}(\mathcal{F})_{\mathrm{si}}$ and $|T \circ \xi\rangle\rangle = T \circ |\xi\rangle\rangle$.*
In particular, if $\xi \in \mathcal{E}_{\mathrm{si}}$ and $T \in \mathcal{L}^{\mathcal{G}}(\mathcal{E}, \mathcal{F})$, then $T(\xi) \in \mathcal{F}_{\mathrm{si}}$ and
$$|T(\xi)\rangle\rangle = T \circ |\xi\rangle\rangle.$$

(iv) *If $T \in \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)_{\mathrm{si}}$ and $\xi \in \mathcal{M}(\mathcal{E})$, then $T \circ \xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ and*
$$|T \circ \xi\rangle\rangle = |T\rangle\rangle \circ \gamma_{\mathcal{E}}(\xi).$$
In particular, if $T \in \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)_{\mathrm{si}}$ and $\xi \in \mathcal{E}$, then $T(\xi) \in \mathcal{E}_{\mathrm{si}}$ and
$$|T(\xi)\rangle\rangle = |T\rangle\rangle \circ \gamma_{\mathcal{E}}(\xi).$$
More generally, if $\pi : A \to \mathcal{L}(\mathcal{E})$ is a $\mathcal{G}$-equivariant nondegenerate $*$-homomorphism, where A is a C^*-algebra with a coaction of $\mathcal{G}$, then for all $a \in \mathcal{M}(A)_{\mathrm{si}}$ and $\xi \in \mathcal{M}(\mathcal{E})$ we have $\pi(a) \circ \xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ and
$$|\pi(a) \circ \xi\rangle\rangle = |\pi(a)\rangle\rangle\gamma_{\mathcal{E}}(\xi) = (\pi \otimes \mathrm{id}_{H^*})(|a\rangle\rangle) \circ \gamma_{\mathcal{E}}(\xi).$$

(v) *If $\xi \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ and $\eta \in \mathcal{M}(\mathcal{E})$, then $\xi \circ \eta^* \in \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)_{\mathrm{si}}$ and*
$$|\xi \circ \eta^*\rangle\rangle = |\xi\rangle\rangle \circ \gamma_{\mathcal{E}}(\eta)^*.$$
In particular, if $\xi \in \mathcal{E}_{\mathrm{si}}$ and $\eta \in \mathcal{E}$, then $|\xi\rangle\langle\eta| \in \mathcal{K}(\mathcal{E})_{\mathrm{si}}$ and
$$||\xi\rangle\langle\eta|\rangle\rangle = |\xi\rangle\rangle \circ \gamma_{\mathcal{E}}(\eta)^*.$$

In (iv) *and* (v) *we are identifying* $\mathcal{M}(\mathcal{E}\otimes\mathcal{G})=\mathcal{L}(B\otimes\mathcal{G},\mathcal{E}\otimes\mathcal{G})\subseteq\mathcal{L}(B\otimes H,\mathcal{E}\otimes H)$ *and* (*hence*) *also* $\mathcal{L}(\mathcal{E}\otimes\mathcal{G},B\otimes\mathcal{G})\subseteq\mathcal{L}(\mathcal{E}\otimes H,B\otimes H)$*. In* (iv) *we are identifying* $\mathcal{L}(\mathcal{E},\mathcal{E}\otimes H)\cong\mathcal{L}(\mathcal{K}(\mathcal{E}),\mathcal{K}(\mathcal{E})\otimes H)$*. All this follows from Proposition* 2.1.11 *and Remark* 2.1.12(2)*.*

Proof. By Proposition 4.1.4, item (i) follows directly from Proposition 2.4.20(i). To prove (ii) we use Proposition 2.4.20(iv) and conclude that $\xi\cdot b\in\mathcal{M}(\mathcal{E})_{\mathrm{si}}$ and

$$\langle\!\langle\xi\cdot b|=(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(\gamma_{\mathcal{E}}(\xi\cdot b)^*)=(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(\gamma_B(b)^*\gamma_{\mathcal{E}}(\xi)^*)=\gamma_B(b)^*\langle\!\langle\xi|.$$

Item (iii) follows directly from Proposition 2.4.20(v) and the $\mathcal{G}$-equivariance of T: $\gamma_{\mathcal{F}}(T\xi)=(T\otimes 1)\gamma_{\mathcal{E}}(\xi)$. Items (iv) and (v) follow from Proposition 2.4.20(vi),(vii), respectively (for the general case in (iv) one also uses Proposition 3.3.1(ii)). □

Proposition 4.1.11. *Let* $(\mathcal{E},\gamma_{\mathcal{E}})_{(B,\gamma_B)}$ *be a Hilbert* B*-module* $\mathcal{G}$*-coaction. If we equip* $\mathcal{E}_{\mathrm{si}}$ *with the following norm*

$$\|\xi\|_{\mathrm{si}}:=\|\xi\|+\||\xi\rangle\!\rangle\|=\|\xi\|+\|\langle\!\langle\xi|\|=\|\langle\xi|\xi\rangle\|^{\frac{1}{2}}+\|\langle\!\langle\xi\,|\,\xi\rangle\!\rangle\|^{\frac{1}{2}},$$

then $\mathcal{E}_{\mathrm{si}}$ *is a Banach* $\mathcal{L}^{\mathcal{G}}(\mathcal{E}),B$*-bimodule, that is,* $\mathcal{E}_{\mathrm{si}}$ *is complete with respect to* $\|\cdot\|_{\mathrm{si}}$ *and for all* $\xi\in\mathcal{E}_{\mathrm{si}}$*,* $T\in\mathcal{L}^{\mathcal{G}}(\mathcal{E})$ *and* $b\in B$*, we have*

$$\|T\xi\|_{\mathrm{si}}\leq\|T\|\|\xi\|_{\mathrm{si}}\quad\textit{and}\quad\|\xi b\|_{\mathrm{si}}\leq\|\xi\|_{\mathrm{si}}\|b\|.$$

Proof. By Proposition 4.1.10(ii),(iii), the left $\mathcal{L}^{\mathcal{G}}(\mathcal{E})$-action and the right B-action are well-defined on $\mathcal{E}_{\mathrm{si}}$, and for all $\xi\in\mathcal{E}_{\mathrm{si}}$, $T\in\mathcal{L}^{\mathcal{G}}(\mathcal{E})$ and $b\in B$, we have

$$\||T\xi\rangle\!\rangle\|=\|T|\xi\rangle\!\rangle\|\leq\|T\|\||\xi\rangle\!\rangle\|\quad\text{and}\quad\||\xi b\rangle\!\rangle\|=\||\xi\rangle\!\rangle\gamma_B(b)\|\leq\||\xi\rangle\!\rangle\|\|b\|.$$

It follows that $\|T\xi\|_{\mathrm{si}}\leq\|T\|\|\xi\|_{\mathrm{si}}$ and $\|\xi b\|_{\mathrm{si}}\leq\|\xi\|_{\mathrm{si}}\|b\|$. To show that $\mathcal{E}_{\mathrm{si}}$ is complete with respect to $\|\cdot\|_{\mathrm{si}}$, take a Cauchy sequence (ξ_n) in $\mathcal{E}_{\mathrm{si}}$. This implies that (ξ_n) converges to some $\xi\in\mathcal{E}$ and $(\langle\!\langle\xi_n|)$ converges to some T in $\mathcal{L}(\mathcal{E},B\otimes H)$ (in the norm topology). If follows from Corollary 4.1.6 that $\xi\in\mathcal{E}_{\mathrm{si}}$ and $\langle\!\langle\xi|=T$. Therefore $\xi_n\to\xi$ in the norm $\|\cdot\|_{\mathrm{si}}$. □

Remark 4.1.12. Suppose that $\mathcal{G}$ is a compact quantum group. We already know (see Example 4.1.7) that in this case every Hilbert B-module $\mathcal{E}$ with a coaction of $\mathcal{G}$ is square-integrable. By Proposition 4.1.10(i), we have

$$\||\xi\rangle\!\rangle\|^2=\||\xi\rangle\!\rangle\langle\!\langle\xi|\|=\|(\mathrm{id}\otimes\varphi)(\gamma_{\mathcal{K}(\mathcal{E})}(|\xi\rangle\langle\xi|))\|\leq\|\varphi\|\|\xi\|$$

for all $\xi\in\mathcal{E}_{\mathrm{si}}=\mathcal{E}$. Thus $\|\xi\|\leq\|\xi\|_{\mathrm{si}}\leq(1+\|\varphi\|)\|\xi\|$. Therefore the si-norm and the norm on $\mathcal{E}$ are equivalent.

This has a converse in the sense that if $\mathcal{G}$ is any locally compact quantum group and if the norms $\|\cdot\|_{\mathrm{si}}$ and $\|\cdot\|_{\mathcal{E}}$ are equivalent on $\mathcal{E}_{\mathrm{si}}$, for any Hilbert $B,\mathcal{G}$-module, then $\mathcal{G}$ must be compact. In fact, this follows by considering $\mathcal{E}=\mathcal{G}$ as a Hilbert $\mathcal{G},\mathcal{G}$-module. To see this, let c be a positive constant with $\|\xi\|_{\mathrm{si}}\leq c\|\xi\|$ for all $\xi\in\mathcal{G}_{\mathrm{si}}=\mathcal{N}_\varphi^*$ (see Proposition 3.2.12). Then, by Propositions 3.2.10 and 3.2.12, we have

$$\|\varphi(\xi\xi^*)\|=\|E_1(\xi\xi^*)\|=\||\xi\rangle\!\rangle\langle\!\langle\xi|\|=\||\xi\rangle\!\rangle\|^2\leq c^2\|\xi\|^2,$$

and therefore φ is bounded, that is, $\mathcal{G}$ is compact.

4.2 The $\mathcal{G}$-equivariance of the bra-ket operators

Let $\mathcal{G}$ be a locally compact quantum group, let B be a C^*-algebra with a coaction γ_B of $\mathcal{G}$ and consider a Hilbert B-module $\mathcal{E}$ with a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$. As in the previous section, we fix a GNS-construction of the form (H, ι, Λ) for the left Haar weight of $\mathcal{G}$.

In this section we prove that the bra-ket operators $\langle\langle\xi| \in \mathcal{L}(\mathcal{E}, B \otimes H)$ and $|\xi\rangle\rangle \in \mathcal{L}(B \otimes H, \mathcal{E})$ are $\mathcal{G}$-equivariant, for any square-integrable element ξ in $\mathcal{E}$. In order to speak of $\mathcal{G}$-equivariance we have to define a $\mathcal{G}$-coaction on $B \otimes H$. The coaction that will work is the balanced tensor product of the coactions γ_B on B and a coaction γ_H on H which comes from the left regular corepresentation W of $\mathcal{G}$.

We define a coaction $\gamma_{B\otimes H}$ of $\mathcal{G}$ on $B \otimes H$ by

$$\gamma_{B\otimes H}(\zeta) := (1 \otimes \Sigma W)(\gamma_B \otimes \mathrm{id})(\zeta) = \Sigma_{23} W_{23}(\gamma_B \otimes \mathrm{id})(\zeta), \quad \zeta \in B \otimes H, \tag{4.2}$$

where $\Sigma : \mathcal{G} \otimes H \to H \otimes \mathcal{G}$ is the flip operator. Recall that $\hat{W}\Sigma = \Sigma W^*$, where $\hat{W}$ is the left regular corepresentation of the dual of $\mathcal{G}$. Thus the coaction defined above is the same already considered in Example 2.6.18(3). If we consider $B = \mathbb{C}$ with the trivial coaction of $\mathcal{G}$, then we get a coaction γ_H of $\mathcal{G}$ on H given by

$$\gamma_H(\eta) = \Sigma W(1 \otimes \eta) = \Sigma W \Sigma^*(\eta \otimes 1) = \hat{W}^*(\eta \otimes 1), \quad \eta \in H.$$

The coaction $\gamma_{B\otimes H}$ on $B \otimes H \cong H \otimes_{\mathbb{C}} B$ can also be seen as the balanced tensor product of γ_H and γ_B (see Definition 2.6.15).

Example 4.2.1. Let us analyze the case $\mathcal{G} = \mathcal{C}_0(G)$ for a locally compact group G. For this we fix a left Haar measure $\mathrm{d}s$ on G and consider the corresponding left Haar weight φ on $\mathcal{G}$ given by integration with respect to $\mathrm{d}s$. We also fix the canonical GNS-construction $(L^2(G), M, \Lambda)$, where $M : \mathcal{C}_0(G) \to \mathcal{L}\big(L^2(G)\big)$ is the multiplication representation and $\Lambda : \mathcal{N}_\varphi = \mathcal{C}_0(G) \cap L^2(G) \to L^2(G)$ is the inclusion map. In this case, W is the unitary in $\mathcal{L}\big(L^2(G) \otimes L^2(G)\big) \cong \mathcal{L}\big(L^2(G \times G)\big)$ given by

$$W\xi(s,t) = \xi(s, s^{-1}t), \quad \text{for all } \xi \in L^2(G \times G) \text{ and } s, t \in G.$$

The unitary $\hat{W} \in \mathcal{L}\big(L^2(G \times G)\big)$ is given by $\hat{W}\xi(s,t) = \xi(ts, t)$ and therefore $\hat{W}^*\xi(s,t) = \xi(t^{-1}s, t)$. From this it is easy to see that the coaction $\gamma_{L^2(G)}$ defined above corresponds to the action λ of G on $L^2(G)$ given by

$$\lambda_t(\eta)(s) = \eta(t^{-1}s), \quad \text{for all } \eta \in L^2(G) \text{ and } s, t \in G,$$

that is, $\gamma_{L^2(G)}$ corresponds to the left regular representation of G. If B is a C^*-algebra with a coaction γ_B of $\mathcal{G}$ corresponding to an action β of G on B, then the coaction $\gamma_{B\otimes L^2(G)}$ in (4.2) corresponds to the action $\beta \otimes \lambda$ of G on $B \otimes L^2(G) \cong L^2(G, B)$ given by

$$(\beta \otimes \lambda)_t(f)(s) = \beta_t\big(f(t^{-1}s)\big) \quad \text{for all } f \in \mathcal{C}_c(G, B) \subseteq L^2(G, B) \text{ and } s, t \in G.$$

If we have a Hilbert B-module $\mathcal{E}$ with a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$ corresponding to an action γ of G, then we already know (see Example 4.1.8) that for $\xi \in \mathcal{E}_{\mathrm{si}}$, the operators $|\xi\rangle\rangle$ and $\langle\langle\xi|$ are given by

$$|\xi\rangle\rangle f = \int_G \gamma_t(\xi) f(t)\,\mathrm{d}t \quad \text{and} \quad (\langle\langle\xi|\eta)(t) = \langle\gamma_t(\xi)|\eta\rangle_B$$

for all $f \in \mathcal{C}_c(G,B)$, $t \in G$ and $\eta \in \mathcal{E}$. In this case, it is an easy exercise to verify that $|\xi\rangle\!\rangle$ and $\langle\!\langle\xi|$ are G-equivariant, when we furnish $L^2(G,B)$ with the action $\beta \otimes \lambda$.

We want to prove that the operators $|\xi\rangle\!\rangle$ and $\langle\!\langle\xi|$ are $\mathcal{G}$-equivariant for any locally compact quantum group. But this is not so easy as in the group case. We need a preliminary result which is a generalization of Equation (3.2).

Proposition 4.2.2. *Let $\mathcal{E}$ be a Hilbert B-module. Then* $(\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\mathrm{id}_\mathcal{G}\otimes\varphi}$ *for all* $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$*, and we have*

$$W_{23}(\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_\mathcal{G} \otimes \Lambda)\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)\big) = \big((\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x)\big)_{13}.$$

Proof. First we prove the proposition in the case of a C^*-algebra A. Take a net (e_j) as in Lemma 2.4.10. By left invariance, $\Delta(e_j) \in \bar{\mathcal{N}}_{\mathrm{id}_\mathcal{G}\otimes\varphi}$ (see Proposition 3.2.12). Thus $(\mathrm{id}_A \otimes \Delta)(x)\big(1_A \otimes \Delta(e_j)\big) \in \bar{\mathcal{N}}_{\mathrm{id}_A\otimes\mathrm{id}_\mathcal{G}\otimes\varphi}$ for all j. Since (e_j) converges to 1 strictly, we have $(\mathrm{id}_A \otimes \Delta)(x)\big(1 \otimes \Delta(e_j)\big) \to (\mathrm{id}_A \otimes \Delta)(x)$ strictly. Now note that the relation $\Delta(a) = W^*(1 \otimes a)W$ yields

$$(\mathrm{id}_A \otimes \Delta)(x)(1_A \otimes W^*) = (1_A \otimes W^*)x_{13} = W_{23}^* x_{13}.$$

Thus

$$\begin{aligned}
(\mathrm{id}_A \otimes \mathrm{id}_\mathcal{G} \otimes \Lambda)\Big((\mathrm{id}_A \otimes \Delta)(x)\big(1_A \otimes \Delta(e_j)\big)\Big)&\\
&= (\mathrm{id}_A \otimes \Delta)(x)\Big(1_A \otimes (\mathrm{id}_\mathcal{G} \otimes \Lambda)\big(\Delta(e_j)\big)\Big)\\
&= (\mathrm{id}_A \otimes \Delta)(x)(1_A \otimes W^*)\big(1_A \otimes 1_\mathcal{G} \otimes \Lambda(e_j)\big)\\
&= W_{23}^* x_{13}\big(1_A \otimes \Lambda(e_j)\big)_{13}\\
&= W_{23}^*\big(x(1_A \otimes \Lambda(e_j)\big)_{13}\\
&= W_{23}^*\Big((\mathrm{id}_A \otimes \Lambda)\big(x(1 \otimes e_j)\big)\Big)_{13}\\
&= W_{23}^*\big(1_A \otimes 1_\mathcal{G} \otimes J\sigma_{\frac{i}{2}}(e_j)^* J\big)\big((\mathrm{id}_A \otimes \Lambda)(x)\big)_{13},
\end{aligned}$$

where the last equality follows from Proposition 2.4.13. The last expression above converges strongly in $\mathcal{L}(A \otimes \mathcal{G}, A \otimes \mathcal{G} \otimes H)$ to $W_{23}^*\big((\mathrm{id}_A \otimes \Lambda)(x)\big)_{13}$. By the closedness of $\mathrm{id}_A \otimes \mathrm{id}_\mathcal{G} \otimes \Lambda$ (Proposition 2.4.6(v)) we get that $(\mathrm{id}_A \otimes \Delta)(x) \in \bar{\mathcal{N}}_{\mathrm{id}_A\otimes\mathrm{id}_\mathcal{G}\otimes\varphi}$ and

$$(\mathrm{id}_A \otimes \mathrm{id}_\mathcal{G} \otimes \Lambda)\big((\mathrm{id}_A \otimes \Delta)(x)\big) = W_{23}^*\big((\mathrm{id}_A \otimes \Lambda)(x)\big)_{13}.$$

We have just proved the proposition for an arbitrary C^*-algebra. Now we use this special case to prove the general in assertion. If $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ then $x^*x \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi}$. Note that if $a \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$, then (writing $a = b^*b$, where $b \in \bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi}$) it follows from what we have proved above that $(\mathrm{id}_A \otimes \Delta)(a) \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\mathrm{id}_\mathcal{G}\otimes\varphi}$. Applying this to $A = \mathcal{K}(\mathcal{E})$ and $a = x^*x$, we get that

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)^*(\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x) = (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Delta)(x^*x) \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\mathrm{id}_\mathcal{G}\otimes\varphi}.$$

Hence $(\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi}$ (by Proposition 2.4.19(i)). By Proposition 2.4.20(iii), we have $(\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)(\eta \otimes 1_{\mathcal{G}} \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi}$, for every $\eta \in \mathcal{E}$, and

$$\begin{aligned}
&(\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \Lambda)\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)\big)(\eta \otimes 1_{\mathcal{G}}) \\
&= (\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{G}} \otimes \Lambda)\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)(\eta \otimes 1_{\mathcal{G}} \otimes 1_{\mathcal{G}})\big) \\
&= (\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{G}} \otimes \Lambda)\Big((\mathrm{id}_B \otimes \Delta)\big(x(\eta \otimes 1_{\mathcal{G}})\big)\Big).
\end{aligned}$$

Since $x(\eta \otimes 1_{\mathcal{G}}) \in \bar{\mathcal{N}}_{\mathrm{id}_B \otimes \varphi}$ (again by Proposition 2.4.20(iii)), we can now apply the first part to the C^*-algebra B. Therefore the above equals

$$\begin{aligned}
W_{23}^*\Big((\mathrm{id}_B \otimes \Lambda)\big(x(\eta \otimes 1_{\mathcal{G}})\big)\Big)_{13} &= W_{23}^*\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x)\eta\big)_{13} \\
&= W_{23}^*\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x)\big)_{13}(\eta \otimes 1_{\mathcal{G}}).
\end{aligned}$$

The result now follows because η is arbitrary. □

As a consequence, we get the following generalization of the left invariance of the Haar weight φ.

Corollary 4.2.3. *Let $\mathcal{E}$ be a Hilbert B-module. Then $(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Delta)(x^*x) \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi}$ for all $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$, and we have*

$$(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi)\big((\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Delta)(x^*x)\big) = (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(x^*x) \otimes 1_{\mathcal{G}}.$$

Proof. We have $(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Delta)(x^*x) = (\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)^*(\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)$. By Propositions 4.2.2 and 2.4.20(i), we get that $(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Delta)(x^*x) \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi}$ and

$$\begin{aligned}
&(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi)\big((\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Delta)(x^*x)\big) \\
&\qquad = (\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \Lambda)\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)\big)^*(\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \Lambda)\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)\big) \\
&\qquad = \big(W_{23}^*(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x)_{13}\big)^*\big(W_{23}^*(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x)_{13}\big) \\
&\qquad = \big((\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x)^*(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x)\big)_{13} \\
&\qquad = \big((\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(x^*x)\big)_{13} \\
&\qquad = (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(x^*x) \otimes 1_{\mathcal{G}}.
\end{aligned}$$

□

Of course, we also have analogous results for the right invariant Haar weight ψ of $\mathcal{G}$ with a GNS-construction of the form (Γ, ι, H) and the right regular corepresentation V. For reference we enunciate them here.

Proposition 4.2.4. *Let $\mathcal{E}$ be a Hilbert B-module.*

(i) *For all $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \psi}$, we have $(\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \psi \otimes \mathrm{id}_{\mathcal{G}}}$ and*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Gamma \otimes \mathrm{id}_{\mathcal{G}})\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)\big) = V_{23}\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Gamma)(x)\big)_{12}.$$

(ii) *For all* $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\psi}$, *we have* $(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Delta)(x^*x) \in \bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\psi\otimes\mathrm{id}_{\mathcal{G}}}$ *and*

$$(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \psi \otimes \mathrm{id}_{\mathcal{G}})\big((\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \Delta)(x^*x)\big) = (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \psi)(x^*x) \otimes 1_{\mathcal{G}}.$$

Note that (i) above is a generalization of the following equation

$$V\big(\Gamma(a) \otimes 1\big) = (\Gamma \otimes \mathrm{id})\big(\Delta(a)\big) \quad \text{for all } a \in \bar{\mathcal{N}}_{\psi}. \tag{4.3}$$

which is equivalent to

$$(\mathrm{id} \otimes \omega)(V)\Gamma(a) = \Gamma\big((\mathrm{id} \otimes \omega)\Delta(a)\big) \quad \text{for all } a \in \bar{\mathcal{N}}_{\psi} \text{ and } \omega \in \mathcal{L}(H)_*. \tag{4.4}$$

And (ii) is a generalization of the right invariance of ψ.

As a consequence of Corollary 4.2.3, we get the following result.

Corollary 4.2.5. *Let A be a C^*-algebra with a coaction of $\mathcal{G}$. If $a \in \mathcal{M}(A)_{\mathrm{i}}$, then the element $E_1(a) = (\mathrm{id}_A \otimes \varphi)\big(\gamma_A(a)\big) \in \mathcal{M}(A)$ belongs to the fixed point algebra*

$$\mathcal{M}_1(A) = \{b \in \mathcal{M}(A) : \gamma_A(b) = b \otimes 1_{\mathcal{G}}\}.$$

Proof. Since $\mathcal{M}(A)_{\mathrm{i}} = \operatorname{span} \mathcal{M}(A)_{\mathrm{si}}\mathcal{M}(A)^*_{\mathrm{si}}$, we may assume that $a = b^*b$, where b belongs to $\mathcal{M}(A)^*_{\mathrm{si}}$. By definition of $\mathcal{M}(A)_{\mathrm{si}}$ we have $\gamma_A(b) \in \bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi}$. Corollary 4.2.3 implies that $(\mathrm{id}_A \otimes \Delta)\big(\gamma_A(a)\big) \in \mathcal{M}^+_{\mathrm{id}_A\otimes\mathrm{id}_{\mathcal{G}}\otimes\varphi}$ and

$$(\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi)\big((\mathrm{id}_A \otimes \Delta)\gamma_A(a)\big) = (\mathrm{id}_A \otimes \varphi)\big(\gamma_A(a)\big) \otimes 1_{\mathcal{G}}.$$

Therefore, by Lemma 2.4.8,

$$\begin{aligned} \gamma_A\big((\mathrm{id}_A \otimes \varphi)\gamma_A(a)\big) &= (\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi)\big((\gamma_A \otimes \mathrm{id}_{\mathcal{G}})\gamma_A(a)\big) \\ &= (\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi)\big((\mathrm{id}_A \otimes \Delta)\gamma_A(a)\big) \\ &= (\mathrm{id}_A \otimes \varphi)\big(\gamma_A(a)\big) \otimes 1_{\mathcal{G}}. \end{aligned}$$ □

Corollary 4.2.6. *Let $\mathcal{E}$ be a Hilbert B-module with a coaction of $\mathcal{G}$. Then for all $\xi, \eta \in \mathcal{M}(\mathcal{E})_{\mathrm{si}}$ we have $|\xi\rangle\!\rangle\langle\!\langle\eta| \in \mathcal{M}_1\big(\mathcal{K}(\mathcal{E})\big)$.*

Proof. This follows from Corollary 4.2.5 and Proposition 4.1.10(i). □

Finally, we prove that the bra-ket operators are $\mathcal{G}$-equivariant.

Proposition 4.2.7. *Let $\mathcal{E}$ be a Hilbert B-module with a coaction of $\mathcal{G}$. Then the operators $|\xi\rangle\!\rangle \in \mathcal{L}(B \otimes H, \mathcal{E})$ and $\langle\!\langle\xi| \in \mathcal{L}(\mathcal{E}, B \otimes H)$ are $\mathcal{G}$-equivariant for all $\xi \in \mathcal{E}_{\mathrm{si}}$, when we define on $B \otimes H$ the coaction $\gamma_{B\otimes H}$ of $\mathcal{G}$ given by Equation* (4.2).

Proof. Since $|\xi\rangle\!\rangle = \langle\!\langle\xi|^*$, it is enough to prove that $\langle\!\langle\xi|$ is $\mathcal{G}$-equivariant. By Proposition 2.2.4, we have the relation

$$(\gamma_B \otimes \mathrm{id})(x^*y) = (\gamma_{\mathcal{E}} \otimes \mathrm{id})(x)^*(\gamma_{\mathcal{E}} \otimes \mathrm{id})(y) \quad \text{for all } x, y \in \mathcal{M}(\mathcal{E} \otimes \mathcal{G}).$$

In particular, for all $\eta \in \mathcal{E}$,

$$\begin{aligned}(\gamma_B \otimes \mathrm{id}_{\mathcal{G}})\big(\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1_{\mathcal{G}})\big) &= \big((\gamma_{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{G}})\gamma_{\mathcal{E}}(\xi)\big)^*\big(\gamma_{\mathcal{E}}(\eta) \otimes 1_{\mathcal{G}}\big)\\ &= \big((\mathrm{id}_{\mathcal{E}} \otimes \Delta)\gamma_{\mathcal{E}}(\xi)\big)^*\big(\gamma_{\mathcal{E}}(\eta) \otimes 1_{\mathcal{G}}\big)\\ &= (\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(\gamma_{\mathcal{E}}(\xi)^*)(\gamma_{\mathcal{E}}(\eta) \otimes 1_{\mathcal{G}}).\end{aligned}$$

Combining this with Lemma 2.4.8(ii) and Propositions 2.4.20(iii) and 4.2.2 we get

$$\begin{aligned}\gamma_{B\otimes H}\big(\langle\!\langle\xi|\eta\big) &= \gamma_{B\otimes H}\Big((\mathrm{id}_B \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1_{\mathcal{G}})\big)\Big)\\ &= \Sigma_{23}W_{23}(\gamma_B \otimes \mathrm{id}_H)\Big((\mathrm{id}_B \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1_{\mathcal{G}})\big)\Big)\\ &= \Sigma_{23}W_{23}(\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{G}} \otimes \Lambda)\Big((\gamma_B \otimes \mathrm{id}_{\mathcal{G}})\big(\gamma_{\mathcal{E}}(\xi)^*(\eta \otimes 1_{\mathcal{G}})\big)\Big)\\ &= \Sigma_{23}W_{23}(\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{G}} \otimes \Lambda)\Big(\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(\gamma_{\mathcal{E}}(\xi)^*)\big)(\gamma_{\mathcal{E}}(\eta) \otimes 1_{\mathcal{G}})\Big)\\ &= \Sigma_{23}W_{23}(\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \Lambda)\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(\gamma_{\mathcal{E}}(\xi)^*)\big)\gamma_{\mathcal{E}}(\eta)\\ &= \Sigma_{23}\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(\gamma_{\mathcal{E}}(\xi)^*)\big)_{13}\gamma_{\mathcal{E}}(\eta)\\ &= \Sigma_{23}\langle\!\langle\xi|_{13}\gamma_{\mathcal{E}}(\eta)\\ &= (\langle\!\langle\xi| \otimes 1_{\mathcal{G}})\gamma_{\mathcal{E}}(\eta).\end{aligned}$$

□

4.3 The L^1-action on square-integrable elements

Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. If $\xi \in \mathcal{E}_{\mathrm{si}}$ and $\omega \in L^1(\mathcal{G})$, then it is natural to ask whether $\omega * \xi \in \mathcal{E}_{\mathrm{si}}$. However, if $\mathcal{G}$ is not unimodular, that is, if the modular element is not trivial, then some problems appear. Let us analyze the group case $\mathcal{G} = \mathcal{C}_0(G)$, where G is some locally compact group. Suppose that $\gamma_{\mathcal{E}}$ corresponds to an action γ of G on $\mathcal{E}$. Then for a function $\omega \in L^1(G)$, the element $\omega * \xi \in \mathcal{E}$ is given by

$$\omega * \xi = \int_G \gamma_t(\xi)\omega(t)\,\mathrm{d}t.$$

Thus, for all $f \in \mathcal{C}_c(G, B)$, we have

$$\begin{aligned}|\omega * \xi\rangle\!\rangle f &= \int_G\int_G \gamma_{st}(\xi)f(s)\omega(t)\,\mathrm{d}t\,\mathrm{d}s\\ &= \int_G\int_G \gamma_t(\xi)f(s)\omega(s^{-1}t)\,\mathrm{d}t\,\mathrm{d}s\\ &= |\xi\rangle\!\rangle(f * \omega),\end{aligned}$$

where $(f * \omega)(t) := \int_G f(s)\omega(s^{-1}t)\,\mathrm{d}s = \int_G f(ts^{-1})\delta_G(s)^{-1}\omega(s)\,\mathrm{d}s$, where δ_G denotes the modular function of G. If ω satisfies $\int_G \delta_G(t)^{-\frac{1}{2}}|\omega(t)|\,\mathrm{d}t < \infty$, then the map $\rho_\omega := [g \mapsto g * \omega]$ defines a bounded operator on $L^2(G)$ with $\|\rho_\omega\| \leq \int_G \delta_G(t)^{-\frac{1}{2}}|\omega(t)|\,\mathrm{d}t$ ([29, Theorem 20.13]). Note that $f * \omega = (1_B \otimes \rho_\omega)(f)$. Thus, if $\xi \in \mathcal{E}_{\mathrm{si}}$ and $\omega \in L^1(G)$ satisfies $\delta_G^{-\frac{1}{2}}\omega \in L^1(G)$, then $\omega * \xi \in \mathcal{E}_{\mathrm{si}}$ and

$$|\omega * \xi\rangle\!\rangle = |\xi\rangle\!\rangle(1_B \otimes \rho_\omega).$$

The hypothesis $\delta_G^{-\frac{1}{2}}\omega \in L^1(G)$ is essential here in order to define the operator ρ_ω. In fact, if G is not unimodular, then there are functions $\omega \in L^1(G)$ and $g \in L^2(G)$ such that $g * \omega \notin L^2(G)$ (see [29, 20.34]).

In order to generalize the results above for a general locally compact quantum group $\mathcal{G}$, we shall need the modular element. As usual the proof in the quantum setting is much more technical. Let us recall that the modular element of $\mathcal{G}$, denoted by δ, is a strictly positive operator affiliated[2] with $\mathcal{G}$ such that $\sigma_t(\delta) = \nu^t\delta$ for all $t \in \mathbb{R}$ and $\psi = \varphi_\delta$ (see [41]), where $\{\sigma_s\}_{s\in\mathbb{R}}$ is the modular group of φ and ν is the scaling constant of $\mathcal{G}$. Roughly speaking, $\psi = \phi_\delta$ means that $\psi(\,\cdot\,) = \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$ and we can define a GNS-construction for ψ of the form (H, ι, Γ) from the GNS-construction (H, ι, Λ) for φ satisfying $\Gamma(\,\cdot\,) = \Lambda(\,\cdot\,\delta^{\frac{1}{2}})$. In order to be more precise, for each $n \in \mathbb{N}$, we define

$$e_n := \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2t^2)\delta^{\mathrm{i}t}\,\mathrm{d}t. \tag{4.5}$$

These elements behave very well with the modular element. Before we state some basic properties, we introduce some terminology.

Definition 4.3.1. Let A be a C^*-algebra and let T be an affiliated element with A.

(i) A multiplier $a \in \mathcal{M}(A)$ is called a *left multiplier* of T if there is $b \in \mathcal{M}(A)$ such that $aT(c) = bc$ for all $c \in \mathcal{D}(T)$. In this case we put $aT := b$.

(ii) A multiplier $a \in \mathcal{M}(A)$ is called *right multiplier* of T if $aA \subseteq \mathcal{D}(T)$. In this case, there is a unique element $b \in \mathcal{M}(A)$ such that $T(ac) = bc$ for all $c \in A$ and we write $Ta = b$.

The following properties hold ([38, Proposition 8.2]):

1. For each $n \in \mathbb{N}$, the element $e_n \in \mathcal{M}(\mathcal{G})$ is strictly analytic with respect to σ.
2. For every $z \in \mathbb{C}$, the sequence $(\sigma_z(e_n))$ is bounded and converges strictly to 1.

[2] A positive operator is called *strictly positive* if it has dense range. Self-adjointness is considered as a part of the definition of positivity. For the notion of elements affiliated with a C^*-algebra $\mathcal{G}$ we refer to [3, 37, 43, 79]. Roughly speaking, these are "unbounded multipliers" of $\mathcal{G}$. If $\mathcal{G}$ is considered as a concrete (nondegenerate) C^*-subalgebra of operators on a Hilbert space H, then any element affiliated with $\mathcal{G}$ can be regarded as a closed operator acting on H (see [79, Example 4]).

3. For any $n \in \mathbb{N}$ and $z \in \mathbb{C}$, e_n is a left and right multiplier of δ^z. Moreover, we have $e_n \delta^z = \delta^z e_n$.
4. For any $n \in \mathbb{N}$ and $y, z \in \mathbb{C}$, $\sigma_y(e_n)$ is a left and right multiplier of δ^z and $\sigma_y(e_n)\delta^z = \delta^z \sigma_y(e_n)$.

Let $\{\sigma'_t\}_{t\in\mathbb{R}}$ be the modular group of ψ. Since $\sigma'_t(x) = \delta^{\mathrm{i}t}\sigma_t(x)\delta^{-\mathrm{i}t}$ for $t \in \mathbb{R}$, it follows that $\sigma_t(\delta^z e_n) = \sigma'_t(\delta^z e_n)$ for all $n \in \mathbb{N}$, $z \in \mathbb{C}$. Thus the same properties above hold if we replace σ by σ'.

The main fact is the following (see [38, Lemma 8.5]):

Lemma 4.3.2. *If $x \in \bar{\mathcal{N}}_\varphi$ and $z \in \mathbb{C}$, then $x(\delta^z e_n) \in \bar{\mathcal{N}}_\psi$ for all $n \in \mathbb{N}$, and*

$$\Gamma\big(x(\delta^z e_n)\big) = \Lambda\big(x(\delta^{z+\frac{1}{2}} e_n)\big).$$

We can generalize this as follows:

Proposition 4.3.3. *Let $\mathcal{E}$ be a Hilbert B-module. If $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$, then $x(1 \otimes \delta^z e_n) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\psi}$ for all $n \in \mathbb{N}$, $z \in \mathbb{C}$, and*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Gamma)\big(x(1 \otimes \delta^z e_n)\big) = (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(x(1 \otimes \delta^{z+\frac{1}{2}} e_n)\big).$$

Proof. Take any $\theta \in \mathcal{K}(\mathcal{E})^*_+$. Then $(\theta \otimes \mathrm{id}_{\mathcal{G}})(x^*x) \in \bar{\mathcal{M}}_\varphi$ and hence Lemma 4.3.2 yields

$$(\theta \otimes \mathrm{id}_{\mathcal{G}})\big((1 \otimes \delta^z e_n)^* x^* x (1 \otimes \delta^z e_n)\big) = \delta^{\bar{z}} e_n (\theta \otimes \mathrm{id}_{\mathcal{G}})(x^*x)\delta^z e_n \in \bar{\mathcal{M}}_\psi$$

and

$$\begin{aligned}
\psi\Big((\theta \otimes \mathrm{id}_{\mathcal{G}})\big((1 \otimes \delta^z e_n)^* x^* x (1\otimes\delta^z e_n)\big)\Big) &= \Big\|\Gamma\big((\theta \otimes \mathrm{id}_{\mathcal{G}})(x^*x)^{\frac{1}{2}}\delta^z e_n\big)\Big\|^2 \\
&= \Big\|\Lambda\big((\theta \otimes \mathrm{id}_{\mathcal{G}})(x^*x)^{\frac{1}{2}}\delta^{z+\frac{1}{2}} e_n\big)\Big\|^2 \\
&= \varphi\Big((\theta \otimes \mathrm{id}_{\mathcal{G}})\big((1 \otimes \delta^{\bar{z}+\frac{1}{2}} e_n) x^* x (1 \otimes \delta^{z+\frac{1}{2}} e_n)\big)\Big) \\
&= \theta\Big((\mathrm{id}_{\mathcal{G}} \otimes \varphi)\big((1 \otimes \delta^{\bar{z}+\frac{1}{2}} e_n) x^* x (1 \otimes \delta^{z+\frac{1}{2}} e_n)\big)\Big).
\end{aligned}$$

It follows from Proposition 2.4.5 that $(1 \otimes \delta^z e_n)^* x^* x (1 \otimes \delta^z e_n) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\psi}$, that is, $x(1 \otimes \delta^z e_n) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\psi}$. Now take any $\omega \in B^*$ and $\eta \in \mathcal{E}$. Then (using again Lemma 4.3.2)

$$\begin{aligned}
(\omega \otimes \mathrm{id}_H)\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \Gamma)\big(x(1 \otimes \delta^z e_n)\big)\eta\Big) &= (\omega \otimes \mathrm{id}_H)\Big((\mathrm{id}_B \otimes \Gamma)\big(x(\eta \otimes \delta^z e_n)\big)\Big) \\
&= \Gamma\Big((\omega \otimes \mathrm{id}_{\mathcal{G}})\big(x(\eta \otimes \delta^z e_n)\big)\Big) \\
&= \Gamma\Big((\omega \otimes \mathrm{id}_{\mathcal{G}})\big(x(\eta \otimes 1)\big)\delta^z e_n\Big) \\
&= \Lambda\Big((\omega \otimes \mathrm{id}_{\mathcal{G}})\big(x(\eta \otimes 1)\big)\delta^{z+\frac{1}{2}} e_n\Big) \\
&= \Lambda\Big((\omega \otimes \mathrm{id}_{\mathcal{G}})\big(x(\eta \otimes \delta^{z+\frac{1}{2}} e_n)\big)\Big) \\
&= (\omega \otimes \mathrm{id}_H)\Big((\mathrm{id}_B \otimes \Lambda)\big(x(\eta \otimes \delta^{z+\frac{1}{2}} e_n)\big)\Big) \\
&= (\omega \otimes \mathrm{id}_H)\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(x(1 \otimes \delta^{z+\frac{1}{2}} e_n)\big)\eta\Big).
\end{aligned}$$

Since ω and η are arbitrary, the result now follows. $\square$

Similarly, since $\varphi = \psi_{\delta^{-1}}$, we also have $x(1 \otimes \delta^z e_n) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ for all $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \psi}$, and

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(x(1 \otimes \delta^z e_n)\big) = (\mathrm{id}_{\mathcal{E}^*} \otimes \Gamma)\big(x(1 \otimes \delta^{z-\frac{1}{2}} e_n)\big). \tag{4.6}$$

Corollary 4.3.4. *Let $\mathcal{E}$ be a Hilbert B-module. If $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ and $x(1 \otimes \delta^{-\frac{1}{2}})$ is bounded (that is, x is a left multiplier of $1 \otimes \delta^{-\frac{1}{2}}$), then $x(1 \otimes \delta^{-\frac{1}{2}}) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \psi}$ and*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Gamma)\big(x(1 \otimes \delta^{-\frac{1}{2}})\big) = (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x).$$

Proof. By 4.3.3 we have $x(1 \otimes \delta^{-\frac{1}{2}} e_n) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \psi}$ and (using Proposition 2.4.22)

$$\begin{aligned}(\mathrm{id}_{\mathcal{E}^*} \otimes \Gamma)\big(x(1 \otimes \delta^{-\frac{1}{2}} e_n)\big) &= (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(x(1 \otimes e_n)\big) \\ &= (1 \otimes J\sigma_{\frac{i}{2}}(e_n)^* J)(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x) \to (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x) \quad (\mathcal{K}\text{-strongly}).\end{aligned}$$

Now because $x(1 \otimes \delta^{-\frac{1}{2}})$ is bounded, we also have that

$$x(1 \otimes \delta^{-\frac{1}{2}} e_n) \to x(1 \otimes \delta^{-\frac{1}{2}}) \quad \text{(bi-strictly)}.$$

The assertion now follows from Proposition 2.4.20(viii). $\square$

Analogously, using Equation (4.6), one proves the following result.

Corollary 4.3.5. *If $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \psi}$ and $x(1 \otimes \delta^{\frac{1}{2}})$ is bounded, then $x(1 \otimes \delta^{\frac{1}{2}}) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$ and*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(x(1 \otimes \delta^{\frac{1}{2}})\big) = (\mathrm{id}_{\mathcal{E}^*} \otimes \Gamma)(x).$$

We shall need a generalization of [73, Proposition 1.9.13]. This result says that for all $a \in \mathcal{N}_\varphi$, $u \in \mathcal{D}(\delta^{\frac{1}{2}})$ and $v \in H$, we have $(\mathrm{id} \otimes \omega_{u,v})\Delta(a) \in \mathcal{N}_\varphi$ and

$$\Lambda\big((\mathrm{id} \otimes \omega_{u,v})\Delta(a)\big) = (\mathrm{id} \otimes \omega_{\delta^{\frac{1}{2}} u,v})(V)\Lambda(a). \tag{4.7}$$

To generalize this formula we first need a lemma:

Lemma 4.3.6. *Let $x \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G})$, $u \in \mathcal{D}(\delta^{\frac{1}{2}})$ and $v \in H$. Then $(\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{u,v})\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)\big(x(1 \otimes e_n)\big)\Big)$ is a left multiplier of $1 \otimes \delta^{-\frac{1}{2}}$ and*

$$\begin{aligned}(\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{u,v})&\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)\big(x(1 \otimes e_n)\big)\Big)(1 \otimes \delta^{-\frac{1}{2}}) \\ &= (\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{\delta^{\frac{1}{2}} u,v})\Big(\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)\big(x(1 \otimes \delta^{-\frac{1}{2}} e_n)\big)\Big).\end{aligned}$$

Proof. For the case $\mathcal{E} = \mathbb{C}$ this follows from the relation $\Delta(\delta) = \delta \otimes \delta$ (see the proof of [73, Proposition 1.9.13]). We use this special case to prove the general one. That is, we use that for any $x \in \mathcal{M}(\mathcal{G})$ we have

$$(\mathrm{id}_{\mathcal{G}} \otimes \omega_{u,v})\big(\Delta(xe_n)\big)\delta^{-\frac{1}{2}} = (\mathrm{id}_{\mathcal{G}} \otimes \omega_{\delta^{\frac{1}{2}}u,v})\big(\Delta(x\delta^{-\frac{1}{2}}e_n)\big).$$

Take $\xi \in \mathcal{E}$, $\eta \in \mathcal{D}(\delta^{-\frac{1}{2}})$ and $\theta \in B^*$. Then

$$\begin{aligned}
&(\theta \otimes \mathrm{id}_{\mathcal{G}})\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{u,v})\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)\big(x(1 \otimes e_n)\big)\Big)(\xi \otimes \delta^{-\frac{1}{2}}\eta)\Big) \\
&= (\theta \otimes \mathrm{id}_{\mathcal{G}})\Big((\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{u,v})\Big((\mathrm{id}_B \otimes \Delta)\big(x(\xi \otimes e_n)\big)\Big)\Big)\delta^{-\frac{1}{2}}\eta \\
&= (\mathrm{id}_{\mathcal{G}} \otimes \omega_{u,v})\Big(\Delta\Big((\theta \otimes \mathrm{id}_{\mathcal{G}})\big(x(\xi \otimes 1)\big)e_n\Big)\Big)\delta^{-\frac{1}{2}}\eta \\
&= (\mathrm{id}_{\mathcal{G}} \otimes \omega_{\delta^{\frac{1}{2}}u,v})\Big(\Delta\Big((\theta \otimes \mathrm{id}_{\mathcal{G}})\big(x(\xi \otimes 1)\big)\delta^{-\frac{1}{2}}e_n\Big)\Big)\eta \\
&= (\theta \otimes \mathrm{id}_{\mathcal{G}})\Big((\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{\delta^{\frac{1}{2}}u,v})\Big((\mathrm{id}_B \otimes \Delta)\big(x(\xi \otimes \delta^{-\frac{1}{2}}e_n)\big)\Big)\Big)\eta \\
&= (\theta \otimes \mathrm{id}_{\mathcal{G}})\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{\delta^{\frac{1}{2}}u,v})\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)\big(x(1 \otimes \delta^{-\frac{1}{2}}e_n)\big)\Big)(\xi \otimes \eta)\Big).
\end{aligned}$$

Since ξ, η, θ are arbitrary the result now follows. $\square$

Now we generalize Equation (4.7).

Proposition 4.3.7. *Let $x \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$, $u \in \mathcal{D}(\delta^{\frac{1}{2}})$ and $v \in H$. Then*

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{u,v})\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)\big) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi} \quad \textit{and}$$

$$\begin{aligned}
&(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{u,v})\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)(x)\big)\Big) \\
&\qquad\qquad = \big(1_B \otimes (\mathrm{id}_{\mathcal{K}(H)} \otimes \omega_{\delta^{\frac{1}{2}}u,v})(V)\big)(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x).
\end{aligned}$$

Proof. The proof is very similar to the one in [73, Proposition 1.9.13]. Since $x(1 \otimes \delta^{-\frac{1}{2}}e_n) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \psi}$, Proposition 4.2.4 yields $(\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)\big(x(1 \otimes \delta^{-\frac{1}{2}}e_n)\big) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \psi \otimes \mathrm{id}_{\mathcal{G}}}$ and hence

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{\delta^{\frac{1}{2}}u,v})\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)\big(x(1 \otimes \delta^{-\frac{1}{2}}e_n)\big)\Big) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \psi}.$$

Corollary 4.3.5 and Lemma 4.3.6 imply that

$$(\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{u,v})\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)\big(x(1 \otimes e_n)\big)\Big) \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*} \otimes \varphi}$$

and (using Proposition 4.2.4(i))

$$\begin{aligned}
(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)&\Big((\mathrm{id}_{\mathcal{E}^*}\otimes\mathrm{id}_{\mathcal{G}}\otimes\omega_{u,v})\big((\mathrm{id}_{\mathcal{E}^*}\otimes\Delta)(x(1\otimes e_n))\big)\Big)\\
&=(\mathrm{id}_{\mathcal{E}^*}\otimes\Gamma)\Big((\mathrm{id}_{\mathcal{E}^*}\otimes\mathrm{id}_{\mathcal{G}}\otimes\omega_{\delta^{\frac{1}{2}}u,v})\big((\mathrm{id}_{\mathcal{E}^*}\otimes\Delta)(x(1\otimes\delta^{-\frac{1}{2}}e_n))\big)\Big)\\
&=(\mathrm{id}_{\mathcal{E}^*}\otimes\mathrm{id}_H\otimes\omega_{\delta^{\frac{1}{2}}u,v})\Big((\mathrm{id}_{\mathcal{E}^*}\otimes\Gamma\otimes\mathrm{id}_{\mathcal{G}})\big((\mathrm{id}_{\mathcal{E}^*}\otimes\Delta)(x(1\otimes\delta^{-\frac{1}{2}}e_n))\big)\Big)\\
&=(\mathrm{id}_{\mathcal{E}^*}\otimes\mathrm{id}_H\otimes\omega_{\delta^{\frac{1}{2}}u,v})\Big(V_{23}(\mathrm{id}_{\mathcal{E}^*}\otimes\Gamma)\big(x(1\otimes\delta^{-\frac{1}{2}}e_n)\big)_{12}\Big)\\
&=(\mathrm{id}_{\mathcal{E}^*}\otimes\mathrm{id}_H\otimes\omega_{\delta^{\frac{1}{2}}u,v})\Big(V_{23}(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)\big(x(1\otimes e_n)\big)_{12}\Big)\\
&=\big(1_B\otimes(\mathrm{id}_{\mathcal{K}(H)}\otimes\omega_{\delta^{\frac{1}{2}}u,v})(V)\big)(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)(x). \qquad\square
\end{aligned}$$

We are now ready to prove the main result of this section. Define

$$L^1_{00}(\mathcal{G}):=\mathrm{span}\{\omega_{u,v}:u\in H,v\in\mathcal{D}(\delta^{\frac{1}{2}})\}$$

Note that $L^1_{00}(\mathcal{G})$ is a dense subspace of $L^1(\mathcal{G})=\overline{\mathrm{span}}\{\omega_{u,v}:u,v\in H\}$. Moreover, by Equation (2.17), we have $L^1(\mathcal{G})=\{\omega_{u,v}:u,v\in H\}$. Thus the difference between $L^1_{00}(\mathcal{G})$ and $L^1(\mathcal{G})$ is essentially the same as the difference between $\mathcal{D}(\delta^{\frac{1}{2}})$ and H. In particular, if $\mathcal{G}$ is unimodular, then $L^1_{00}(\mathcal{G})$ is equal to $L^1(\mathcal{G})$. We also define a map

$$\rho:L^1_{00}(\mathcal{G})\to\mathcal{L}(H),\quad \rho_{\omega_{u,v}}:=(\mathrm{id}\otimes\omega_{u,\delta^{\frac{1}{2}}v})(V^*)$$

for all $u,\in H$ and $v\in\mathcal{D}(\delta^{\frac{1}{2}})$, and extended linearly to $L^1_{00}(\mathcal{G})$. Note that if $\mathcal{G}$ is unimodular, then $\rho_\omega=(\mathrm{id}\otimes\omega)(V^*)$ for all $\omega\in L^1(\mathcal{G})$.

Proposition 4.3.8. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. Then, for all $\xi\in\mathcal{E}_{\mathrm{si}}$ and $\omega\in L^1_{00}(\mathcal{G})$, we have $\omega*\xi\in\mathcal{E}_{\mathrm{si}}$ and*

$$|\omega*\xi\rangle\!\rangle=|\xi\rangle\!\rangle(1_B\otimes\rho_\omega).$$

In particular, $\|\omega*\xi\|_{\mathrm{si}}\leq\|\omega\|_\rho\|\xi\|_{\mathrm{si}}$, where $\|\omega\|_\rho:=\max\{\|\omega\|,\|\rho_\omega\|\}$. Here $\|\omega\|$ denotes the norm of ω in $L^1(\mathcal{G})$ and $\|\rho_\omega\|$ denotes the norm of the operator $\rho_\omega\in\mathcal{L}(H)$.

Proof. We may assume that $\omega=\omega_{u,v}$, for $u\in H$ and $v\in\mathcal{D}(\delta^{\frac{1}{2}})$. We have

$$\begin{aligned}
\gamma_{\mathcal{E}}(\omega*\xi)&=\gamma_{\mathcal{E}}\big((\mathrm{id}_{\mathcal{E}}\otimes\omega)\gamma_{\mathcal{E}}(\xi)\big)\\
&=(\mathrm{id}_{\mathcal{E}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\omega)\big((\gamma_{\mathcal{E}}\otimes\mathrm{id}_{\mathcal{G}})\gamma_{\mathcal{E}}(\xi)\big)\\
&=(\mathrm{id}_{\mathcal{E}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\omega)\big((\mathrm{id}_{\mathcal{E}}\otimes\Delta)\gamma_{\mathcal{E}}(\xi)\big).
\end{aligned}$$

Hence $\gamma_{\mathcal{E}}(\omega*\xi)^*=(\mathrm{id}_{\mathcal{E}^*}\otimes\mathrm{id}_{\mathcal{G}}\otimes\omega_{v,u})\big((\mathrm{id}_{\mathcal{E}^*}\otimes\Delta)\gamma_{\mathcal{E}}(\xi)^*\big)$. Since $\xi\in\mathcal{E}_{\mathrm{si}}$ we have $\gamma_{\mathcal{E}}(\xi)^*\in$

$\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ and hence, by Proposition 4.3.7, $\gamma_{\mathcal{E}}(\omega * \xi)^* \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$, that is, $\omega * \xi \in \mathcal{E}_{\mathrm{si}}$ and

$$\begin{aligned}
\langle\!\langle \omega * \xi| &= (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\omega * \xi)^*\big)\\
&= (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\Big((\mathrm{id}_{\mathcal{E}^*} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega_{v,u})\big((\mathrm{id}_{\mathcal{E}^*} \otimes \Delta)\gamma_{\mathcal{E}}(\xi)^*\big)\Big)\\
&= \big(1_B \otimes (\mathrm{id}_{\mathcal{K}(H)} \otimes \omega_{\delta^{\frac{1}{2}}v,u})(V)\big)(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*\big)\\
&= \big(1_B \otimes (\mathrm{id}_{\mathcal{K}(H)} \otimes \omega_{\delta^{\frac{1}{2}}v,u})(V)\big)\langle\!\langle \xi|.
\end{aligned}$$

The formula $|\omega * \xi\rangle\!\rangle = |\xi\rangle\!\rangle(1_B \otimes \rho_\omega)$ now follows by taking adjoints. $\square$

Corollary 4.3.9. *Suppose that $\mathcal{G}$ is unimodular. Then for all $\xi \in \mathcal{E}_{\mathrm{si}}$ and $\omega \in L^1(\mathcal{G})$, we have $\omega * \xi \in \mathcal{E}_{\mathrm{si}}$ and*

$$|\omega * \xi\rangle\!\rangle = |\xi\rangle\!\rangle(1_B \otimes \rho_\omega),$$

where $\rho_\omega = (\mathrm{id} \otimes \omega)(V^*)$.

If $\mathcal{G}$ is unimodular then, using Proposition 4.2.4(i), one can also prove directly that $\omega * \xi \in \mathcal{E}_{\mathrm{si}}$ for all $\omega \in \mathcal{G}^*$, $\xi \in \mathcal{E}_{\mathrm{si}}$, and $|\omega * \xi\rangle\!\rangle = |\xi\rangle\!\rangle(1_B \otimes \rho_\omega)$, where $\rho_\omega := (\mathrm{id} \otimes \omega)(V^*)$. Using this formula we see that $\|\omega * \xi\|_{\mathrm{si}} \leq \|\omega\|\|\xi\|_{\mathrm{si}}$, so that $\mathcal{E}_{\mathrm{si}}$ is a Banach left $\mathcal{G}^*$-module with the restricted action of $\mathcal{E}$. In particular, $\mathcal{E}_{\mathrm{si}}$ is also a Banach left $L^1(\mathcal{G})$-module. In order to obtain a Banach left module also in the general non-unimodular case, we define following subspace of $L^1(\mathcal{G})$:

$$L^1_0(\mathcal{G}) := \{\omega \in L^1(\mathcal{G}) : \delta^{\frac{1}{2}}\omega \in L^1(\mathcal{G})\},$$

where $(\delta^{\frac{1}{2}}\omega)(x) := \omega(x\delta^{\frac{1}{2}})$ for all left multipliers x of $\delta^{\frac{1}{2}}$. The condition $\delta^{\frac{1}{2}}\omega \in L^1(\mathcal{G})$ means that there is $\theta \in L^1(\mathcal{G})$ such that $\theta(x) = \omega(x\delta^{\frac{1}{2}})$ for all left multipliers x of $\delta^{\frac{1}{2}}$, and in this case we put $\delta^{\frac{1}{2}}\omega = \theta$.

Proposition 4.3.10. *$L^1_0(\mathcal{G})$ is a subalgebra of $L^1(\mathcal{G})$.*

Proof. Take $\omega_1, \omega_2 \in L^1_0(\mathcal{G})$. Then, for every left multiplier x of $\delta^{\frac{1}{2}}$, we have

$$\begin{aligned}
(\omega_1 \cdot \omega_2)(x\delta^{\frac{1}{2}}) &= (\omega_1 \otimes \omega_2)\big(\Delta(x\delta^{\frac{1}{2}})\big)\\
&= (\omega_1 \otimes \omega_2)\big(\Delta(x)(\delta^{\frac{1}{2}} \otimes \delta^{\frac{1}{2}})\big)\\
&= (\delta^{\frac{1}{2}}\omega_1 \otimes \delta^{\frac{1}{2}}\omega_2)\Delta(x)\\
&= \big((\delta^{\frac{1}{2}}\omega_1) \cdot (\delta^{\frac{1}{2}}\omega_2)\big)(x).
\end{aligned}$$

Thus $\delta^{\frac{1}{2}}(\omega_1 \cdot \omega_2) \in L^1(\mathcal{G})$, that is, $\omega_1 \cdot \omega_2 \in L^1_0(\mathcal{G})$, and

$$\delta^{\frac{1}{2}}(\omega_1 \cdot \omega_2) = (\delta^{\frac{1}{2}}\omega_1) \cdot (\delta^{\frac{1}{2}}\omega_2). \tag{4.8}$$

This finishes the proof. $\square$

Now define the following norm on $L^1_0(\mathcal{G})$,

$$\|\omega\|_0 := \max\left\{\|\omega\|, \|\delta^{\frac{1}{2}}\omega\|\right\}.$$

Proposition 4.3.11. *The space $L^1_0(\mathcal{G})$ endowed with the norm $\|\cdot\|_0$ (and the product of $L^1(\mathcal{G})$) is a Banach algebra.*

Proof. By Equation (4.8), we have

$$\|\omega_1 \cdot \omega_2\|_0 \leq \|\omega_1\|_0\|\omega_2\|_0$$

for all $\omega_1, \omega_2 \in L^1_0(\mathcal{G})$. Thus all we have to prove is that $L^1_0(\mathcal{G})$ is a Banach space with the norm $\|\cdot\|_0$. Take a Cauchy sequence (ω_n) in $L^1_0(\mathcal{G})$ (with respect to $\|\cdot\|_0$). Then, by definition of the norm $\|\cdot\|_0$, both (ω_n) and $(\delta^{\frac{1}{2}}\omega_n)$ are Cauchy sequences in $L^1(\mathcal{G})$. Let ω and θ be the respective limits in $L^1(\mathcal{G})$. Then, for every left multiplier x of $\delta^{\frac{1}{2}}$, we have

$$(\delta^{\frac{1}{2}}\omega)(x) = \omega(x\delta^{\frac{1}{2}}) = \lim_{n\to\infty}\omega_n(x\delta^{\frac{1}{2}}) = \lim_{n\to\infty}\delta^{\frac{1}{2}}\omega_n(x) = \theta(x).$$

Hence $\delta^{\frac{1}{2}}\omega = \theta \in L^1(\mathcal{G})$, that is, $\omega \in L^1_0(\mathcal{G})$, and therefore $\|\omega_n - \omega\|_0 \to 0$. □

Note that $L^1_{00}(\mathcal{G})$ is contained in $L^1_0(\mathcal{G})$. If fact, if $u \in L^2(\mathcal{G})$ and $v \in \mathcal{D}(\delta^{\frac{1}{2}})$, then

$$\delta^{\frac{1}{2}}\omega_{u,v}(x) = \omega_{u,v}(x\delta^{\frac{1}{2}}) = \langle u\,|\,x\delta^{\frac{1}{2}}v\rangle = \omega_{u,\delta^{\frac{1}{2}}v}(x) \tag{4.9}$$

for every left multiplier x of $\delta^{\frac{1}{2}}$. This means that $\delta^{\frac{1}{2}}\omega_{u,v} = \omega_{u,\delta^{\frac{1}{2}}v} \in L^1(\mathcal{G})$.

Proposition 4.3.12. *The subspace $L^1_{00}(\mathcal{G})$ is dense in $L^1_0(\mathcal{G})$ (with respect to $\|\cdot\|_0$).*

Proof. Take any $\omega \in L^1_0(\mathcal{G})$. Let $u, v \in H$ such that $\omega = \omega_{u,v}$ (see Equation (2.13)). Take a sequence $(v_k) \subseteq \mathcal{D}(\delta^{\frac{1}{2}})$ such that $v_k \to v$, and define $v_{n,k} := e_n v_k$. Since e_n commutes with $\delta^{\frac{1}{2}}$, it follows that $v_{n,k} \in \mathcal{D}(\delta^{\frac{1}{2}})$. Observe that $\omega_{u,v_{n,k}} \in L^1_{00}(\mathcal{G})$ for all $n, k \in \mathbb{N}$. Since $v_{n,k} \to v$ as $n, k \to \infty$, we have $\omega_{u,v_{n,k}} \to \omega_{u,v}$ in $L^1(\mathcal{G})$ as $n, k \to \infty$. Now note that

$$\begin{aligned}
\|\delta^{\frac{1}{2}}\omega_{u,v_{n,k}} - \delta^{\frac{1}{2}}\omega_{u,v}\| &= \|\delta^{\frac{1}{2}}\omega_{u,e_n v_k} - \delta^{\frac{1}{2}}\omega_{u,v}\| \\
&= \|\delta^{\frac{1}{2}}e_n\omega_{u,v_k} - \delta^{\frac{1}{2}}\omega_{u,v}\| \\
&\leq \|\delta^{\frac{1}{2}}e_n\omega_{u,v_k} - \delta^{\frac{1}{2}}e_n\omega_{u,v}\| + \|\delta^{\frac{1}{2}}e_n\omega_{u,v} - \delta^{\frac{1}{2}}\omega_{u,v}\|.
\end{aligned}$$

For the second term above, we use $e_n\delta^{\frac{1}{2}} = \delta^{\frac{1}{2}}e_n$ to get

$$\|\delta^{\frac{1}{2}}e_n\omega_{u,v} - \delta^{\frac{1}{2}}\omega_{u,v}\| = \|e_n\delta^{\frac{1}{2}}\omega - \delta^{\frac{1}{2}}\omega\| \to 0, \quad \text{as } n \to \infty$$

For the first term, note that, for each fixed n, we have

$$\|\delta^{\frac{1}{2}}e_n\omega_{u,v_k} - \delta^{\frac{1}{2}}e_n\omega_{u,v}\| \to 0, \quad \text{as } k \to \infty.$$

Thus we can find a sequence (k_n) of natural numbers such that $k_1 < k_2 < \ldots$ and

$$\|\delta^{\frac{1}{2}} e_n \omega_{u,v_k} - \delta^{\frac{1}{2}} e_n \omega_{u,v}\| < 1/n.$$

Finally, defining $v_n := v_{n,k_n}$, we conclude that $\omega_n := \omega_{u,v_n} \in L^1_{00}(\mathcal{G})$ and

$$\|\delta^{\frac{1}{2}}\omega_n - \delta^{\frac{1}{2}}\omega\| \leq 1/n + \|e_n\delta^{\frac{1}{2}}\omega - \delta^{\frac{1}{2}}\omega\| \to 0.$$

Therefore $\|\omega_n - \omega\|_0 \leq \|\omega_n - \omega\| + \|\delta^{\frac{1}{2}}\omega_n - \delta^{\frac{1}{2}}\omega\| \to 0$. □

Define

$$\rho : L^1_0(\mathcal{G}) \to \mathcal{L}(H), \quad \rho(\omega) := (\mathrm{id} \otimes \delta^{\frac{1}{2}}\omega)(V^*).$$

Note that ρ is, in fact, an extension of the map $\rho : L^1_{00}(\mathcal{G}) \to \mathcal{L}(H)$ previously defined, so that there is no problem of notation.

Proposition 4.3.13. *$\rho : L^1_0(\mathcal{G}) \to \mathcal{L}(H)$ is an injective, contractive, algebra anti-homomorphism whose image is dense in $\widehat{\mathcal{G}}^{\mathrm{c}}$.*

Proof. Consider the opposite $\mathcal{G}^{\mathrm{op}}$ of $\mathcal{G}$. The left regular corepresentation W^{op} of $\mathcal{G}^{\mathrm{op}}$ is equal to $\Sigma V^*\Sigma$ (see [73, Proposition 1.14.10]). It follows that

$$\rho(\omega) = (\mathrm{id} \otimes \delta^{\frac{1}{2}}\omega)(V^*) = (\delta^{\frac{1}{2}}\omega \otimes \mathrm{id})(W^{\mathrm{op}}) = \lambda^{\mathrm{op}}(\delta^{\frac{1}{2}}\omega)$$

for all $\omega \in L^1_0(\mathcal{G})$. Since $L^1(\mathcal{G}^{\mathrm{op}})$ equals the opposite algebra of $L^1(\mathcal{G})$, we get

$$\begin{aligned}\rho(\omega_1 \cdot \omega_2) &= \lambda^{\mathrm{op}}\big(\delta^{\frac{1}{2}}(\omega_1 \cdot \omega_2)\big)\\ &= \lambda^{\mathrm{op}}\big((\delta^{\frac{1}{2}}\omega_1) \cdot (\delta^{\frac{1}{2}}\omega_2)\big)\\ &= \lambda^{\mathrm{op}}(\delta^{\frac{1}{2}}\omega_2)\lambda^{\mathrm{op}}(\delta^{\frac{1}{2}}\omega_1)\\ &= \rho(\omega_2)\rho(\omega_1).\end{aligned}$$

Thus ρ is an anti-homomorphism. Note also that $\|\rho(\omega)\| \leq \|\delta^{\frac{1}{2}}\omega\| \leq \|\omega\|_0$. Hence ρ is contractive. If $\rho(\omega) = \lambda^{\mathrm{op}}(\delta^{\frac{1}{2}}\omega) = 0$, then $\delta^{\frac{1}{2}}\omega = 0$ because λ^{op} is injective. This implies $\omega(x\delta^{\frac{1}{2}}) = 0$ for every left multiplier x of $\delta^{\frac{1}{2}}$. Taking $x = ye_n\delta^{-\frac{1}{2}}$ we get $\omega(ye_n) = 0$ for all $n \in \mathbb{N}$ and $y \in \mathcal{G}$ and hence $\omega = 0$ because $e_n \to 1$ strictly. Therefore $\rho : L^1_0(\mathcal{G}) \to \mathcal{L}(H)$ is an injective, contractive, algebra anti-homomorphism.

Finally, note that $\rho\big(L^1_0(\mathcal{G})\big) = \lambda^{\mathrm{op}}\big(\delta^{\frac{1}{2}}L^1_0(\mathcal{G})\big) \subseteq \widehat{\mathcal{G}^{\mathrm{op}}} = \widehat{\mathcal{G}}^{\mathrm{c}}$. Since $\delta^{\frac{1}{2}}L^1_0(\mathcal{G})$ contains $\delta^{\frac{1}{2}}L^1_{00}(\mathcal{G})$, which contains elements of the form $\omega_{u,\delta^{\frac{1}{2}}v}$, where $u \in H$ and $v \in \mathcal{D}(\delta^{\frac{1}{2}})$, and since such elements span a dense subspace of $L^1(\mathcal{G})$, we conclude that $\rho\big(L^1_0(\mathcal{G})\big)$ is dense in $\widehat{\mathcal{G}}^{\mathrm{c}}$ as well (remember that the image of λ^{op} is dense in $\widehat{\mathcal{G}^{\mathrm{op}}} = \widehat{\mathcal{G}}^{\mathrm{c}}$). □

The next result implies that $\mathcal{E}_{\mathrm{si}}$ is a Banach left $L^1_0(\mathcal{G})$-module.

Proposition 4.3.14. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. If $\omega \in L^1_0(\mathcal{G})$ and $\xi \in \mathcal{E}_{\mathrm{si}}$, then $\omega \in \mathcal{E}_{\mathrm{si}}$ and*

$$|\omega * \xi\rangle\!\rangle = |\xi\rangle\!\rangle(1_B \otimes \rho_\omega).$$

In particular, $\|\omega * \xi\|_{\mathrm{si}} \leq \|\omega\|_0\|\xi\|_{\mathrm{si}}$ for all $\xi \in \mathcal{E}_{\mathrm{si}}$ and $\omega \in L^1_0(\mathcal{G})$.

Proof. Let (ω_n) be a sequence in $L^1_{00}(\mathcal{G})$ converging to ω (with respect to $\|\cdot\|_0$). In particular, $\omega_n \to \omega$ in $L^1(\mathcal{G})$, and hence $\omega_n * \xi \to \omega * \xi$ in $\mathcal{E}$. Since $\rho_{\omega_n} \to \rho_\omega$, we also have

$$|\omega_n * \xi\rangle\!\rangle = |\xi\rangle\!\rangle(1_B \otimes \rho_{\omega_n}) \to |\xi\rangle\!\rangle(1_B \otimes \rho_\omega).$$

This implies that $(\omega_n*\xi)$ is a Cauchy sequence with respect to $\|\cdot\|_{\mathrm{si}}$. By Proposition 4.1.11, this sequence converges to some $\eta \in \mathcal{E}_{\mathrm{si}}$. In particular, $\omega_n * \xi \to \eta$ in $\mathcal{E}$. It follows that $\omega * \xi = \eta \in \mathcal{E}_{\mathrm{si}}$. Moreover,

$$|\omega * \xi\rangle\!\rangle = |\eta\rangle\!\rangle = \lim_n |\omega_n * \xi\rangle\!\rangle = \lim_n |\xi\rangle\!\rangle(1_B \otimes \rho_{\omega_n}) = |\xi\rangle\!\rangle(1_B \otimes \rho_\omega). \qquad \square$$

Remark 4.3.15. Let us return to the group case, that is, $\mathcal{G} = \mathcal{C}_0(G)$, where G is some locally compact group. There is a small difference of convention with respect to the modular element δ of $\mathcal{G} = \mathcal{C}_0(G)$, in the sense that it is not given by the modular function δ_G of G, but by its inverse, that is, by the function $t \mapsto \delta_G(t)^{-1}$ (see comments after Definition 1.9.1 in [73]). It follows that $L^1_0(\mathcal{G})$ corresponds to

$$L^1_0(G) = \{\omega \in L^1(G) : \delta_G^{-\frac{1}{2}} \cdot \omega \in L^1(G)\},$$

where $\cdot$ denotes pointwise multiplication. Given $\omega \in L^1_0(G)$, the operator $\rho_\omega \in \mathcal{L}\big(L^2(G)\big)$ corresponds to the operator given by right convolution with ω. Thus, for groups, Proposition 4.3.14 says exactly what have already seen in the beginning of this section.

Before finishing this section, we want to prove two more properties of the Banach algebra $L^1_0(\mathcal{G})$ that we are going to need later.

We already know (Proposition 2.5.5) that $L^1(\mathcal{G})$ is a nondegenerate Banach algebra. In particular, if $\mathcal{G}$ is unimodular, we get that $L^1_0(\mathcal{G})$ is also a nondegenerate Banach algebra. Now we prove that this holds in general.

Proposition 4.3.16. *Let $\mathcal{G}$ be a locally compact quantum group. Then $L^1_0(\mathcal{G})$ is a nondegenerate Banach algebra, that is, the linear span of $L^1_0(\mathcal{G}) \cdot L^1_0(\mathcal{G})$ is dense in $L^1_0(\mathcal{G})$ (of course, with respect to the norm $\|\cdot\|_0$).*

Proof. The proof is essentially the same as for Proposition 2.5.5. We only have to be careful with the modular element. As already noted in the proof of Proposition 2.5.5, we have

$$\omega_{\xi,\eta} \cdot \omega_{f,g}(x) = \langle W(\xi \otimes f) \,|\, (1 \otimes x) W(\eta \otimes g)\rangle, \tag{4.10}$$

for all $\xi, \eta, f, g \in H$ and $x \in \mathcal{G}$, where W is the left regular corepresentation of $\mathcal{G}$. Now, the relations $\Delta(\delta^{\frac{1}{2}}) = \delta^{\frac{1}{2}} \otimes \delta^{\frac{1}{2}}$ and $\Delta(x) = W^*(1 \otimes x)W$, imply that $(1 \otimes \delta^{\frac{1}{2}})W = W(\delta^{\frac{1}{2}} \otimes \delta^{\frac{1}{2}})$. Since $\mathcal{D}(\delta^{\frac{1}{2}}) \odot \mathcal{D}(\delta^{\frac{1}{2}})$ is a core for $\delta^{\frac{1}{2}} \otimes \delta^{\frac{1}{2}}$, the space $W\big(\mathcal{D}(\delta^{\frac{1}{2}}) \odot \mathcal{D}(\delta^{\frac{1}{2}})\big)$ is a core for $1 \otimes \delta^{\frac{1}{2}}$. Thus, given $\eta \in H$ and $g \in \mathcal{D}(\delta^{\frac{1}{2}})$, there is a sequence (ζ_n) contained in $\mathcal{D}(\delta^{\frac{1}{2}}) \odot \mathcal{D}(\delta^{\frac{1}{2}})$ such that $W\zeta_n \to \eta \otimes g$ and $(1 \otimes \delta^{\frac{1}{2}})W\zeta_n \to \eta \otimes \delta^{\frac{1}{2}} g$. Take any $\xi, f \in H$ and choose a sequence (ζ'_n) in $H \odot H$ such that $W\zeta'_n \to \xi \otimes f$. Each ζ_n has the form $\sum_k \xi_{k,n} \otimes f_{k,n}$ and each ζ'_n has the form $\sum_k \eta_{k,n} \otimes g_{k,n}$, where both sums are finite and $\xi_{k,n}, f_{k,n} \in H$ and

$\eta_{k,n}, g_{k,n} \in \mathcal{D}(\delta^{\frac{1}{2}})$ for all k, n. By adding zeros, if necessary, we may assume that both sums have the same number of terms, say $N(n)$, for all n. For each n, we define

$$\omega_n := \sum_{k,l=1}^{N(n)} \omega_{\xi_{l,n},\eta_{k,n}} \cdot \omega_{f_{l,n},g_{k,n}} \in \operatorname{span}\bigl(L^1_0(\mathcal{G}) \cdot L^1_0(\mathcal{G})\bigr).$$

Note that, by Equation (4.10), we have

$$\omega_n(x) = \sum_{k,l} \bigl\langle W(\xi_{k,n} \otimes f_{k,n}) \big| (1 \otimes x) W(\eta_{l,n} \otimes g_{l,n}) \bigr\rangle = \bigl\langle W\zeta'_n \big| (1 \otimes x) W\zeta_n \bigr\rangle .$$

Analogously, using also Equations (4.8) and (4.9), we get

$$\delta^{\frac{1}{2}}\omega_n(x) = \sum_{k,l} \Bigl\langle W(\xi_{k,n} \otimes f_{k,n}) \Big| (1 \otimes x) W(\delta^{\frac{1}{2}}\eta_{l,n} \otimes \delta^{\frac{1}{2}} g_{l,n}) \Bigr\rangle = \Bigl\langle W\zeta'_n \Big| (1 \otimes x\delta^{\frac{1}{2}}) W\zeta_n \Bigr\rangle .$$

It follows that $\omega_n \to \langle \xi \mid \eta \rangle \omega_{f,g}$ and $\delta^{\frac{1}{2}}\omega_n \to \langle \xi \mid \eta \rangle \omega_{f,\delta^{\frac{1}{2}} g}$ in $L^1(\mathcal{G})$. In other words, $\omega_n \to \langle \xi \mid \eta \rangle \omega_{f,g}$ in $L^1_0(\mathcal{G})$. Therefore the closed linear span of $L^1_0(\mathcal{G}) \cdot L^1_0(\mathcal{G})$ in $L^1_0(\mathcal{G})$ contains $L^1_{00}(\mathcal{G})$. The assertion now follows from Proposition 4.3.12. □

We also need to know when $L^1_0(\mathcal{G})$ has a bounded approximate unit. Recall that $L^1(\mathcal{G})$ has a bounded approximate unit if and only if $\mathcal{G}$ is co-amenable. Since the inclusion $L^1_0(\mathcal{G}) \hookrightarrow L^1(\mathcal{G})$ is contractive and has dense image, the existence of a bounded approximate unit for $L^1_0(\mathcal{G})$ also implies the existence for $L^1(\mathcal{G})$, that is, $\mathcal{G}$ is co-amenable. The converse also holds:

Proposition 4.3.17. *The Banach algebra $L^1_0(\mathcal{G})$ has a bounded approximate unit if and only if $\mathcal{G}$ is co-amenable.*

Proof. Suppose that $\mathcal{G}$ is co-amenable. Then one can find an approximate unit (ω_i) for $L^1(\mathcal{G}) \cong M_*$ consisting of normal states, where $M := \mathcal{G}''$ (see [30, Theorem 2]). Since M is in standard form, each ω_i has the form $\omega_i = \omega_{\xi_i,\xi_i}$, where $\xi_i \in H$ are unit vectors. By the Banach–Alaoglu Theorem, we may assume that $\omega_i(x) \to \epsilon(x)$ for all $x \in M$, where $\epsilon \in M^*$ is some state whose restriction to $\mathcal{G}$ is (necessarily) the counit of $\mathcal{G}$ (see the proof of [8, Theorem 3.1]). In particular,

$$\epsilon(x) = \lim_i \omega_i(x) = \lim_i \langle \xi_i \,|\, x\xi_i \rangle \quad \text{for all } x \in \mathcal{M}(\mathcal{G}).$$

Let $e \in \mathcal{M}(\mathcal{G})$ with $\epsilon(e) = 1$. We claim that $\|e\omega_i - \omega_i\| \to 0$. In fact, recall that ϵ is a $*$-homomorphism. Thus

$$\|e\xi_i - \xi_i\|^2 = \langle \xi_i \,|\, e^*e\xi_i \rangle - \langle \xi_i \,|\, e^*\xi_i \rangle - \langle \xi_i \,|\, e\xi_i \rangle + 1 \to \epsilon(e^*e) - \epsilon(e^*) - \epsilon(e) + 1 = 0.$$

Hence

$$\|e\omega_i - \omega_i\| = \|\omega_{\xi_i,e\xi_i} - \omega_{\xi_i,\xi_i}\| \leq \|e\xi_i - \xi_i\| \to 0.$$

Note that this implies that $(e\omega_i)$ is also a (bounded) approximate unit for $L^1(\mathcal{G})$. Now suppose, in addition, that e is a right multiplier of $\delta^{\frac{1}{2}}$ (for instance, one can take $e = e_n$ defined by Equation (4.5), for any $n \in \mathbb{N}$). Then, for all $\omega \in L^1(\mathcal{G})$, we have $e\omega \in L^1_0(\mathcal{G})$ and

$$\|e\omega\|_0 \leq \max\left\{\|e\|, \|\delta^{\frac{1}{2}}e\|\right\}\|\omega\|.$$

In other words, $\omega \mapsto e\omega$ is a bounded linear map $L^1(\mathcal{G}) \to L^1_0(\mathcal{G})$. Note that $\epsilon(\delta^{\frac{1}{2}}e) = 1$ (this follows from the relations $\Delta(\delta) = \delta \otimes \delta$ and $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$). By the claim we have just proved above (applied to $\delta^{\frac{1}{2}}e$), we get $\|\delta^{\frac{1}{2}}e\omega_i - \omega_i\| \to 0$ and therefore $(\delta^{\frac{1}{2}}e\omega_i)$ is also a (bounded) approximate unit for $L^1(\mathcal{G})$. To complete the proof, we show that the (bounded) net $(e\omega_i)$ is an approximate unit for $L^1_0(\mathcal{G})$. In fact, by Equation (4.8) and the fact that the nets $(e\omega_i)$ and $(\delta^{\frac{1}{2}}e\omega_i)$ are approximate units for $L^1(\mathcal{G})$, we get

$$\|(e\omega_i)\cdot\omega - \omega\|_0 \leq \|(e\omega_i)\cdot\omega - \omega\| + \|(\delta^{\frac{1}{2}}e\omega_i)\cdot(\delta^{\frac{1}{2}}\omega) - \delta^{\frac{1}{2}}\omega\| \to 0$$

for any $\omega \in L^1_0(\mathcal{G})$. Analogously, $\|\omega \cdot (e\omega_i) - \omega\|_0 \to 0$ for all $\omega \in L^1_0(\mathcal{G})$. $\square$

4.4 Square-integrability of $L^2(\mathcal{G})$

Let $\mathcal{G}$ be al locally compact quantum group. Recall that $H = L^2(\mathcal{G})$ denotes the L^2-space of $\mathcal{G}$. Let B be a C^*-algebra with a coaction of $\mathcal{G}$. One of the main features in the group case is that the coaction $\gamma_{B\otimes H}$ given by Equation (4.2) (or equivalently, the corresponding action $\beta \otimes \lambda$ of the underlying group; see Example 4.2.1) is square-integrable ([47, 48]). In this section we prove that this is still true in the general quantum setting. In fact, note that if $\mathcal{G}$ is regular, then this follows from Corollaries 3.3.6 and 4.1.3. Therefore, we already have this result in the regular case. In this section we give another proof where regularity is not necessary.

We shall use in this section the following slight modification of Proposition 3.3.1.

Proposition 4.4.1. *Consider a Hilbert B-module $\mathcal{E}$ with a coaction of $\mathcal{G}$ and suppose that $\pi : A \to \mathcal{L}(\mathcal{E})$ is a $\mathcal{G}$-equivariant nondegenerate $*$-homomorphism, where A is a C^*-algebra with a coaction of $\mathcal{G}$. If A is integrable, then $\mathcal{E}$ is square-integrable.*

Proof. Since $\mathcal{L}(\mathcal{E}) \cong \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$, this follows from Proposition 3.3.1 and Corollary 4.1.3. $\square$

First we show that the coaction γ_H of $\mathcal{G}$ on $H = L^2(\mathcal{G})$ is square-integrable or, equivalently, that the induced coaction $\gamma_{\mathcal{K}(H)}$ on $\mathcal{K}(H)$ is integrable. Recall that $\gamma_H(\eta) = \hat{W}^*(\eta \otimes 1)$, where $\hat{W}$ is the left regular corepresentation of $\widehat{\mathcal{G}}$. Hence

$$\gamma_{\mathcal{K}(H)}(x) = \hat{W}^*(x \otimes 1)\hat{W}, \quad x \in \mathcal{K}(H).$$

Recall that we also have a coaction $\tilde{\gamma}_H$ of $\mathcal{G}$ on H coming from the right regular corepresentation $V \in \mathcal{L}(H \otimes \mathcal{G})$. It is given by the formula

$$\tilde{\gamma}_H(\eta) = V(\eta \otimes 1), \quad \eta \in H.$$

The corresponding coaction $\tilde{\gamma}_{\mathcal{K}(H)}$ of $\mathcal{G}$ on $\mathcal{K}(H)$ is therefore given by

$$\tilde{\gamma}_{\mathcal{K}(H)}(x) = V(x \otimes 1)V^*, \quad x \in \mathcal{K}(H).$$

Proposition 4.4.2. *The coaction $\tilde{\gamma}_H$ (or, equivalently, $\tilde{\gamma}_{\mathcal{K}(H)}$) is square-integrable.*

Proof. Recall that $\Delta(x) = V(x \otimes 1)V^*$ for all $x \in \mathcal{G}$. From this relation it is obvious that the inclusion map of $\mathcal{G}$ into $\mathcal{M}(\mathcal{K}) = \mathcal{L}(H)$ is $\mathcal{G}$-equivariant with respect to the coaction $\tilde{\gamma}_H$ on H. Since $(\mathcal{G}, \Delta)$ is an integrable $\mathcal{G}$-C^*-algebra (Proposition 3.2.12), the assertion follows from Proposition 4.4.1. □

As already mentioned in Example 2.6.18(3), the coactions γ_H and $\tilde{\gamma}_H$ are equivalent. Therefore, we get as a consequence the desired result:

Corollary 4.4.3. *The coaction γ_H (or, equivalently, $\gamma_{\mathcal{K}(H)}$) is square-integrable.*

In the case of a locally compact group G, that is, when $\mathcal{G} = \mathcal{C}_0(G)$, the unitary $V \in \mathcal{L}\big(L^2(G \times G)\big)$ is given by $V\xi(s,t) = \delta_G(t)^{\frac{1}{2}}\xi(st,t)$. In this case $\tilde{\gamma}_H$ corresponds to the right regular representation $\rho_t(\xi)(s) = \delta_G(t)^{\frac{1}{2}}\xi(st)$ of G on $L^2(G)$. And as we saw in Example 4.2.1, the coaction γ_H corresponds to the left regular representation λ of G on $L^2(G)$. As mentioned above, the coactions γ_H and $\tilde{\gamma}_H$ are equivalent. This means that the left and right regular representations of G are equivalent. The equivalence is implemented by the unitary $U \in \mathcal{L}\big(L^2(G)\big)$ given by $U\xi(s) = \delta_G(s)^{-1}\xi(s^{-1})$ for all $\xi \in L^2(G)$ and $s \in G$. Note that $U = J\hat{J}$, where J and $\hat{J}$ are the modular conjugations of $\mathcal{C}_0(G)$ and its dual $C^*_{\mathrm{r}}(G)$. They are given by $J\xi(s) = \overline{\xi(s)}$ and $\hat{J}\xi(s) = \delta_G(s)^{-1}\overline{\xi(s^{-1})}$.

As already noted in Example 2.6.18(3), for a general locally compact quantum group $\mathcal{G}$ the equivalence between γ_H and $\tilde{\gamma}_H$ is also implemented by the unitary $U = J\hat{J}$. Since the unitaries associated to γ_H and $\tilde{\gamma}_H$ are $\hat{W}^*$ and V, respectively, the equivalence means $V = (U^* \otimes 1)\hat{W}^*(U \otimes 1)$. And this relation follows from Equations (2.15) and (2.16).

The square-integrability of H also implies the square-integrability of $B \otimes H$:

Corollary 4.4.4. *Let B be C^*-algebra with a coaction γ_B of $\mathcal{G}$. Then the Hilbert B-module $B \otimes H$ equipped with the coaction $\gamma_{B\otimes H}$ is square-integrable.*

Proof. The map $\mathcal{K}(H) \to \mathcal{L}(B \otimes H)$, $T \mapsto 1 \otimes T$ is a $\mathcal{G}$-equivariant nondegenerate $*$-homomorphism. Since H is square-integrable, $\mathcal{K}(H)$ is integrable (Corollary 4.1.3). Thus $\gamma_{B\otimes H}$ is square-integrable by Proposition 4.4.1. □

In particular, the corresponding coaction $\gamma_{B\otimes\mathcal{K}(H)}$ of $\mathcal{G}$ on the C^*-algebra of compact operators $\mathcal{K}(B \otimes H) \cong B \otimes \mathcal{K}(H)$ is integrable. This generalizes Corollary 3.3.6.

Of course, one can also use the equivalent coaction $\tilde{\gamma}_{B\otimes H}$ (see Example 2.6.18(3)) of $\mathcal{G}$ on $B \otimes H$ in the corollary above.

More generally, if $\mathcal{E}$ is a Hilbert B-module with a γ_B-compatible coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$, then we can equip $\mathcal{E} \otimes H$ with the coaction $\gamma_{\mathcal{E}\otimes H}$ (or, equivalently, $\tilde{\gamma}_{\mathcal{E}\otimes H}$) as in Example 2.6.18(3). As above, the canonical map $\mathcal{K}(H) \to \mathcal{L}(\mathcal{E} \otimes H)$ is $\mathcal{G}$-equivariant. Therefore, we also have:

Corollary 4.4.5. *The Hilbert B-module $\mathcal{E} \otimes H$ equipped with the coaction $\gamma_{\mathcal{E}\otimes H}$ is square-integrable.*

4.5 The Kasparov Stabilization Theorem

In this section we prove one of the main results of this chapter, namely, a quantum version of the equivariant Kasparov Stabilization Theorem for square-integrable Hilbert modules proved in the group case by Meyer ([47, Theorem 8.5]).

We need some preliminary results. First we show that a direct summand of a square-integrable Hilbert module is also square-integrable. In the group case, this holds because the projection onto the direct summand maps square-integrable elements to square-integrable elements. We are going to extend this to the general quantum setting.

Throughout this section we fix a locally compact quantum group $\mathcal{G}$ and a C^*-algebra B with a coaction γ_B of $\mathcal{G}$. As in the previous sections, $H = L^2(\mathcal{G})$ denotes the L^2-space of $\mathcal{G}$.

Proposition 4.5.1. *Let $\mathcal{F}$ be a Hilbert B-module with a coaction of $\mathcal{G}$, let $\mathcal{E}$ be a $\mathcal{G}$-invariant direct summand of $\mathcal{F}$ and $\mathcal{E}^\perp$ be its complement. Then $\eta \in \mathcal{F}_{\mathrm{si}}$ if and only if $P_{\mathcal{E}}(\eta) \in \mathcal{E}_{\mathrm{si}}$ and $P_{\mathcal{E}^\perp}(\eta) \in \mathcal{E}^\perp_{\mathrm{si}}$, where $P_{\mathcal{E}}$ and $P_{\mathcal{E}^\perp}$ denote the projections of $\mathcal{F}$ onto $\mathcal{E}$ and $\mathcal{E}^\perp$, respectively. In other words, we have $\mathcal{F}_{\mathrm{si}} = \mathcal{E}_{\mathrm{si}} \oplus \mathcal{E}^\perp_{\mathrm{si}}$. In particular, $\mathcal{F}$ is square-integrable if and only if $\mathcal{E}$ and $\mathcal{E}^\perp$ are square-integrable. Moreover, under the canonical identification $\mathcal{L}(B \otimes H, \mathcal{F}) \cong \mathcal{L}(B \otimes H, \mathcal{E}) \oplus \mathcal{L}(B \otimes H, \mathcal{E}^\perp)$, we have*

$$|\eta\rangle\!\rangle = |P_{\mathcal{E}}(\eta)\rangle\!\rangle \oplus |P_{\mathcal{E}^\perp}(\eta)\rangle\!\rangle \quad \textit{for all } \eta \in \mathcal{F}_{\mathrm{si}}.$$

Proof. Let $\eta \in \mathcal{F}_{\mathrm{si}}$ and write $\eta = \xi + \xi^\perp$, with $\xi \in \mathcal{E}$ and $\xi^\perp \in \mathcal{E}^\perp$. Then for every $\zeta \in \mathcal{E}$ and $\zeta^\perp \in \mathcal{E}^\perp$ we have

$$\gamma_{\mathcal{F}}(\eta)^*((\zeta + \zeta^\perp) \otimes 1_{\mathcal{G}}) = \gamma_{\mathcal{E}}(\xi)^*(\zeta \otimes 1_{\mathcal{G}}) + \gamma_{\mathcal{E}^\perp}(\xi^\perp)^*(\zeta^\perp \otimes 1_{\mathcal{G}}).$$

It follows that $\eta \in \mathcal{F}_{\mathrm{si}}$ if and only if $\xi \in \mathcal{E}_{\mathrm{si}}$ and $\xi^\perp \in \mathcal{E}^\perp_{\mathrm{si}}$ and

$$\langle\!\langle \eta|(\zeta + \zeta^\perp) = \langle\!\langle \xi|\zeta + \langle\!\langle \xi|\zeta^\perp,$$

or equivalently $|\eta\rangle\!\rangle = |\xi\rangle\!\rangle \oplus |\xi^\perp\rangle\!\rangle$. ☐

Given a Hilbert B-module $\mathcal{E}$ we define the Hilbert B-module $\mathcal{E}^\infty := \oplus_{n\in\mathbb{N}}\mathcal{E} \cong l^2(\mathbb{N})\otimes\mathcal{E}$. If we have a coaction $\gamma_{\mathcal{E}}$ on $\mathcal{E}$ and we equip $l^2(\mathbb{N})$ with the trivial coaction γ_{tr} of $\mathcal{G}$, then we can consider on $\mathcal{E}^\infty = l^2(\mathbb{N}) \otimes_{\mathbb{C}} \mathcal{E}$ the coaction $\gamma_{\mathcal{E}^\infty}$ which is the (balanced) tensor product of γ_{tr} and $\gamma_{\mathcal{E}}$ (Definition 2.6.15). It is given by the formula

$$\gamma_{\mathcal{E}^\infty}(f \otimes \xi) = f \otimes \gamma_{\mathcal{E}}(\xi) \in l^2(\mathbb{N}) \otimes \mathcal{M}(\mathcal{E} \otimes \mathcal{G}) \subseteq \mathcal{M}(\mathcal{E}^\infty \otimes \mathcal{G}), \quad f \in l^2(\mathbb{N}), \xi \in \mathcal{E}.$$

Proposition 4.5.2. *Let $\mathcal{E}$ be a Hilbert B-module with a coaction of $\mathcal{G}$. Then $\mathcal{E}$ is square-integrable if and only if $\mathcal{E}^\infty$ is square-integrable.*

Proof. If $\mathcal{E}$ is square-integrable, then so is $\mathcal{E}^\infty$ because the map $T \mapsto 1 \otimes T$ from $\mathcal{K}(\mathcal{E})$ into $\mathcal{L}(\mathcal{E}^\infty) \cong \mathcal{M}\big(\mathcal{K}(l^2\mathbb{N}) \otimes \mathcal{K}(\mathcal{E})\big)$ is a nondegenerate $\mathcal{G}$-equivariant $*$-homomorphism (see Proposition 4.4.1). The other direction follows from Proposition 4.5.1. $\square$

Recall from Equation (2.5) that $\mathcal{T}_\varphi$ denotes the Tomita $*$-algebra of φ.

Lemma 4.5.3. *Let $\mathcal{E}$ be a Hilbert B-module with a coaction of $\mathcal{G}$. Let $a \in \mathcal{N}_\varphi$, $b \in \mathcal{T}_\varphi$ and $\xi \in \mathcal{E}_{\mathrm{si}}$. Define $\omega := \omega_{\Lambda(b),\Lambda(a)} = a\varphi b^* \in L^1(\mathcal{G})$ and $x_\omega := a\sigma_{-\mathrm{i}}(b^*) \in \mathcal{N}_\varphi$. Then*

$$\omega * \xi = |\xi\rangle\!\rangle\big(1_B \otimes \Lambda(x_\omega)\big).$$

Proof. Since $\xi \in \mathcal{E}_{\mathrm{si}}$ and $a \in \mathcal{N}_\varphi$ we have $\gamma_{\mathcal{E}}(\xi)^* \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ and $1_B \otimes a \in \bar{\mathcal{N}}_{\mathrm{id}_B\otimes\varphi}$. By Proposition 2.4.20(iv), $\gamma_{\mathcal{E}}(\xi)(1_B \otimes a) \in \bar{\mathcal{M}}_{\mathrm{id}_{\mathcal{E}}\otimes\varphi}$. Using Proposition 2.4.22(ii) we get

$$\begin{aligned}
\omega * \xi &= (a\varphi b^*) * \xi \\
&= (\mathrm{id}_{\mathcal{E}} \otimes a\varphi b^*)\big(\gamma_{\mathcal{E}}(\xi)\big) \\
&= (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big((1_{\mathcal{E}} \otimes b^*)\gamma_{\mathcal{E}}(\xi)(1_B \otimes a)\big) \\
&= (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)(1_B \otimes a)(1_B \otimes \sigma_{-\mathrm{i}}(b^*))\big) \\
&= (\mathrm{id}_{\mathcal{E}} \otimes \varphi)\big(\gamma_{\mathcal{E}}(\xi)(1_B \otimes x_\omega))\big) \\
&= |\xi\rangle\!\rangle\big(1_B \otimes \Lambda(x_\omega)\big).
\end{aligned}$$

$\square$

In the group case $\mathcal{G} = \mathcal{C}_0(G)$, it was proved in [47, Lemma 8.1(iii)] that for every Hilbert B,G-module, we have $\xi \in \overline{\operatorname{Ran}|\xi\rangle\!\rangle}$ for all $\xi \in \mathcal{E}_{\mathrm{si}}$. This was used to prove the Kasparov Stabilization Theorem in this case [47, Theorem 8.5]. For general quantum groups, this is not true because coactions need not be injective, and if ξ is in the kernel of the coaction, then ξ is square-integrable and $|\xi\rangle\!\rangle = 0$. Note that this does not happen if $\mathcal{G}$ is co-amenable (in particular, this does not happen in the group case). In fact, we can prove the following generalization of [47, Lemma 8.1(iii)].

Lemma 4.5.4. *Let $\mathcal{E}$ be a Hilbert B-module with a weakly continuous coaction of G and suppose that $\mathcal{D}$ is a subset of $\mathcal{E}_{\mathrm{si}}$ which is dense in $\mathcal{E}$. Then, for every $\epsilon > 0$ and every $\xi \in \mathcal{E}$, there are $\xi_1, \dots, \xi_n \in \mathcal{D}$, $u \in B$ and $x_1, \dots, x_n \in \mathcal{N}_\varphi$ such that*

$$\left\| \xi - \sum_{k=1}^{n} |\xi_k\rangle\!\rangle(u \otimes \Lambda(x_k)) \right\| < \epsilon.$$

Moreover, if $\mathcal{G}$ is co-amenable, then, for every $\epsilon > 0$ and every $\xi \in \mathcal{E}$, there are $x \in \mathcal{N}_\varphi$ and $u \in B$ such that

$$\big\|\xi - |\xi\rangle\!\rangle\big(u \otimes \Lambda(x)\big)\big\| < \epsilon.$$

In particular, if $\mathcal{G}$ is co-amenable, we have $\xi \in \overline{\operatorname{Ran}|\xi\rangle\!\rangle}$.

Proof. Let $\xi \in \mathcal{E}$ and $\epsilon > 0$. Let $\mathcal{T}_\varphi$ be the Tomita $*$-algebra of φ. Note that, by Lemma 2.4.11, we have

$$L^1(\mathcal{G}) = \overline{\operatorname{span}}\{a\varphi b^* : a, b \in \mathcal{T}_\varphi\}. \tag{4.11}$$

Since the coaction of $\mathcal{E}$ is weakly continuous, the linear span of $L^1(\mathcal{G}) * \mathcal{E}$ is dense in $\mathcal{E}$. It follows from Equation (4.11) that there are $a_1, b_1, \ldots, a_n, b_n \in \mathcal{T}_\varphi$ and $\xi_1, \ldots, \xi_n \in \mathcal{D}$ such that

$$\left\| \xi - \sum_{k=1}^{n} (a_k \varphi b_k^*) * \xi_k \right\| < \frac{\epsilon}{2}.$$

By Lemma 4.5.3, we have

$$\sum_{k=1}^{n} (a_k \varphi b_k^*) * \xi_k = \sum_{k=1}^{n} |\xi_k\rangle\!\rangle \big(1_B \otimes \Lambda(x_k)\big),$$

where $x_k := a_k \sigma_{-\mathrm{i}}(b_k^*) \in \mathcal{N}_\varphi$. Now take $u \in B$ such that $\|u\| \le 1$ and

$$\|\xi - \xi u\| < \frac{\epsilon}{2}.$$

Then

$$\left\| \xi - \sum_{k=1}^{n} |\xi_k\rangle\!\rangle \big(u \otimes \Lambda(x_k)\big) \right\| \le \|\xi - \xi u\| + \left\| \left(\xi - \sum_{k=1}^{n} |\xi_k\rangle\!\rangle \big(1_B \otimes \Lambda(x_k)\big) \right) u \right\| < \epsilon.$$

Now suppose that $\mathcal{G}$ is co-amenable, that is, $L^1(\mathcal{G})$ has a bounded approximate unit. Then, by the weak continuity of $\gamma_\mathcal{E}$, we have $\omega_i * \xi \to \xi$ for all $\xi \in \mathcal{E}$, where (ω_i) is a bounded approximate unit for $L^1(\mathcal{G})$. Combining this with Equation (4.11), we can find $a_1, b_1, \ldots, a_n, b_n \in \mathcal{T}_\varphi$ such that

$$\left\| \xi - \sum_{k=1}^{n} (a_k \varphi b_k^*) * \xi \right\| \le \frac{\epsilon}{2}.$$

Again, by Lemma 4.5.3, we have

$$\sum_{k=1}^{n} (a_k \varphi b_k^*) * \xi = \sum_{k=1}^{n} |\xi\rangle\!\rangle \big(1_B \otimes \Lambda(x_k)\big) = |\xi\rangle\!\rangle \big(1_B \otimes \Lambda(x)\big),$$

where $x_k := a_k \sigma_{-\mathrm{i}}(b_k^*)$ and $x := \sum_{k=1}^{n} x_k \in \mathcal{N}_\varphi$. Therefore, if $u \in B$ is such that $\|u\| \le 1$ and $\|\xi - \xi u\| < \frac{\epsilon}{2}$, we get as above that

$$\big\| \xi - |\xi\rangle\!\rangle \big(u \otimes \Lambda(x)\big) \big\| < \epsilon. \qquad \square$$

The next result says that a Hilbert $B,\mathcal{G}$-module is square-integrable if and only if there are enough equivariant adjointable operators $B \otimes H \to \mathcal{E}$. This is basically what will be used to prove the Kasparov Stabilization Theorem (Theorem 4.5.6 below) for square-integrable Hilbert modules.

Corollary 4.5.5. *Let $\mathcal{E}$ be a Hilbert B-module with a weakly continuous coaction of $\mathcal{G}$. Then the following assertions are equivalent:*

(i) *$\mathcal{E}$ is square-integrable,*

(ii) *there is a subset $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E})$ such that $\overline{\operatorname{span}}\,\mathcal{F}(B \otimes H) = \mathcal{E}$*

(iii) *there is Hilbert B-module $\mathcal{E}'$ with a square-integrable coaction of $\mathcal{G}$ and a subset $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{E}', \mathcal{E})$ such that $\overline{\operatorname{span}}\,\mathcal{F}(\mathcal{E}') = \mathcal{E}$.*

Proof. By Lemma 4.5.4 and Proposition 4.2.7, (i) implies (ii) for $\mathcal{F} := \{|\xi\rangle\!\rangle : \xi \in \mathcal{E}_{\mathrm{si}}\}$. It follows from Corollary 4.4.4 that (ii) implies (iii) by taking $\mathcal{E}' = B \otimes H$. Finally, suppose that (iii) is true. Since $\mathcal{E}'$ is square-integrable, the linear span of $\mathcal{F}(\mathcal{E}'_{\mathrm{si}})$ is dense in $\mathcal{E}$. And by Proposition 4.1.10(iii), $\mathcal{F}(\mathcal{E}'_{\mathrm{si}})$ is contained in $\mathcal{E}_{\mathrm{si}}$. □

Recall that a Hilbert B-module $\mathcal{E}$ is *countably generated* if there is a countable subset $\mathcal{D} \subseteq \mathcal{E}$ which generates $\mathcal{E}$. We now prove the main result of this chapter.

Given a C^*-algebra B with a coaction of $\mathcal{G}$, we shall use the notation $\mathcal{H}_B$ for $(B \otimes H)^\infty$, where we equip $B \otimes H$ with the coaction (4.2).

Theorem 4.5.6 (Kasparov's Stabilization Theorem). *Let B be a C^*-algebra with a coaction γ_B of $\mathcal{G}$ and let $\mathcal{E}$ be a countably generated Hilbert B-module with a weakly continuous γ_B-compatible coaction of $\mathcal{G}$. The following statements are equivalent:*

(i) *$\mathcal{E}$ is square-integrable,*

(ii) *$\mathcal{K}(\mathcal{E})$ is integrable,*

(iii) *$\mathcal{E} \oplus \mathcal{H}_B \cong \mathcal{H}_B$ as Hilbert $B,\mathcal{G}$-modules,*

(iv) *$\mathcal{E}$ is a $\mathcal{G}$-invariant direct summand of $\mathcal{H}_B$.*

Proof. The proof is almost the same as the one in [47, Theorem 8.5], the basic difference being the use of Lemma 4.5.4 instead of [47, Lemma 8.1(iii)]. The equivalence between (i) and (ii) is the content of Corollary 4.1.3. It is trivial that (iii) implies (iv). By Proposition 4.5.1, to prove that (iv) implies (i) one has only to check that $\mathcal{H}_B$ is square-integrable. But this follows from Proposition 4.5.2 and Corollary 4.4.4. It remains to show that (i) implies (iii). Suppose that $\mathcal{E}$ is square-integrable. Since $\mathcal{E}$ is countably generated, there is a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of square-integrable elements of $\mathcal{E}$ which generates $\mathcal{E}$. We may assume that each ξ_n is repeated infinitely often. For each $n \in \mathbb{N}$, we define $T_n := \langle\!\langle \xi_n| \in \mathcal{L}^{\mathcal{G}}(\mathcal{E}, B \otimes H)$. We may also assume that $\|T_n\| \leq 1$ for all n. We identify each element of $\mathcal{H}_B$ as a sequence (f_n) with $f_n \in B \otimes H$ for all n. We formally write $\sum f_n\delta_n$ for this sequence. We define an adjointable operator $T : \mathcal{H}_B \to \mathcal{E} \oplus \mathcal{H}_B$ by

$$T\left(\sum_{n=1}^{\infty} f_n\delta_n\right) := \sum_{n=1}^{\infty} 2^{-n}T_n^*(f_n) \oplus \sum_{n=1}^{\infty} 4^{-n} f_n\delta_n,$$

$$T^*|_{\mathcal{E}}(\eta) := \sum_{n=1}^{\infty} 2^{-n} T_n(\eta)\delta_n, \quad T^*|_{\mathcal{H}_A}\left(\sum_{n=1}^{\infty} f_n\delta_n\right) := \sum_{n=1}^{\infty} 4^{-n} f_n\delta_n.$$

Since each T_n is $\mathcal{G}$-equivariant, the operator T is also $\mathcal{G}$-equivariant.

Of course, T^* has dense range. We claim that T has dense range as well. Let $f \in B \otimes H$. By definition of T we have $T_n^*(f) \oplus 2^{-n} f\delta_n \in \mathrm{Ran}(T)$ for all n. Since each T_n^* is repeated infinitely often, we have $T_n^*(f) \oplus 2^{-k} f\delta_k \in \mathrm{Ran}\, T$ for infinitely many $k \in \mathbb{N}$. Hence $|\xi_n\rangle\!\rangle(f) \oplus 0 = T_n^*(f) \oplus 0 \in \overline{\mathrm{Ran}\, T}$ for all $f \in B \otimes H$ and $n \in \mathbb{N}$. By Proposition 4.1.10, $|\xi_n b\rangle\!\rangle f = |\xi_n\rangle\!\rangle \gamma_B(b) f \in \overline{\mathrm{Ran}(T)}$ for all $b \in B$. Applying Lemma 4.5.4 to $\mathcal{D} := \mathrm{span}\{\xi_n b : n \in \mathbb{N}, b \in B\}$, we get $\xi \oplus 0 \in \overline{\mathrm{Ran}\, T}$ for all $\xi \in \mathcal{E}$. Finally, we get $0 \oplus f\delta_n = 2^n(T^*(f) \oplus 2^{-n} f\delta_n) - 2^n T_n^*(f) \oplus 0 \in \overline{\mathrm{Ran}\, T}$ for all $f \in B \otimes H$ and $n \in \mathbb{N}$. Therefore $\overline{\mathrm{Ran}\, T} = \mathcal{E} \oplus \mathcal{H}_B$.

Now we use polar decomposition to construct the desired unitary. Since both T and T^* have dense range, the composition T^*T has dense range. Thus $|T| := (T^*T)^{\frac{1}{2}}$ has dense range because $|T|(\mathcal{E}) \supseteq |T|\big(|T|(\mathcal{E})\big) = T^*T(\mathcal{E})$. Since $\langle |T|\eta, |T|\eta\rangle_B = \langle T^*T\eta, \eta\rangle_B = \langle T\eta, T\eta\rangle_B$, the formula $U(|T|\eta) := T\eta$ well-defines an isometry U from $\mathrm{Ran}\,|T|$ onto $\mathrm{Ran}\, T$. Extending U continuously, we obtain a unitary $U : \mathcal{H}_B \to \mathcal{E} \oplus \mathcal{H}_B$ which is $\mathcal{G}$-equivariant because T is $\mathcal{G}$-equivariant. □

If $\mathcal{G}$ is compact, then every Hilbert $B, \mathcal{G}$-module is square-integrable. Thus we get the following well-known consequence (see [76, Theorem 3.2]).

Corollary 4.5.7. *Let $\mathcal{E}$ be a countably generated Hilbert $B, \mathcal{G}$-module, for a compact quantum group $\mathcal{G}$. Then $\mathcal{E}$ is a $\mathcal{G}$-invariant direct summand of $\mathcal{H}_B$, that is,*

$$\mathcal{E} \oplus \mathcal{H}_B \cong \mathcal{H}_B.$$

The strategy used by Vergnioux in [76] to prove Kasparov's Stabilization Theorem for compact quantum groups is similar to the idea used by Mingo and Phillips in [49] for compact groups where they use the non-equivariant version of the Kasparov Stabilization Theorem in order to prove the equivariant version. In fact, if $\mathcal{E}, \mathcal{F}$ are Hilbert $B, \mathcal{G}$-modules, and if $\mathcal{E}$ and $\mathcal{F}$ are isomorphic as Hilbert B-modules, then $\mathcal{E} \otimes H$ and $\mathcal{F} \otimes H$ are isomorphic as Hilbert $B, \mathcal{G}$-modules (this is exactly Theorem 3.2(1) in [76]). The non-equivariant version of Kasparov's Stabilization Theorem tell us that $\mathcal{E} \oplus B^\infty \cong B^\infty$ as Hilbert B-modules, whenever $\mathcal{E}$ is countably generated. Tensoring with H and using the fact we have just mentioned, we get

$$(\mathcal{E} \otimes H) \oplus \mathcal{H}_B \cong \mathcal{H}_B,$$

as Hilbert $B, \mathcal{G}$-modules. This is true for any locally compact quantum group $\mathcal{G}$ and for any countably generated Hilbert $B, \mathcal{G}$-module $\mathcal{E}$. In fact, this is just saying that $\mathcal{E} \otimes H$ is square-integrable, and this is exactly Corollary 4.4.5.

In the case of a compact quantum group $\mathcal{G}$, the point is that $\mathcal{E}$ is always a $\mathcal{G}$-invariant direct summand of $\mathcal{E} \otimes H$. In fact, it follows from Equation (3.2) that the vector $\delta_1 := \Lambda(1) \in H$ is $\mathcal{G}$-invariant in the sense that $\gamma_H(\delta_1) = \delta_1 \otimes 1$. Thus the map $\xi \mapsto \xi \otimes \delta_1$ is a $\mathcal{G}$-equivariant isomorphism of $\mathcal{E}$ onto a $\mathcal{G}$-invariant direct summand of $\mathcal{E} \otimes H$. Therefore $\mathcal{E}$ is a $\mathcal{G}$-invariant direct summand of $\mathcal{E} \otimes H$ and so also of $\mathcal{H}_B$ by the argument above.

Chapter 5

Continuously square-integrable Hilbert modules

5.1 Concrete Hilbert modules

In this section we recall some definitions and constructions due to Meyer [48, Section 5]. Throughout this section, we fix a locally compact quantum group $\mathcal{G}$, a C^*-algebra B with a coaction γ_B of $\mathcal{G}$ and a Hilbert B-module $\mathcal{X}$ with a γ_B-compatible coaction of $\mathcal{G}$. We also fix a nondegenerate C^*-subalgebra $A \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X})$. We are mainly interested in the case $\mathcal{X} = B \otimes L^2(\mathcal{G})$ and $A = B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.[1]

Definition 5.1.1. Let $\mathcal{E}$ be a Hilbert B-module with a γ_B-compatible coaction of $\mathcal{G}$. A *concrete* Hilbert A-module is a closed subspace $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E})$ such that

$$\mathcal{F} \circ A \subseteq \mathcal{F} \quad \text{and} \quad \mathcal{F}^* \circ \mathcal{F} \subseteq A.$$

We say that $\mathcal{F}$ is *essential* if the linear span of $\mathcal{F}(\mathcal{X})$ is dense in $\mathcal{E}$.

Given a concrete Hilbert A-module $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E})$ we can turn it canonically into an abstract Hilbert A-module, defining the right A-module structure by

$$\xi \cdot a := \xi \circ a, \quad \xi \in \mathcal{F},\, a \in A, \tag{5.1}$$

and the A-valued inner product by

$$\langle \xi \,|\, \eta \rangle_A := \xi^* \circ \eta, \quad \xi, \eta \in \mathcal{F}. \tag{5.2}$$

Conversely, given an (abstract) Hilbert A-module $\mathcal{F}$, define $\mathcal{E} := \mathcal{F} \otimes_A \mathcal{X}$. We equip $\mathcal{F}$ with the trivial coaction of $\mathcal{G}$ and $\mathcal{E}$ with the balanced tensor product coaction of $\mathcal{G}$. More explicitly, the coaction on $\mathcal{F} \otimes_A \mathcal{X}$ is given by

$$\gamma_{\mathcal{F}\otimes_A \mathcal{X}}(\xi \otimes_A f) = \xi \otimes_A \gamma_{\mathcal{X}}(f) \in \mathcal{F} \otimes_A \mathcal{M}(\mathcal{X} \otimes \mathcal{G}) \subseteq \mathcal{M}\big((\mathcal{F} \otimes_A \mathcal{X}) \otimes \mathcal{G}\big),$$

[1] Recall that, by definition, $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is a C^*-subalgebra of $\mathcal{L}\big(B \otimes L^2(\mathcal{G})\big)$ (see Section 2.7.1).

where we are using the canonical homomorphism $A \to \mathcal{L}(\mathcal{X} \otimes \mathcal{G})$, $a \mapsto a \otimes 1_{\mathcal{G}}$ to define the balanced tensor product above. Now define the map

$$T : \mathcal{F} \cong \mathcal{K}(A, \mathcal{F}) \to \mathcal{L}(A \otimes_A \mathcal{X}, \mathcal{F} \otimes_A \mathcal{X}) \cong \mathcal{L}(\mathcal{X}, \mathcal{E}) \tag{5.3}$$

given by $T(\xi)(f) = \xi \otimes_A f$ and $T(\xi)^*(\eta \otimes_A f) = \langle \xi \mid \eta \rangle (f)$. It is easy to see that each operator $T(\xi)$ is $\mathcal{G}$-equivariant, that is, $T(\mathcal{F}) \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E})$.

Theorem 5.1.2 (Theorem 5.1 in [48])**.** *With the notations above, $T(\mathcal{F}) \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E})$ is an essential, concrete Hilbert A-module, and $T : \mathcal{F} \to T(\mathcal{F})$ is an isomorphism of Hilbert A-modules. Moreover, if $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E})$ is already an essential, concrete Hilbert A-module, then the map*

$$U : \mathcal{F} \otimes_A \mathcal{X} \to \mathcal{E}, \quad \xi \otimes_A f \mapsto \xi(f)$$

is a $\mathcal{G}$-equivariant unitary satisfying $U \circ (T(\xi)) = \xi$ for all $\xi \in \mathcal{F}$. In other words, $\mathcal{F}$ and $T(\mathcal{F})$ are isomorphic as concrete Hilbert A-modules via U.

The only difference between our version of the theorem above and Meyer's version in [48] is that we are working with quantum groups instead of classical groups. The proof goes exactly as in the classical case. Of course, one has to check that the constructions above are equivariant also in the quantum version, but this is easy. For example, to show that the unitary U above is $\mathcal{G}$-equivariant one just uses that each $\xi \in \mathcal{F}$ is an equivariant operator in $\mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E})$ to get the desired result:

$$\gamma_{\mathcal{E}}\big(U(\xi \otimes_A f)\big) = \gamma_{\mathcal{E}}\big(\xi(f)\big) = (\xi \otimes 1_{\mathcal{G}})\big(\gamma_{\mathcal{X}}(f)\big) = (U \otimes 1_{\mathcal{G}})\gamma_{\mathcal{F} \otimes_A \mathcal{X}}(\xi \otimes_A f).$$

Theorem 5.1.2 says that any abstract Hilbert A-module $\mathcal{F}$ can be represented as an essential, concrete Hilbert A-module, and this representation is unique up to canonical isomorphism. In this picture, the underlying Hilbert $B,\mathcal{G}$-module $\mathcal{E}$ is canonically isomorphic to $\mathcal{F} \otimes_A \mathcal{X}$. Observe that the assignment $\mathcal{F} \mapsto \mathcal{F} \otimes_A \mathcal{X}$ is functorial. An adjointable operator $S : \mathcal{F}_1 \to \mathcal{F}_2$ induces the equivariant adjointable operator $S \otimes_A \mathrm{id}_{\mathcal{X}} : \mathcal{F}_1 \otimes_A \mathcal{X} \to \mathcal{F}_2 \otimes_A \mathcal{X}$.

Theorem 5.1.3 (Theorem 5.2 in [48])**.** *Let $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E})$ be a concrete Hilbert A-module. Then the map*

$$|\xi\rangle\langle\eta| \mapsto \xi \circ \eta^* \in \mathcal{F} \circ \mathcal{F}^* \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{E})$$

extends to a $*$-isomorphism from $\mathcal{K}(\mathcal{F})$ onto the closed linear span of $\mathcal{F} \circ \mathcal{F}^*$ in $\mathcal{L}^{\mathcal{G}}(\mathcal{E})$. This representation is nondegenerate if and only if $\mathcal{F}$ is essential.

If $\mathcal{F}$ is essential, then this representation can be extended to a strictly continuous, injective, unital $*$-homomorphism $\phi : \mathcal{L}(\mathcal{F}) \to \mathcal{L}^{\mathcal{G}}(\mathcal{E})$, whose range is the space

$$\mathcal{M} := \{x \in \mathcal{L}^{\mathcal{G}}(\mathcal{E}) : x \circ \mathcal{F} \subseteq \mathcal{F},\, x^* \circ \mathcal{F} \subseteq \mathcal{F}\}.$$

If $\mathcal{E} = \mathcal{F} \otimes_A \mathcal{X}$ and $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E})$ is the standard representation (5.3)*, then $\phi(S) = S \otimes_A \mathrm{id}_{\mathcal{X}}$ for all $S \in \mathcal{L}(\mathcal{F})$.*

Remark 5.1.4. **(1)** In general, $\phi : \mathcal{L}(\mathcal{F}) \to \mathcal{L}^{\mathcal{G}}(\mathcal{E})$ is not surjective. This happens if and only if $u \circ \mathcal{F} = \mathcal{F}$ for all unitaries $u \in \mathcal{L}^{\mathcal{G}}(\mathcal{E})$ or, equivalently, if $\phi(\mathcal{K}(\mathcal{F})) = \overline{\operatorname{span}}\, \mathcal{F} \circ \mathcal{F}^*$ is an ideal in $\mathcal{L}^{\mathcal{G}}(\mathcal{E})$ ([48, Corollary 5.1]). In this case, $\mathcal{F}$ is called *ideal*.

(2) Let $\mathcal{F}_1 \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E}_1)$ and $\mathcal{F}_2 \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E}_2)$ be concrete Hilbert A-modules. Using the canonical representation of $\mathcal{F} := \mathcal{F}_1 \oplus \mathcal{F}_2 \subseteq \mathcal{L}^{\mathcal{G}}(\mathcal{X}, \mathcal{E})$, where $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2$, one can use Theorem 5.1.3 to get a map

$$\phi_{21} : \mathcal{K}(\mathcal{F}_1, \mathcal{F}_2) \to \mathcal{L}^{\mathcal{G}}(\mathcal{E}_1, \mathcal{E}_2),$$

In fact, if $\phi : \mathcal{K}(\mathcal{F}) \to \mathcal{L}^{\mathcal{G}}(\mathcal{E})$ is the map in Theorem 5.1.3, then, under the canonical identifications

$$\mathcal{K}(\mathcal{F}) \cong \begin{pmatrix} \mathcal{K}(\mathcal{F}_1) & \mathcal{K}(\mathcal{F}_2, \mathcal{F}_1) \\ \mathcal{K}(\mathcal{F}_1, \mathcal{F}_2) & \mathcal{K}(\mathcal{F}_2) \end{pmatrix}$$

and

$$\mathcal{L}^{\mathcal{G}}(\mathcal{E}) \cong \begin{pmatrix} \mathcal{K}^{\mathcal{G}}(\mathcal{E}_1) & \mathcal{K}^{\mathcal{G}}(\mathcal{E}_2, \mathcal{E}_1) \\ \mathcal{K}^{\mathcal{G}}(\mathcal{E}_1, \mathcal{E}_2) & \mathcal{K}^{\mathcal{G}}(\mathcal{E}_2) \end{pmatrix},$$

the map ϕ_{21} is the lower left corner of the decomposition

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}.$$

Since ϕ is a $*$-isomorphism onto

$$\mathcal{D} := \overline{\operatorname{span}}\, \mathcal{F} \circ \mathcal{F}^* \cong \begin{pmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{D}_{21} & \mathcal{D}_{22} \end{pmatrix} := \begin{pmatrix} \overline{\operatorname{span}}\, \mathcal{F}_1 \circ \mathcal{F}_1^* & \overline{\operatorname{span}}\, \mathcal{F}_1 \circ \mathcal{F}_2^* \\ \overline{\operatorname{span}}\, \mathcal{F}_2 \circ \mathcal{F}_1^* & \overline{\operatorname{span}}\, \mathcal{F}_2 \circ \mathcal{F}_2^* \end{pmatrix},$$

it follows that ϕ_{21} is a ϕ_{22}, ϕ_{11}-compatible Hilbert bimodule isomorphism of the Hilbert $\mathcal{K}(\mathcal{F}_2), \mathcal{K}(\mathcal{F}_1)$-bimodule $\mathcal{K}(\mathcal{F}_1, \mathcal{F}_2)$ onto the Hilbert $\mathcal{D}_{22}, \mathcal{D}_{11}$-bimodule $\mathcal{D}_{21}$ (defined in the canonical way).

Moreover, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are essential or, equivalently, if $\mathcal{F}$ is essential, then, by Theorem 5.1.3, ϕ extends to a strictly continuous, injective, unital $*$-homomorphism of $\mathcal{L}(\mathcal{F})$ onto

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix},$$

where, for example, $\mathcal{M}_{11} := \{x \in \mathcal{L}^{\mathcal{G}}(\mathcal{E}_1) : x \circ \mathcal{F}_1 \subseteq \mathcal{F}_1,\, x^* \circ \mathcal{F}_1 \subseteq \mathcal{F}_1\}$ and $\mathcal{M}_{21} := \{x \in \mathcal{L}^{\mathcal{G}}(\mathcal{E}_1, \mathcal{E}_2) : x \circ \mathcal{F}_1 \subseteq \mathcal{F}_2,\, x^* \circ \mathcal{F}_2 \subseteq \mathcal{F}_1\}$. It follows that ϕ_{21} extends to an injective Hilbert bimodule homomorphism from the Hilbert $\mathcal{L}(\mathcal{F}_2), \mathcal{L}(\mathcal{F}_1)$-bimodule $\mathcal{L}(\mathcal{F}_1, \mathcal{F}_2)$ onto the Hilbert $\mathcal{M}_{22}, \mathcal{M}_{11}$-bimodule $\mathcal{M}_{21}$ (defined in the canonical way).

Before finishing this section, we want to consider some special examples of concrete Hilbert modules. Given a Hilbert $B, \mathcal{G}$-module $\mathcal{E}$, we define

$$\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} := \overline{\operatorname{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})(1_B \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big) \subseteq \mathcal{L}(B \otimes H, \mathcal{E} \otimes H).$$

Here we are identifying $\mathcal{M}(\mathcal{E} \otimes \mathcal{G}) \subseteq \mathcal{M}(\mathcal{E} \otimes \mathcal{K}(H)) \cong \mathcal{L}(B \otimes H, \mathcal{E} \otimes H)$ (this last identification follows from Proposition 2.1.11 and Remark 2.1.12(2)).

On the Hilbert B-module $\mathcal{E} \otimes H$ we have a coaction defined by

$$\gamma_{\mathcal{E}\otimes H}(\zeta) = \Sigma_{23}W_{23}(\gamma_{\mathcal{E}} \otimes \mathrm{id})(\zeta), \quad \zeta \in \mathcal{E} \otimes H,$$

where $W \in \mathcal{L}(\mathcal{G} \otimes H)$ is the left regular corepresentation of $\mathcal{G}$ and $\Sigma : \mathcal{G} \otimes H \to H \otimes \mathcal{G}$ is the flip map. Since $\hat{W}^*\Sigma = \Sigma W$, the coaction defined above is the same coaction considered in Example 2.6.18(3). In particular, replacing $\mathcal{E}$ by B above, we also get a coaction of $\mathcal{G}$ on the Hilbert B-module $B \otimes H$.

Proposition 5.1.5. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module and consider on $B \otimes H$ and on $\mathcal{E} \otimes H$ the coactions of $\mathcal{G}$ defined above. Then $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E} \otimes H)$ is a concrete Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-module. Moreover, we have a canonical isomorphism*

$$\mathcal{E} \underset{\gamma_B}{\otimes} (B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \cong \mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.$$

If, in addition, the coaction of $\mathcal{G}$ on $\mathcal{E}$ is continuous, then

$$\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big((1_{\mathcal{K}(\mathcal{E})} \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{E}}(\mathcal{E})\big).$$

Moreover, in this case we have $\mathcal{K}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \cong \mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.

Proof. Note that

$$(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})^*(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \subseteq \overline{\mathrm{span}}(1_B \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{E}}(\mathcal{E})^*\gamma_{\mathcal{E}}(\mathcal{E})(1_B \otimes \widehat{\mathcal{G}}^{\mathrm{c}}) \subseteq B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$$

and

$$\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E} \cdot B)(1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big) = \overline{\mathrm{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big).$$

It remains to show that $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E} \otimes H)$. Let $\xi \in \mathcal{E}$ and $x \in \widehat{\mathcal{G}}^{\mathrm{c}}$. Then, for all $\zeta \in B \otimes H$, we have

$$\begin{aligned}
\gamma_{\mathcal{E}\otimes H}(\gamma_{\mathcal{E}}(\xi)(1_B \otimes x)\zeta) &= \Sigma_{23}W_{23}(\gamma_{\mathcal{E}} \otimes \mathrm{id})(\gamma_{\mathcal{E}}(\xi)(1_B \otimes x)\zeta) \\
&= \Sigma_{23}W_{23}(\gamma_{\mathcal{E}} \otimes \mathrm{id})(\gamma_{\mathcal{E}}(\xi))(\gamma_B \otimes \mathrm{id})((1_B \otimes x)\zeta) \\
&= \Sigma_{23}W_{23}(\mathrm{id} \otimes \Delta)(\gamma_{\mathcal{E}}(\xi))(1_B \otimes 1_{\mathcal{G}} \otimes x)(\gamma_B \otimes \mathrm{id})(\zeta) \\
&= \Sigma_{23}\gamma_{\mathcal{E}}(\xi)_{13}W_{23}(1_B \otimes 1_{\mathcal{G}} \otimes x)(\gamma_B \otimes \mathrm{id})(\zeta) \\
&= (\gamma_{\mathcal{E}}(\xi) \otimes 1_{\mathcal{G}})\Sigma_{23}(1_B \otimes 1_{\mathcal{G}} \otimes x)W_{23}(\gamma_B \otimes \mathrm{id})(\zeta) \\
&= (\gamma_{\mathcal{E}}(\xi)(1_B \otimes x) \otimes 1_{\mathcal{G}})\gamma_{B\otimes H}(\zeta).
\end{aligned}$$

In the above calculation, we have used the relation $W_{23}(\mathrm{id} \otimes \Delta)(\gamma_{\mathcal{E}}(\xi)) = \gamma_{\mathcal{E}}(\xi)_{13}W_{23}$ (which follows from $\Delta(y) = W^*(1 \otimes y)W$) and the fact that $W \in \mathcal{M}(\mathcal{G} \otimes \widehat{\mathcal{G}})$ (which implies that $W(1 \otimes x) = (1 \otimes x)W$ for all $x \in \widehat{\mathcal{G}}^{\mathrm{c}}$). This shows that the operators $\gamma_{\mathcal{E}}(\xi)(1_B \otimes x) \in \mathcal{L}(B \otimes H, \mathcal{E} \otimes H)$ are $\mathcal{G}$-equivariant for all $\xi \in \mathcal{E}$ and $x \in \widehat{\mathcal{G}}^{\mathrm{c}}$. Thus $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E} \otimes H)$ is a concrete Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-module. Finally, it is easy to see that the map

$$\xi \underset{\gamma_B}{\otimes} x \mapsto \gamma_{\mathcal{E}}(\xi)x, \quad \xi \in \mathcal{E},\, x \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$$

induces an isomorphism $\mathcal{E} \underset{\gamma_B}{\otimes} (B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \cong \mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.

For the second part suppose that $\gamma_{\mathcal{E}}$ is continuous. Recall that $\widehat{\mathcal{G}}^{\mathrm{c}}$ is the closure of $(\mathrm{id} \otimes \mathcal{L}(H)_*)(V)$, where V is the right regular corepresentation of $\mathcal{G}$. Combining this with the relation $V_{23}(\gamma_{\mathcal{E}}(\xi) \otimes 1) = (\mathrm{id} \otimes \Delta)(\gamma_{\mathcal{E}}(\xi))V_{23} = (\gamma_{\mathcal{E}} \otimes \mathrm{id})(\gamma_{\mathcal{E}}(\xi))V_{23}$ (which follows from $\Delta(x) = V(x \otimes 1)V^*$) and the fact that any element $\omega \in \mathcal{L}(H)_*$ can be written in the form $\omega = \theta a$, where $\theta \in \mathcal{L}(H)_*$ and $a \in \mathcal{G}$, we get

$$\begin{aligned}
\overline{\mathrm{span}}\big((1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{E}}(\mathcal{E})\big) &= \overline{\mathrm{span}}\big((1 \otimes (\mathrm{id} \otimes \mathcal{L}(H)_*)(V))\gamma_{\mathcal{E}}(\mathcal{E})\big) \\
&= \overline{\mathrm{span}}\big((\mathrm{id} \otimes \mathrm{id} \otimes \mathcal{L}(H)_*)(V_{23}(\gamma_{\mathcal{E}}(\mathcal{E}) \otimes 1))\big) \\
&= \overline{\mathrm{span}}\big((\mathrm{id} \otimes \mathrm{id} \otimes \mathcal{L}(H)_*)((\gamma_{\mathcal{E}} \otimes \mathrm{id})(\gamma_{\mathcal{E}}(\mathcal{E}))V_{23})\big) \\
&= \overline{\mathrm{span}}\big((\mathrm{id} \otimes \mathrm{id} \otimes \mathcal{L}(H)_*)((\gamma_{\mathcal{E}} \otimes \mathrm{id})((1 \otimes \mathcal{G})\gamma_{\mathcal{E}}(\mathcal{E}))V_{23})\big) \\
&= \overline{\mathrm{span}}\big((\mathrm{id} \otimes \mathrm{id} \otimes \mathcal{L}(H)_*)((\gamma_{\mathcal{E}} \otimes \mathrm{id})(\mathcal{E} \otimes \mathcal{G})V_{23})\big) \\
&= \overline{\mathrm{span}}\big((\mathrm{id} \otimes \mathrm{id} \otimes \mathcal{L}(H)_*)((\gamma_{\mathcal{E}}(\mathcal{E}) \otimes \mathcal{G})V_{23})\big) \\
&= \overline{\mathrm{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})(1 \otimes (\mathrm{id} \otimes \mathcal{L}(H)_*)(V))\big) \\
&= \overline{\mathrm{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})(1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big).
\end{aligned}$$

Therefore $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big((1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{E}}(\mathcal{E})\big)$. Using this together with Theorem 5.1.3 we conclude that

$$\begin{aligned}
\mathcal{K}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) &\cong \overline{\mathrm{span}}\big((\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})^*\big) \\
&= \overline{\mathrm{span}}\big((1_{\mathcal{K}(\mathcal{E})} \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{E}}(\mathcal{E})\gamma_{\mathcal{E}}(\mathcal{E})^*(1_{\mathcal{K}(\mathcal{E})} \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big) \\
&= \overline{\mathrm{span}}\big((1_{\mathcal{K}(\mathcal{E})} \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{K}(\mathcal{E})}(\mathcal{K}(\mathcal{E}))(1_{\mathcal{K}(\mathcal{E})} \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big) \\
&= \mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.
\end{aligned}$$

□

Remark 5.1.6. **(1)** Recall that for a regular quantum group $\mathcal{G}$ the coaction on a Hilbert $B,\mathcal{G}$-module $\mathcal{E}$ is automatically continuous (see Remark 2.6.9(3)). Thus in this case the hypothesis above is redundant.

(2) Suppose that $\mathcal{E}$ is a Hilbert $B,\mathcal{G}$-module with a continuous coaction of $\mathcal{G}$. Under the canonical identification $\mathcal{K}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \cong \mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ we get as a consequence that $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is a Hilbert $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-bimodule. One can also define a *dual coaction* of $\widehat{\mathcal{G}}^{\mathrm{c}}$ on $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. It is the coaction

$$\widehat{\gamma}_{\mathcal{E}}^{\mathrm{c}} : \mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \to \mathcal{M}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \otimes \widehat{\mathcal{G}}^{\mathrm{c}}),$$

given by $\widehat{\gamma}_{\mathcal{E}}^{\mathrm{c}}(\gamma_{\mathcal{E}}(\xi)(1 \otimes \hat{x})) := (\gamma_{\mathcal{E}}(\xi) \otimes 1)(1 \otimes \hat{\Delta}^{\mathrm{c}}(\hat{x}))$ for all $\xi \in \mathcal{E}$ and $\hat{x} \in \widehat{\mathcal{G}}^{\mathrm{c}}$. This coaction is compatible with the dual coactions on $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ and on $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. Thus $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is a $\widehat{\mathcal{G}}^{\mathrm{c}}$-equivariant Hilbert $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-bimodule.

More generally, if $\mathcal{E}$ is a $\mathcal{G}$-equivariant right-Hilbert A,B-bimodule, that is, if there is a left action $\pi : A \to \mathcal{L}(\mathcal{E})$ of a $\mathcal{G}$-C^*-algebra A, then there is an induced left action of $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ on $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ turning it into a $\widehat{\mathcal{G}}^{\mathrm{c}}$-equivariant right-Hilbert $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-bimodule. The left $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-action is given by $\pi \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} : A \rtimes_{\mathrm{r}} \mathcal{G}^{\mathrm{c}} \to \mathcal{M}(\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \mathcal{G}^{\mathrm{c}}) \cong \mathcal{L}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})$. In particular, if A and B are two Morita equivalent $\mathcal{G}$-C^*-algebras, then so are the $\widehat{\mathcal{G}}^{\mathrm{c}}$-$C^*$-algebras $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ and $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.

(3) Consider the dual $\mathcal{E}^*$ of $\mathcal{E}$. If $\mathcal{E}$ is a Hilbert $B, \mathcal{G}$-module with a continuous coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$, then there is a canonical coaction on $\mathcal{E}^*$ given by $\gamma_{\mathcal{E}^*}(\xi^*) := \gamma_{\mathcal{E}}(\xi)^*$. Here we identify $\gamma_{\mathcal{E}}(\xi)^* \in \mathcal{L}(\mathcal{E} \otimes \mathcal{G}, B \otimes \mathcal{G}) \hookrightarrow \mathcal{M}(\mathcal{E}^* \otimes \mathcal{G})$. The continuity of $\gamma_{\mathcal{E}}$ ensures that $\gamma_{\mathcal{E}^*}$ is a continuous coaction as well. Thus we can apply the results above to get the $\widehat{\mathcal{G}}^{\mathrm{c}}$-equivariant Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, \mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-bimodule

$$\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})^*(1_{\mathcal{E}} \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big) = \overline{\mathrm{span}}\big((1_B \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{E}}(\mathcal{E})^*\big) \subseteq \mathcal{L}(\mathcal{E} \otimes H, B \otimes H).$$

We also have

$$\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = \overline{\mathrm{span}}\big(\gamma_{\mathcal{E}}(\mathcal{E})^*(\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big)$$

and the map

$$\xi^* \otimes_{\gamma_{\mathcal{K}(\mathcal{E})}} y \mapsto \gamma_{\mathcal{E}}(\xi)^* y, \quad \xi \in \mathcal{E},\, y \in \mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$$

induces an isomorphism

$$\mathcal{E}^* \otimes_{\gamma_{\mathcal{K}(\mathcal{E})}} (\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \cong \mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.$$

Note that the dual $(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})^*$ of $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is canonically isomorphic to $\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.

5.2 Relative continuity and generalized fixed point algebras

Suppose that $\mathcal{G}$ is a locally compact quantum group. Let B be a $\mathcal{G}$-C^*-algebra and let $\mathcal{E}$ be a Hilbert $B, \mathcal{G}$-module. Recall that $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is a C^*-subalgebra of $\mathcal{L}(B \otimes H)$, where $H = L^2(\mathcal{G})$. Also recall that, given $\xi, \eta \in \mathcal{E}_{\mathrm{si}}$, the operator $\langle\!\langle \xi \mid \eta \rangle\!\rangle = \langle\!\langle \xi | \circ |\eta \rangle\!\rangle$ is the composition of the operators $\langle\!\langle \xi | \in \mathcal{L}(\mathcal{E}, B \otimes H)$ and $|\eta\rangle\!\rangle \in \mathcal{L}(B \otimes H, \mathcal{E})$. Thus $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle$ is also an element of $\mathcal{L}(B \otimes H)$. Moreover, we know (see Equation (2.27)) that $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is, in fact, a C^*-subalgebra of $\mathcal{L}^{\mathcal{G}}(B \otimes H)$, the space of $\mathcal{G}$-equivariant operators on $B \otimes H$, where we always consider on $B \otimes H$ the coaction $\gamma_{B \otimes H}$ defined by Equation (4.2). On the other hand, we also know (Proposition 4.2.7) that the operators $\langle\!\langle \xi |$ and $|\eta\rangle\!\rangle$ are $\mathcal{G}$-equivariant with respect to the same coaction on $B \otimes H$. Thus we also have $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in \mathcal{L}^{\mathcal{G}}(B \otimes H)$. Therefore it is natural to ask whether $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. This turns out to be a crucial question and, as we will see, this is not true in general.

Definition 5.2.1. Let B be a $\mathcal{G}$-C^*-algebra, let $\mathcal{E}$ be a Hilbert $B, \mathcal{G}$-module, and suppose that $\xi, \eta \in \mathcal{E}_{\mathrm{si}}$. We say that the pair (ξ, η) is *relatively continuous*, and write $\xi \overset{rc}{\sim} \eta$, if $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. A subset $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$ is called *relatively continuous* if $\xi \overset{rc}{\sim} \eta$ for all $\xi, \eta \in \mathcal{R}$, that is,

$$\langle\!\langle \mathcal{R} \,|\, \mathcal{R} \rangle\!\rangle := \{\langle\!\langle \xi \,|\, \eta \rangle\!\rangle : \xi, \eta \in \mathcal{R}\} \subseteq B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.$$

For a relatively continuous subset $\mathcal{R}$ of $\mathcal{E}$, we define the following subspaces

$$\mathcal{F}(\mathcal{E}, \mathcal{R}) := \overline{\mathrm{span}}(|\mathcal{R}\rangle\!\rangle \circ B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \subseteq \mathcal{L}(B \otimes H, \mathcal{E}),$$

$$\mathcal{I}(\mathcal{E}, \mathcal{R}) := \overline{\mathrm{span}}(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \circ \langle\!\langle \mathcal{R} \,|\, \mathcal{R} \rangle\!\rangle \circ B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \subseteq B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}},$$

and the *generalized fixed point algebra*

$$\mathrm{Fix}(\mathcal{E},\mathcal{R}) := \overline{\mathrm{span}}(|\mathcal{R}\rangle\!\rangle \circ B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \circ \langle\!\langle \mathcal{R}|) \subseteq \mathcal{L}(\mathcal{E}).$$

We say that $\mathcal{R}$ is *saturated* if $\mathcal{I}(\mathcal{E},\mathcal{R}) = B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.

Remark 5.2.2. (1) Relative continuity was first defined by Exel [19] in the case of Abelian groups and was generalized to non-Abelian groups by Meyer in [48]. Our definition generalizes Meyer's definition to quantum groups. Exel's definition involves integrable elements instead of square-integrable elements as above, but this turns out to be equivalent (see [9]).

(2) Relative continuity is *not* an equivalence relation. For instance, it is not true, in general, that $\xi \overset{rc}{\sim} \xi$. Thus it is not reflexive in general. Of course, it is symmetric, that is, if $\xi \overset{rc}{\sim} \eta$, then $\eta \overset{rc}{\sim} \xi$. Note that it is not transitive, that is, the conditions $\xi \overset{rc}{\sim} \eta$ and $\eta \overset{rc}{\sim} \zeta$ do not imply that $\xi \overset{rc}{\sim} \zeta$. In fact, we always have $\xi \overset{rc}{\sim} 0$ and $0 \overset{rc}{\sim} \zeta$.

(3) Observe that we assume continuity of the coaction of $\mathcal{G}$ on B in the definition above. The reason is because we need the reduced crossed product $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, and continuity is a very natural condition in connection with crossed products. Remember, however, that given a Hilbert $B,\mathcal{G}$-module $\mathcal{E}$, we do not assume that the coaction of $\mathcal{G}$ on $\mathcal{E}$ is continuous (only the coaction of $\mathcal{G}$ on B is assumed to be continuous).

Recall that the canonical coaction of $\mathcal{G}$ on $\mathcal{K}\big(L^2(\mathcal{G})\big)$ is continuous if and only if $\mathcal{G}$ is regular (Proposition 2.7.11). Thus the concept of relative continuity is not defined in this case, unless $\mathcal{G}$ is regular. However, it is defined for the coaction of $\mathcal{G}$ on $L^2(\mathcal{G})$ or, more generally, for the coaction of $\mathcal{G}$ on $B \otimes L^2(\mathcal{G})$, where B is any $\mathcal{G}$-C^*-algebra, and this will turn out to be the most important example in order to develop the theory.

(4) Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module, and assume that $\mathcal{R} \subseteq \mathcal{E}$ is a relatively continuous subset. It is clear from the definition above that $\mathcal{I}(\mathcal{E},\mathcal{R})$ is an ideal of $A := B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ and $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{E})$. Let $\mathcal{F} := \mathcal{F}(\mathcal{E},\mathcal{R})$. Since $\langle\!\langle \mathcal{R} \,|\, \mathcal{R} \rangle\!\rangle \subseteq A$, we have

$$\overline{\mathrm{span}}\, |\mathcal{R}\rangle\!\rangle \subseteq \mathcal{F}. \tag{5.4}$$

In fact, let (e_i) be an approximate unit for A. If T is an operator on $B \otimes H$ such that $T^*T \in A$, then we have

$$\|Te_i - T\|^2 = \|e_iT^*Te_i - e_iT^*T - T^*Te_i + T^*T\| \to 0.$$

Note that, by definition, we have

$$\mathcal{I}(\mathcal{E},\mathcal{R}) = \overline{\mathrm{span}}\, \mathcal{F}^* \circ \mathcal{F} \quad \text{and} \quad \mathrm{Fix}(\mathcal{E},\mathcal{R}) = \overline{\mathrm{span}}\, \mathcal{F} \circ \mathcal{F}^*. \tag{5.5}$$

In particular, we get

$$\overline{\mathrm{span}}\langle\!\langle \mathcal{R} \,|\, \mathcal{R} \rangle\!\rangle \subseteq \mathcal{I}(\mathcal{E},\mathcal{R}) \quad \text{and} \quad \overline{\mathrm{span}}\, |\mathcal{R}\rangle\!\rangle\langle\!\langle \mathcal{R}| \subseteq \mathrm{Fix}(\mathcal{E},\mathcal{R}). \tag{5.6}$$

We are going to see later that the inclusions (5.4) and (5.6) become equalities if we impose more conditions on $\mathcal{R}$.

Proposition 5.2.3. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module, let $\mathcal{R}\subseteq\mathcal{E}_{\mathrm{si}}$ be a relatively continuous subset and denote $A:=B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$. Then $\mathcal{F}:=\mathcal{F}(\mathcal{E},\mathcal{R})\subseteq\mathcal{L}^{\mathcal{G}}(B\otimes H,\mathcal{E})$ is a concrete Hilbert A-module. Moreover, if $\mathcal{R}$ is dense in $\mathcal{E}$, then $\mathcal{F}$ is essential.*

Proof. Since $A\subseteq\mathcal{L}^{\mathcal{G}}(B\otimes H)$, and since the bra-ket operators are $\mathcal{G}$-equivariant (see Proposition 4.2.7), we have $\mathcal{F}\subseteq\mathcal{L}^{\mathcal{G}}(B\otimes H,\mathcal{E})$. By definition, $\mathcal{F}\circ A\subseteq\mathcal{F}$ and $\mathcal{F}^*\circ\mathcal{F}\subseteq A$, and hence $\mathcal{F}$ is a concrete Hilbert A-module. Suppose that $\mathcal{R}$ is dense in $\mathcal{E}$. Then, by Lemma 4.5.4 and because A is a nondegenerate C^*-subalgebra of $\mathcal{L}(B\otimes H)$, we get

$$\overline{\operatorname{span}}\big(\mathcal{F}(B\otimes H)\big)=\overline{\operatorname{span}}\big(|\mathcal{R}\rangle\!\rangle\circ A(B\otimes H)\big)=\overline{\operatorname{span}}\big(|\mathcal{R}\rangle\!\rangle(B\otimes H)\big)=\mathcal{E}.$$

Therefore $\mathcal{F}$ is essential. $\square$

Proposition 5.2.3 and Equation (5.5) show that $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ is contained in $\mathcal{L}^{\mathcal{G}}(\mathcal{E})$. Since $\mathcal{L}^{\mathcal{G}}(\mathcal{E})$ is (under the canonical identification $\mathcal{L}(\mathcal{E})\cong\mathcal{M}(\mathcal{K}(\mathcal{E}))$) the fixed point algebra $\mathcal{M}_1\big(\mathcal{K}(\mathcal{E})\big)=\{x\in\mathcal{M}(\mathcal{K}(\mathcal{E})):\gamma_{\mathcal{K}(\mathcal{E})}(x)=x\otimes 1\}$ (see Proposition 2.6.13), we see that the elements of $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ are fixed by the coaction of $\mathcal{K}(\mathcal{E})$ and $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ is a C^*-subalgebra of $\mathcal{M}_1\big(\mathcal{K}(\mathcal{E})\big)$. Note that, by Theorem 5.1.3, $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ is a nondegenerate C^*-subalgebra of $\mathcal{L}(\mathcal{E})$ if and only if $\mathcal{F}(\mathcal{E},\mathcal{R})$ is essential. For instance, this is the case if $\mathcal{R}$ is dense.

Proposition 5.2.4. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module, and let $\mathcal{R}\subseteq\mathcal{E}_{\mathrm{si}}$ be a relatively continuous subset of $\mathcal{E}$. Then $\mathcal{F}(\mathcal{E},\mathcal{R})$ is a Morita equivalence between the generalized fixed point algebra $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ and the ideal $\mathcal{I}(\mathcal{E},\mathcal{R})$ in $B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$.*

Proof. Theorem 5.1.3 yields a canonical identification $\mathcal{K}(\mathcal{F})\cong\overline{\operatorname{span}}\,\mathcal{F}\circ\mathcal{F}^*=\mathrm{Fix}(\mathcal{E},\mathcal{R})$, where $\mathcal{F}:=\mathcal{F}(\mathcal{E},\mathcal{R})$. And by definition of the A-valued inner product on $\mathcal{F}$ (see Equation (5.2)), we have $\overline{\operatorname{span}}\{\langle x|y\rangle:x,y\in\mathcal{F}\}=\overline{\operatorname{span}}\,\mathcal{F}^*\mathcal{F}=\mathcal{I}(\mathcal{E},\mathcal{R})$. $\square$

In the situation above, $\mathcal{R}$ is, by definition, saturated if and only if $\mathcal{I}(\mathcal{E},\mathcal{R})$ is the entire reduced crossed product $B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$. Thus, in this case, $\mathcal{F}(\mathcal{E},\mathcal{R})$ is a Morita equivalence between $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ and $B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$.

Example 5.2.5 (The group case). Let G be a locally compact group and consider the corresponding quantum group $\mathcal{G}=\mathcal{C}_0(G)$. Let $\mathcal{E}$ be a Hilbert B,G-module, that is, a Hilbert B-module with a (continuous) action γ of G compatible with a (continuous) action β of G on B. Given $\xi,\eta\in\mathcal{E}_{\mathrm{si}}$, we already know (see Example 4.1.8) that the operators $\langle\!\langle\xi|\in\mathcal{L}^G\big(L^2(G,B),\mathcal{E}\big)$ and $|\eta\rangle\!\rangle\in\mathcal{L}^G\big(\mathcal{E},L^2(G,B)\big)$ are given by the formulas

$$\langle\!\langle\xi|\zeta|_t=\langle\gamma_t(\xi)\,|\,\zeta\rangle_B\quad\text{and}\quad|\eta\rangle\!\rangle f=\int_G\gamma_t(\eta)f(t)\,\mathrm{d}t$$

for all $\zeta\in\mathcal{E}$, $t\in G$ and $f\in\mathcal{C}_c(G,B)\subseteq L^2(G,B)$. It follows from the formulas above that the operator $\langle\!\langle\xi\,|\,\eta\rangle\!\rangle\in\mathcal{L}^G\big(L^2(G,B)\big)$ is given by

$$\langle\!\langle\xi\,|\,\eta\rangle\!\rangle f|_t=\int_G\langle\gamma_t(\xi)\,|\,\gamma_s(\eta)\rangle f(s)\,\mathrm{d}s=\int_G\beta_t\big(\langle\xi\,|\,\gamma_{t^{-1}s}(\eta)\rangle\big)f(s)\,\mathrm{d}s,\tag{5.7}$$

for all $f \in \mathcal{C}_c(G,B)$ and $t \in G$. Now recall from Remark 2.7.1 that the reduced crossed product $C^*_{\mathrm{r}}(G,B) \subseteq \mathcal{L}^G\big(L^2(G,B)\big)$ is generated by the operators ρ_K, with $K \in \mathcal{C}_c(G,B)$, where

$$\rho_K(f)|_t = \int_G \beta_t\big(K(t^{-1}s)\big) f(s)\,\mathrm{d}s$$

for all $f \in \mathcal{C}_c(G,B)$ and $t \in G$. An operator in the above form for a (not necessarily compactly supported) continuous function $K : G \to B$ is called a *Laurent operator* with *symbol* K ([19, 48]). Comparing with Equation (5.7), we see that $\langle\!\langle \xi \mid \eta \rangle\!\rangle$ is a Laurent operator with symbol (also denoted by) $\langle\!\langle \xi \mid \eta \rangle\!\rangle(t) := \langle \xi \mid \gamma_t(\eta)\rangle$. In particular, we see that if $\langle\!\langle \xi \mid \eta \rangle\!\rangle \in \mathcal{C}_c(G,B)$, then $\xi \overset{rc}{\sim} \eta$. In particular, if G is compact, then every subset of $\mathcal{E}$ is relatively continuous. In general, of course, the problem is that we only have $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in \mathcal{C}_b(G,B)$ and it may happen that $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \notin C^*_{\mathrm{r}}(G,B)$.

Let us consider a special situation which is, in fact, the motivation for all the theory of relative continuity and generalized fixed point algebras. Namely, we consider G acting on a locally compact (Hausdorff) space X. Let α be the induced action of G on $B = \mathcal{C}_0(X)$: $\alpha_t(f)(p) := f(t^{-1} \cdot p)$ for all $t \in G$, $f \in \mathcal{C}_0(X)$ and $p \in X$.

Let $\xi, \eta \in \mathcal{C}_c(X)$ and consider the kernel function $\langle\!\langle \xi \mid \eta \rangle\!\rangle \in \mathcal{C}_b(G,B)$ given as above by $\langle\!\langle \xi \mid \eta \rangle\!\rangle(t) := \langle \xi | \alpha_t(\eta)\rangle = \overline{\xi}\alpha_t(\eta)$ for $t \in G$. Since $B = \mathcal{C}_0(X)$, we can regard $\langle\!\langle \xi \mid \eta \rangle\!\rangle$ as a function of two variables given by $\langle\!\langle \xi \mid \eta \rangle\!\rangle(t,p) = \overline{\xi(p)}\eta(t^{-1} \cdot p)$ for all $t \in G$ and $p \in X$. Now suppose that the action of G on X is proper (as already mentioned, the action on B is integrable if and only if X is a proper G-space). Then it follows that $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in \mathcal{C}_c(G \times X) \subseteq \mathcal{C}_c(G,B)$ and therefore the corresponding Laurent operator $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle$ belongs to $C^*_{\mathrm{r}}(G,B)$. By Proposition 6.5 in [48], this implies that $\mathcal{R} := \mathcal{C}_c(X)$ consists of square-integrable elements and is relatively continuous.

Now we describe the Hilbert $C^*_{\mathrm{r}}(G,B)$-module $\mathcal{F} := \mathcal{F}(B,\mathcal{R})$. Given $\xi \in \mathcal{R}$ and $K \in \mathcal{C}_c(G,B)$, we define

$$\xi * K := \int_G \alpha_t\big(\xi K(t^{-1})\big)\,\mathrm{d}t.$$

Then it is easy to see that $|\xi * K\rangle\!\rangle = |\xi\rangle\!\rangle \circ \rho_K$ (in fact, this is true for any Hilbert B,G-module; see [48]). Note that $\xi * K$ is represented by the function

$$(\xi * K)(p) = \int_G \xi(t^{-1} \cdot p) K(t^{-1}, t^{-1} \cdot p)\,\mathrm{d}t = \int_G \xi(t \cdot p) K(t, t \cdot p)\delta_G(t)^{-1}\,\mathrm{d}t.$$

Thus $\xi * K$ belongs to $\mathcal{C}_c(X)$ and hence the operation above turns $\mathcal{C}_c(X)$ into a right $\mathcal{C}_c(G,B)$-module. Since $C^*_{\mathrm{r}}(G,B)$ is the closure of the operators ρ_K, where $K \in \mathcal{C}_c(G,B)$, we see that $\mathcal{F}$ is the closure of $|\mathcal{C}_c(X)\rangle\!\rangle$ in $\mathcal{L}^G(L^2(G,B),B)$. If we forget the representation $K \mapsto \rho_K$ and so identify $\mathcal{C}_c(G,B) \subseteq C^*_{\mathrm{r}}(G,B)$, then $\mathcal{C}_c(X)$ can be seen as a pre-Hilbert $C^*_{\mathrm{r}}(G,B)$-module with respect to the inner product $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle(r,p) = \overline{\xi(p)}\eta(r^{-1} \cdot p)$. Moreover, it follows that the completion of $C_c(X)$ is isomorphic to $\mathcal{F}$ via the map $\xi \mapsto |\xi\rangle\!\rangle$.

Finally, we describe the generalized fixed point algebra $\mathrm{Fix}(B,\mathcal{R})$. We claim that $\mathrm{Fix}(B,\mathcal{R}) \cong \mathcal{C}_0(G\backslash X)$, where $G\backslash X$ denotes the quotient space. Note that $\mathrm{Fix}(B,\mathcal{R})$ is the closed linear span of the operators $E_1(\xi\overline{\eta}) = |\xi\rangle\!\rangle\langle\!\langle \eta|$, where $\xi, \eta \in \mathcal{C}_c(X)$. Thus

$$\mathrm{Fix}(B,\mathcal{R}) = \overline{\{E_1(f) : f \in \mathcal{C}_c(X)\}}.$$

By Corollary 3.2.19 and Proposition 4.1.10(i), the element $E_1(f) \in \mathcal{M}(B)$ is equal to the strict unconditional integral $\int_G^{\mathrm{su}} \alpha_t(f)\,\mathrm{d}t$, and under the canonical identification $\mathcal{M}(B) \cong \mathcal{C}_b(X)$, it is represented by the function

$$E_1(f)(p) = \int_G f(t^{-1}\cdot p)\,\mathrm{d}t.$$

Note that $E_1(f)(s\cdot p) = E_1(f)(p)$ for all $s \in G$ and $p \in X$. Thus $E_1(f) \in \mathcal{C}_b(G\backslash X)$ (here we identify $\mathcal{C}_b(G\backslash X) \subseteq \mathcal{C}_b(X)$ via the quotient map $X \to G\backslash X$). Moreover, since $f \in \mathcal{C}_c(X)$, we have $E_1(f) \in \mathcal{C}_c(G\backslash X) \subseteq \mathcal{C}_0(G\backslash X)$. Hence $\mathrm{Fix}(B,\mathcal{R}) \subseteq \mathcal{C}_0(G\backslash X)$. This inclusion has dense image by the Stone–Weierstrass theorem.

We conclude that $\mathrm{Fix}(B,\mathcal{R}) \cong \mathcal{C}_0(G\backslash X)$. In particular, $\mathcal{F}$ is a Morita equivalence between $\mathcal{C}_0(G\backslash X)$ and the ideal of $C^*_{\mathrm{r}}(G,B)$ generated by the functions $\langle\!\langle \xi\,|\,\eta\rangle\!\rangle \in \mathcal{C}_c(G,B)$, $\xi,\eta \in \mathcal{C}_c(X)$.

Before finishing this example, let us mention that in the situation above the full and the reduced crossed products $C^*_{\mathrm{r}}(G,B)$ and $C^*(G,B)$ are isomorphic (see [60, Theorem 6.1]).

Finally, we mention that $\mathcal{R}$ is saturated if and only if the action of G on X is free and therefore, in this case, $\mathcal{C}_0(G\backslash X)$ is Morita equivalent to $C^*_{\mathrm{r}}(G,B)$. Thus we can think of saturation as a generalization of freeness.

For a further discussion of this example, see [64, 65, 66] and the references therein.

From the previous example, it follows, in particular, that $\mathcal{C}_c(G)$ is a relatively continuous subspace of the G-C^*-algebra $\mathcal{C}_0(G)$ endowed with the action $\alpha_t(f)|_s = f(t^{-1}s)$ for all $t,s \in G$ and $f \in \mathcal{C}_0(G)$. This action is equivalent to the action obtained by considering $\mathcal{C}_0(G)$ as a quantum group and letting it coact on itself by the comultiplication.

Thus, it is natural to ask what happens in the general case of a locally compact quantum group $\mathcal{G}$ coacting on itself by the comultiplication. By Proposition 3.2.12, we know that $\mathcal{G}$ is an integrable $\mathcal{G}$-C^*-algebra in this way. Is there some (dense) relatively continuous subspace of $\mathcal{G}$? The following result answers this question.

Proposition 5.2.6. *Let $\mathcal{G}$ be a locally compact quantum group and let $\mathcal{G}$ coact on itself by the comultiplication Δ. Then there is a non-zero relatively continuous subset of $\mathcal{G}$ if and only if $\mathcal{G}$ is semi-regular. In this case, any subset $\mathcal{R} \subseteq \mathcal{G}_{\mathrm{si}}$ is relatively continuous (in particular, $\mathcal{G}_{\mathrm{si}}$ itself is relatively continuous) and*

$$\mathcal{F}(\mathcal{G},\mathcal{R}) = (1_{\mathcal{G}} \otimes H_0^*)W \subseteq \mathcal{L}(\mathcal{G}\otimes H,\mathcal{G}),^{2}$$

where $H_0 := \overline{\mathrm{span}}\big(\widehat{\mathcal{G}}^{\mathrm{c}}\Lambda(\mathcal{G}\mathcal{R}^*)\big) \subseteq H$ and $W \in \mathcal{L}(\mathcal{G}\otimes H)$ is the left regular corepresentation of $\mathcal{G}$. In particular, if $\mathcal{R} \neq \{0\}$, then

$$\mathrm{Fix}(\mathcal{G},\mathcal{R}) = \mathbb{C}1_{\mathcal{G}} \cong \mathbb{C} \quad \textit{and} \quad \mathcal{I}(\mathcal{G},\mathcal{R}) = W^*(1\otimes\mathcal{K}(H_0))W \cong \mathcal{K}(H_0).$$

There is a saturated, relatively continuous subset of $\mathcal{G}$ if and only if $\mathcal{G}$ is regular.

[2] Here H_0^* denotes the set of all $\xi^* \in \mathcal{L}(H,\mathbb{C})$, with $\xi \in H_0$. Recall that ξ^* is the element of $\mathcal{L}(H,\mathbb{C})$ given by $\xi^*(\eta) = \langle \xi\,|\,\eta\rangle$ for all $\eta \in H$.

Proof. By Proposition 3.2.12, we have $\mathcal{G}_{\mathrm{si}} = \mathcal{N}_\varphi^*$ and, for all $\xi, \eta \in \mathcal{N}_\varphi^* = \mathcal{G}_{\mathrm{si}}$,

$$\langle\!\langle \xi \,|\, \eta \rangle\!\rangle = W^*(1 \otimes |\Lambda(\xi^*)\rangle\langle\Lambda(\eta^*)|)W \in W^*(1 \otimes \mathcal{K}(H))W. \tag{5.8}$$

Furthermore, we have

$$\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = W^*(1 \otimes C)W, \tag{5.9}$$

where $C := \overline{\operatorname{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}})$ (see Section 2.7.4). Thus the existence of a non-zero relatively continuous subset implies $C \cap \mathcal{K}(H) \neq \{0\}$, so that $\mathcal{G}$ is semi-regular by Proposition 2.7.9. Conversely, if $\mathcal{G}$ is semi-regular, then $\mathcal{K}(H) \subseteq C$ and hence any subset $\mathcal{R} \subseteq \mathcal{G}_{\mathrm{si}} = \mathcal{N}_\varphi^*$ is relatively continuous by the same calculation as above. Moreover, by Proposition 3.2.12, we have $|\xi\rangle\!\rangle = (1 \otimes \Lambda(\xi^*)^*)W$ for all $\xi \in \mathcal{G}_{\mathrm{si}}$. Thus, Equation (5.9) implies

$$\mathcal{F}(\mathcal{G}, \mathcal{R}) = \overline{\operatorname{span}}\big((1 \otimes \Lambda(\mathcal{R}^*)^*)W(\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big) = \overline{\operatorname{span}}\big((1 \otimes \Lambda(\mathcal{R}^*)^*C)W\big) = (1 \otimes H_0^*)W.$$

Finally, we prove that there is a saturated relatively continuous subset $\mathcal{R} \subseteq \mathcal{G}$ if and only if $\mathcal{G}$ is regular. In fact, if $\mathcal{G}$ is regular, then $\mathcal{K}(H) = C$ and hence for $\mathcal{R} = \mathcal{N}_\varphi^*$ we get

$$\begin{aligned}\overline{\operatorname{span}}\langle\!\langle \mathcal{R} \,|\, \mathcal{R} \rangle\!\rangle &= W^*(1 \otimes (\overline{\operatorname{span}}\{|\Lambda(\xi)\rangle\langle\Lambda(\eta)| : \xi, \eta \in \mathcal{N}_\varphi\}))W \\ &= W^*(1 \otimes \mathcal{K}(H))W = \mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.\end{aligned}$$

It follows from Equation (5.6) that $\mathcal{I}(\mathcal{G}, \mathcal{R}) = \mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, that is, $\mathcal{R}$ is saturated. Conversely, suppose that $\mathcal{R} \subseteq \mathcal{G}_{\mathrm{si}} = \mathcal{N}_\varphi^*$ is relatively continuous and saturated. In particular, $\mathcal{R} \neq \{0\}$ and hence $\mathcal{G}$ is semi-regular. It follows from Equations (5.8) and (5.9) that

$$\begin{aligned}\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} &= \mathcal{I}(\mathcal{G}, \mathcal{R}) = \overline{\operatorname{span}}\big(\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}(\langle\!\langle \mathcal{R} \,|\, \mathcal{R} \rangle\!\rangle)\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}\big) \\ &= \overline{\operatorname{span}}\big(W^*(1 \otimes C)(|\Lambda(\mathcal{R}^*)\rangle\langle\Lambda(\mathcal{R}^*)|)(1 \otimes C)W\big) \\ &\subseteq W^*(1 \otimes \mathcal{K}(H))W \subseteq \mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.\end{aligned}$$

In the last inclusion above we have used the semi-regularity of $\mathcal{G}$. We conclude that $W^*(1 \otimes \mathcal{K}(H))W = \mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ and this is equivalent to the regularity of $\mathcal{G}$. ☐

Next, we analyze the $\mathcal{G}$-Hilbert space $H = L^2(\mathcal{G})$. Recall that the coaction on H is given by $\gamma_H(\xi) = \hat{W}^*(\xi \otimes 1)$ for all $\xi \in H$, where $\hat{W}$ is the left regular corepresentation of the dual $\widehat{\mathcal{G}}$. We already know that H is square-integrable. In fact, this will follow again from the result below where we show that we can also always find a dense, relatively continuous subspace of H.

Before stating the result we need some preparation. Recall that $\mathcal{G}$ is equal to the closure in $\mathcal{L}(H)$ of the space of the operators $\hat{\lambda}(\omega) = (\omega \otimes \mathrm{id})(\hat{W})$ with $\omega \in \mathcal{L}(H)_*$. Similarly, the dual $\widehat{\mathcal{G}}$ of $\mathcal{G}$ is given by the closure of the operators $\lambda(\omega) = (\omega \otimes \mathrm{id})(W)$ with $\omega \in \mathcal{L}(H)_*$. By Theorem 1.11.13 in [73], the dual left Haar weight $\hat{\varphi}$ of $\widehat{\mathcal{G}}$ has a GNS-construction of the form $(H, \hat{\iota}, \hat{\Lambda})$, where $\hat{\iota}$ denotes the inclusion $\widehat{\mathcal{G}} \hookrightarrow \mathcal{L}(H)$.

Let $\mathcal{T}_{\hat{\varphi}} \subseteq \widehat{\mathcal{G}}$ be the Tomita $*$-algebra of the dual left Haar weight $\hat{\varphi}$. We need the following result from [73, Proposition 1.11.25] (applied to the dual $\widehat{\mathcal{G}}$):[3]

[3]Recall that our convention for Hilbert spaces is that the inner product is linear in the second variable. In this way we get some modified results in comparison with results using the other convention as, for example, in [73]. The next result is one example.

Lemma 5.2.7. *For every $a \in \mathcal{T}_{\hat{\varphi}}$ and $\eta \in H$, we have*

$$\hat{\lambda}(\omega_{\hat{\Lambda}(a),\eta}) \in \mathcal{N}_\varphi \quad \textit{and} \quad \Lambda\big(\hat{\lambda}(\omega_{\hat{\Lambda}(a),\eta})\big) = \hat{J}\hat{\sigma}_{\frac{\mathrm{i}}{2}}(a)\hat{J}\eta,$$

where $\hat{\sigma}$ is the modular group of $\hat{\varphi}$ and $\hat{J}$ is the modular conjugation of $\hat{\varphi}$ in the GNS-construction $(H, \hat{\iota}, \hat{\Lambda})$.

Recall that $\widehat{\mathcal{G}}^{\mathrm{c}} = \hat{J}\widehat{\mathcal{G}}\hat{J}$ denotes the C^*-commutant of the dual $\widehat{\mathcal{G}}$.

Proposition 5.2.8. *Let $\mathcal{G}$ be a locally compact quantum group and consider $H = L^2(\mathcal{G})$ with the coaction γ_H defined above. Then $\mathcal{R} := \hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})$ is a dense, relatively continuous subspace of H and we have $|\xi\rangle\!\rangle = \hat{J}\hat{\sigma}_{\frac{\mathrm{i}}{2}}(a)^*\hat{J}$ for all $\xi = \hat{\Lambda}(a) \in \mathcal{R}$. Moreover,*

$$\mathcal{F}(H, \mathcal{R}) = \mathcal{I}(H, \mathcal{R}) = \mathrm{Fix}(H, \mathcal{R}) = \widehat{\mathcal{G}}^{\mathrm{c}}.$$

Proof. By definition, we have

$$\xi \in H_{\mathrm{si}} \iff \gamma_H(\xi)^*(\eta \otimes 1) \in \bar{\mathcal{N}_\varphi},\ \forall\, \eta \in H \iff$$

$$(\xi^* \otimes 1)\hat{W}(\eta \otimes 1) = (\omega_{\xi,\eta} \otimes \mathrm{id})(\hat{W}) = \hat{\lambda}(\omega_{\xi,\eta}) \in \bar{\mathcal{N}_\varphi},\ \forall\, \eta \in H.$$

Lemma 5.2.7 implies that $\xi := \hat{\Lambda}(a) \in H_{\mathrm{si}}$ for all $a \in \mathcal{T}_{\hat{\varphi}}$, and

$$\langle\!\langle \xi | \eta = \hat{J}\hat{\sigma}_{\frac{\mathrm{i}}{2}}(a)\hat{J}\eta, \quad \forall\, \eta \in H.$$

In other words, we have $|\xi\rangle\!\rangle = \hat{J}\hat{\sigma}_{\frac{\mathrm{i}}{2}}(a)^*\hat{J}$. Moreover, since $\hat{J}\hat{\sigma}_{\frac{\mathrm{i}}{2}}(a)^*\hat{J} \in \widehat{\mathcal{G}}^{\mathrm{c}} = \mathbb{C} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, we get that $\mathcal{R} = \hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})$ is a dense, relatively continuous subspace of H. Since $\hat{J}\hat{\sigma}_{\frac{\mathrm{i}}{2}}(\mathcal{T}_{\hat{\varphi}})^*\hat{J}$ is dense in $\widehat{\mathcal{G}}^{\mathrm{c}}$, we conclude that

$$\mathcal{F}(H, \mathcal{R}) = \overline{\mathrm{span}}(|\mathcal{R}\rangle\!\rangle\widehat{\mathcal{G}}^{\mathrm{c}}) = \overline{\mathrm{span}}(\hat{J}\hat{\sigma}_{\frac{\mathrm{i}}{2}}(\mathcal{T}_{\hat{\varphi}})^*\hat{J}\widehat{\mathcal{G}}^{\mathrm{c}}) = \widehat{\mathcal{G}}^{\mathrm{c}}.$$

And hence

$$\mathrm{Fix}(H, \mathcal{R}) = \mathcal{I}(H, \mathcal{R}) = \widehat{\mathcal{G}}^{\mathrm{c}}. \qquad \square$$

Next, we consider one of the most important examples, namely, the Hilbert $B,\mathcal{G}$-module $B \otimes L^2(\mathcal{G})$, where B is some fixed $\mathcal{G}$-C^*-algebra. Recall that we always consider $B \otimes L^2(\mathcal{G})$ endowed with the coaction defined by Equation (4.2):

$$\gamma_{B\otimes H}(\zeta) = \Sigma_{23}W_{23}(\gamma_B \otimes \mathrm{id})(\zeta), \quad \zeta \in B \otimes H.$$

Proposition 5.2.9. *Let B be a $\mathcal{G}$-C^*-algebra. Then $\mathcal{R} := B \odot \hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})$ is a dense, relatively continuous subspace of the Hilbert $B,\mathcal{G}$-module $B \otimes H$, and*

$$|b \otimes \xi\rangle\!\rangle = \big(1_B \otimes \hat{J}\hat{\sigma}_{\frac{\mathrm{i}}{2}}(a)^*\hat{J}\big)\gamma_B(b) \quad \textit{for all } b \in B \textit{ and } \xi = \hat{\Lambda}(a) \in \hat{\Lambda}(\mathcal{T}_{\hat{\varphi}}).$$

Moreover, $\mathcal{F}(B \otimes H, \mathcal{R}) = \mathcal{I}(B \otimes H, \mathcal{R}) = \mathrm{Fix}(B \otimes H) = B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.

Proof. Note that $1_B\otimes\xi\in\mathcal{M}(B\otimes H)_{\mathrm{si}}$ for all $\xi\in H_{\mathrm{si}}$. In fact, we have $1_B\otimes\xi\in\mathcal{M}(B\otimes H)_{\mathrm{si}}$ if and only if $1_B\otimes|\xi\rangle\langle\xi|\in\mathcal{M}(B\otimes\mathcal{K}(H))_{\mathrm{i}}$, and this follows from the fact that the map $\mathcal{K}(H)\ni T\mapsto 1_B\otimes T\in\mathcal{M}(B\otimes\mathcal{K}(H))$ is $\mathcal{G}$-equivariant (see Proposition 3.3.1). Moreover, we claim that

$$|1_B\otimes\xi\rangle\!\rangle = 1_B\otimes|\xi\rangle\!\rangle,\quad \xi\in H_{\mathrm{si}}.$$

Indeed, it is easy to see that $\gamma_{B\otimes H}(1_B\otimes\xi)=1_B\otimes\gamma_H(\xi)$ and hence, for all $c\in B,\eta\in H$,

$$\begin{aligned}\langle\!\langle 1_B\otimes\xi|(c\otimes\eta) &= (\mathrm{id}_B\otimes\Lambda)\big(\gamma_{B\otimes H}(1_B\otimes\xi)^*(c\otimes\eta\otimes 1_{\mathcal{G}})\big)\\ &= (\mathrm{id}_B\otimes\Lambda)\big(c\otimes\gamma_H(\xi)^*(\eta\otimes 1_{\mathcal{G}})\big)\\ &= c\otimes\Lambda\big(\gamma_H(\xi)^*(\eta\otimes 1_{\mathcal{G}})\big)\\ &= c\otimes\langle\!\langle\xi|\eta.\end{aligned}$$

Therefore $\langle\!\langle 1_B\otimes\xi| = 1_B\otimes\langle\!\langle\xi|$ or, equivalently, $|1_B\otimes\xi\rangle\!\rangle = 1_B\otimes|\xi\rangle\!\rangle$, for all $\xi\in H_{\mathrm{si}}$, as claimed. It follows from Proposition 4.1.10(ii) that $b\otimes\xi=(1_B\otimes\xi)b\in(B\otimes H)_{\mathrm{si}}$ for all $b\in B$ and $\xi\in H_{\mathrm{si}}$ and

$$|b\otimes\xi\rangle\!\rangle = (1_B\otimes|\xi\rangle\!\rangle)\gamma_B(b).$$

In particular, if $\xi=\hat\Lambda(a)$, where $a\in\mathcal{T}_{\hat\varphi}$, then it follows from Proposition 5.2.8 that $b\otimes\xi\in(B\otimes H)_{\mathrm{si}}$ and

$$|b\otimes\xi\rangle\!\rangle = \big(1_B\otimes\hat J\hat\sigma_{\frac{\mathrm{i}}{2}}(a)^*\hat J\big)\gamma_B(b)\in(1\otimes\widehat{\mathcal{G}}^{\mathrm{c}})\gamma_B(B)\subseteq B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}. \tag{5.10}$$

Thus $\mathcal{R}:=B\odot\hat\Lambda(\mathcal{T}_{\hat\varphi})$ is a dense, relatively continuous subspace of $B\otimes H$, and

$$\begin{aligned}\mathcal{F}(B\otimes H,\mathcal{R}) &= \overline{\mathrm{span}}\big(|\mathcal{R}\rangle\!\rangle\circ B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}\big)\\ &= \overline{\mathrm{span}}\Big(\big(1_B\otimes\hat J\hat\sigma_{\frac{\mathrm{i}}{2}}(\mathcal{T}_{\hat\varphi})^*\hat J\big)\gamma_B(B)\big(B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}\big)\Big)\\ &= \overline{\mathrm{span}}\Big(\big(1_B\otimes\widehat{\mathcal{G}}^{\mathrm{c}}\big)\gamma_B(B)\big(B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}\big)\Big)\\ &= \overline{\mathrm{span}}\big(B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}\big)\big(B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}\big)\\ &= B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}.\end{aligned}$$

Hence

$$\mathrm{Fix}(B\otimes H,\mathcal{R})=\mathcal{I}(B\otimes H,\mathcal{R})=B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}.$$ □

Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module and consider on the Hilbert B-module $\mathcal{E}\otimes H$ the following coaction of $\mathcal{G}$:

$$\gamma_{\mathcal{E}\otimes H}(\zeta)=\Sigma_{23}W_{23}(\gamma_{\mathcal{E}}\otimes\mathrm{id}_H)(\zeta),\quad \zeta\in\mathcal{E}\otimes H,$$

where $\Sigma:\mathcal{G}\otimes H\to H\otimes\mathcal{G}$ is the flip operator. This is the same coaction considered in Example 2.6.18(3), and if $\mathcal{E}=B$, we get the coaction $\gamma_{B\otimes H}$ of $\mathcal{G}$ on $B\otimes H$ defined by Equation (4.2). Thus, the following result generalizes Proposition 5.2.9.

Proposition 5.2.10. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module and consider on $\mathcal{E}\otimes H$ the coaction of $\mathcal{G}$ defined above. If $\xi\in\mathcal{E}$ and $v\in H_{\mathrm{si}}$, then $\xi\otimes v\in(\mathcal{E}\otimes H)_{\mathrm{si}}$ and*

$$|\xi\otimes v\rangle\!\rangle=(1_{\mathcal{E}}\otimes|v\rangle\!\rangle)\gamma_{\mathcal{E}}(\xi).^4$$

Suppose that the coaction of $\mathcal{G}$ on $\mathcal{E}$ is continuous. Then $\mathcal{R}:=\mathcal{E}\odot\hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})$ is a dense, relatively continuous subspace of $\mathcal{E}\otimes H$, and we have

$$\mathcal{F}(\mathcal{E}\otimes H,\mathcal{R})=\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}},\quad \mathrm{Fix}(\mathcal{E},\mathcal{R})=\mathcal{K}(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\cong\mathcal{K}(\mathcal{E})\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}},\quad \mathcal{I}(\mathcal{E},\mathcal{R})=I\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}},$$

where $I:=\overline{\mathrm{span}}\langle\mathcal{E}\,|\,\mathcal{E}\rangle_B$.[5] In particular, if $\mathcal{E}$ is full, then $\mathcal{R}$ is saturated.

Proof. First note that

$$\begin{aligned}\gamma_{\mathcal{E}\otimes H}(\xi\otimes v)&=\Sigma_{23}W_{23}(\gamma_{\mathcal{E}}\otimes\mathrm{id}_H)(\xi\otimes v)\\&=\Sigma_{23}W_{23}(\gamma_{\mathcal{E}}(\xi)\otimes v)\\&=\Sigma_{23}W_{23}(1_{\mathcal{E}}\otimes 1_{\mathcal{G}}\otimes v)\gamma_{\mathcal{E}}(\xi)\\&=(1_{\mathcal{E}}\otimes\Sigma W(1_{\mathcal{G}}\otimes v))\gamma_{\mathcal{E}}(\xi)\\&=(1_{\mathcal{E}}\otimes\gamma_H(v))\gamma_{\mathcal{E}}(\xi).\end{aligned}$$

Since $v\in H_{\mathrm{si}}$, we have $z:=\gamma_H(v)^*\in\bar{\mathcal{N}}_{\mathrm{id}_{H^*}\otimes\varphi}$ (Proposition 4.1.4). It follows that $1_{\mathcal{E}}\otimes z\in\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$, where $\mathcal{X}:=\mathcal{K}(\mathcal{E}\otimes H,\mathcal{E})$ (considered as a Hilbert $\mathcal{K}(\mathcal{E}),\mathcal{K}(\mathcal{E}\otimes H)$-bimodule). In fact, $z^*z\in\bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(H)}\otimes\varphi}$ and hence $1_{\mathcal{E}}\otimes z^*z\in\bar{\mathcal{M}}^+_{\mathrm{id}_{\mathcal{K}(\mathcal{E}\otimes H)}\otimes\varphi}$, that is, $1_{\mathcal{E}}\otimes z\in\bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{X}}\otimes\varphi}$. Moreover, by Proposition 2.4.21(iv), we have, for all $\eta\in\mathcal{E}$ and $\zeta\in H$,

$$\begin{aligned}(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(1_{\mathcal{E}}\otimes z)(\eta\otimes\zeta)&=(\mathrm{id}_{\mathcal{E}}\otimes\Lambda)\big(\eta\otimes z(\zeta\otimes 1_{\mathcal{G}})\big)\\&=\eta\otimes\Lambda\big(z(\zeta\otimes 1_{\mathcal{G}})\big)\\&=\eta\otimes(\mathrm{id}_{H^*}\otimes\Lambda)(z)\zeta\\&=\big(1_{\mathcal{E}}\otimes(\mathrm{id}_{H^*}\otimes\Lambda)(z)\big)(\eta\otimes\zeta).\end{aligned}$$

Thus $(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)(1_{\mathcal{E}}\otimes z)=1_{\mathcal{E}}\otimes(\mathrm{id}_{H^*}\otimes\Lambda)(z)$. It follows now from Proposition 2.4.21(v) that $\gamma_{\mathcal{E}\otimes H}(\xi\otimes v)^*=\gamma_{\mathcal{E}}(\xi)^*(1_{\mathcal{E}}\otimes\gamma_H(v)^*)\in\bar{\mathcal{N}}_{\mathrm{id}_{(\mathcal{E}\otimes H)^*}\otimes\varphi}$, that is, $\xi\otimes v\in(\mathcal{E}\otimes H)_{\mathrm{si}}$, and

$$\begin{aligned}\langle\!\langle\xi\otimes v|&=(\mathrm{id}_{(\mathcal{E}\otimes H)^*}\otimes\Lambda)\big(\gamma_{\mathcal{E}\otimes H}(\xi\otimes v)^*\big)\\&=(\mathrm{id}_{(\mathcal{E}\otimes H)^*}\otimes\Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*(1_{\mathcal{E}}\otimes\gamma_H(v)^*)\big)\\&=\gamma_{\mathcal{E}}(\xi)^*(\mathrm{id}_{\mathcal{X}}\otimes\Lambda)\big(1_{\mathcal{E}}\otimes\gamma_H(v)^*\big)\\&=\gamma_{\mathcal{E}}(\xi)^*\big(1_{\mathcal{E}}\otimes(\mathrm{id}_{H^*}\otimes\Lambda)(\gamma_H(v)^*)\big)\\&=\gamma_{\mathcal{E}}(\xi)^*(1_{\mathcal{E}}\otimes\langle\!\langle v|).\end{aligned}$$

In other words, we have $|\xi\otimes v\rangle\!\rangle=(1_{\mathcal{E}}\otimes|v\rangle\!\rangle)\gamma_{\mathcal{E}}(\xi)$. Now, if $v\in\hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})$, then we know from Proposition 5.2.8 that $|v\rangle\!\rangle\in\widehat{\mathcal{G}}^{\mathrm{c}}$. Thus, if $\gamma_{\mathcal{E}}$ is continuous, then Proposition 5.1.5 implies that

$$|\xi\otimes v\rangle\!\rangle=(1_{\mathcal{E}}\otimes|v\rangle\!\rangle)\gamma_{\mathcal{E}}(\xi)\in(1\otimes\widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{E}}(\mathcal{E})\subseteq\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}.$$

[4]Here we identify $\gamma_{\mathcal{E}}(\xi)\in\mathcal{M}(\mathcal{E}\otimes\mathcal{G})\subseteq\mathcal{L}(B\otimes H,\mathcal{E}\otimes H)$. See coments before Proposition 5.1.5.

[5]Recall that I is a $\mathcal{G}$-invariant ideal of B (see Proposition 2.6.24).

Since $\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$ is a (concrete) Hilbert $B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$-module, it follows that $\mathcal{R}=\mathcal{E}\odot\hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})$ is a (dense) relatively continuous subspace of $\mathcal{E}\otimes H$ and

$$\begin{aligned}\mathcal{F}(\mathcal{E}\otimes H,\mathcal{R})&=\overline{\operatorname{span}}\big(|\mathcal{R}\rangle\!\rangle(B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\big)\\&=\overline{\operatorname{span}}\big((1_{\mathcal{E}}\otimes|\hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})\rangle\!\rangle)\gamma_{\mathcal{E}}(\mathcal{E})(B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\big)\\&=\overline{\operatorname{span}}\big((1_{\mathcal{E}}\otimes\widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{E}}(\mathcal{E})(B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\big)\\&=\overline{\operatorname{span}}\big((\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})(B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\big)=\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}.\end{aligned}$$

It follows from Proposition 5.1.5 that

$$\operatorname{Fix}(\mathcal{E},\mathcal{R})=\mathcal{K}(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\cong\mathcal{K}(\mathcal{E})\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}\quad\text{and}\quad\mathcal{I}(\mathcal{E},\mathcal{R})=I\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}.\qquad\square$$

Remark 5.2.11. Let notation be as in Proposition 5.2.10. For each $\xi\in\mathcal{E}$, the operator $\gamma_{\mathcal{E}}(\xi)\in\mathcal{M}(\mathcal{E}\otimes\mathcal{G})$ considered as an element of $\mathcal{L}(B\otimes H,\mathcal{E}\otimes H)$ is $\mathcal{G}$-equivariant, that is, for all $\zeta\in B\otimes H$ we have (see the proof of Proposition 5.1.5)

$$\gamma_{\mathcal{E}\otimes H}(\gamma_{\mathcal{E}}(\xi)\zeta)=(\gamma_{\mathcal{E}}(\xi)\otimes 1_{\mathcal{G}})\gamma_{B\otimes H}(\zeta).$$

Therefore, by Proposition 4.1.10(iii), $\gamma_{\mathcal{E}}(\xi)\zeta\in(\mathcal{E}\otimes H)_{\mathrm{si}}$ for all $\zeta\in(B\otimes H)_{\mathrm{si}}$, and

$$|\gamma_{\mathcal{E}}(\xi)\zeta\rangle\!\rangle=\gamma_{\mathcal{E}}(\xi)|\zeta\rangle\!\rangle.$$

By Proposition 5.2.9, $\mathcal{R}:=B\odot\hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})$ is a relatively continuous subspace of $B\otimes H$ and $|\mathcal{R}\rangle\!\rangle$ is dense in $B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$. It follows that $\gamma_{\mathcal{E}}(\mathcal{E})\mathcal{R}$ is a relatively continuous subset of $\mathcal{E}\otimes H$ and

$$\mathcal{F}(\mathcal{E}\otimes H,\gamma_{\mathcal{E}}(\mathcal{E})\mathcal{R})=\overline{\operatorname{span}}\,\gamma_{\mathcal{E}}(\mathcal{E})(B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})=\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}.$$

Since the linear span of $\gamma_{\mathcal{E}}(\mathcal{E})(B\otimes\mathcal{G})$ is dense in $\mathcal{E}\otimes\mathcal{G}$, it follows that the linear span of $\gamma_{\mathcal{E}}(\mathcal{E})\mathcal{R}$ is a dense, relatively continuous subspace of $\mathcal{E}\otimes H$. Note that this argument does not use continuity of the coaction $\gamma_{\mathcal{E}}$.

We have seen in the Example 5.2.5 that in the case of a compact group G, every subset of a Hilbert B,G-module is relatively continuous. Now we show that this remains the case for arbitrary compact quantum groups.

Proposition 5.2.12. *Let $\mathcal{G}$ be a compact quantum group and let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. Then any subset of $\mathcal{E}$ is relatively continuous. In particular, $\mathcal{E}$ itself is relatively continuous. Moreover, we have*

$$\mathcal{F}_{\mathcal{E}}:=\mathcal{F}(\mathcal{E},\mathcal{E})=(1_{\mathcal{E}}\otimes\delta_1^*)\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}},$$

where $\delta_1:=\Lambda(1)\in H$.[6] *The generalized fixed point algebra* $\operatorname{Fix}(\mathcal{E}):=\operatorname{Fix}(\mathcal{E},\mathcal{E})$ *is the usual fixed point algebra*

$$\operatorname{Fix}(\mathcal{E})=(1\otimes\delta_1^*)\mathcal{K}(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})(1\otimes\delta_1)=\{x\in\mathcal{K}(\mathcal{E}):\gamma_{\mathcal{K}(\mathcal{E})}(x)=x\otimes 1_{\mathcal{G}}\},$$

[6]Recall that δ_1^* denotes the element of $\mathcal{L}(H,\mathbb{C})$ given by $\delta_1(\eta)=\langle\delta_1\,|\,\eta\rangle$ for all $\eta\in H$. Thus $1_{\mathcal{E}}\otimes\delta_1^*$ is an element of $\mathcal{L}(\mathcal{E}\otimes H,\mathcal{E})$.

and it is Morita equivalent to the ideal $\mathcal{I}_{\mathcal{E}} := \mathcal{I}(\mathcal{E},\mathcal{E}) \subseteq B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ *given by*

$$\mathcal{I}_{\mathcal{E}} = \overline{\operatorname{span}}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})^*(1 \otimes p_1)(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) = \overline{\operatorname{span}}\, \gamma_{\mathcal{E}}(\mathcal{E})^*(1_{\mathcal{E}} \otimes p_1)\gamma_{\mathcal{E}}(\mathcal{E}),$$

where $p_1 := |\delta_1\rangle\langle\delta_1| \in \mathcal{K}(H)$.

Proof. We already know that $\mathcal{E} = \mathcal{E}_{\mathrm{si}}$. Thus we have to show that, for any $\xi, \eta \in \mathcal{E}$, the element $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle$ belongs to $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. Recall from Example 4.1.7 that

$$\langle\!\langle \xi | = \gamma_{\mathcal{E}}(\xi)^*(1_{\mathcal{K}(\mathcal{E})} \otimes \delta_1) \quad \text{for all } \xi \in \mathcal{E}.$$

Thus

$$\langle\!\langle \xi \,|\, \eta \rangle\!\rangle = \gamma_{\mathcal{E}}(\xi)^*(1_{\mathcal{K}(\mathcal{E})} \otimes p_1)\gamma_{\mathcal{E}}(\eta).$$

We may assume that φ is a state, that is, $\varphi(1) = 1$. Thus δ_1 is a unitary vector and hence p_1 is a projection. Note also that $\varphi = \omega_{\delta_1,\delta_1} \in L^1(\mathcal{G})$. We claim that $p_1 = \rho(\varphi)$ (recall that $\rho(\omega) = (\mathrm{id} \otimes \omega)(V^*)$, where V is the right regular corepresentation of $\mathcal{G}$). In fact, by Equation (4.4), we have (using that compact quantum groups are unimodular, so that $\Gamma = \Lambda$)

$$(\mathrm{id} \otimes \varphi)(V)\Lambda(b) = \Lambda\big((\mathrm{id} \otimes \varphi)\Delta(b)\big) = \Lambda\big(1\varphi(b)\big) = \delta_1\varphi(b)$$

for all $b \in \mathcal{G}$. On the other hand

$$p_1\Lambda(b) = |\delta_1\rangle\langle\delta_1|\Lambda(b) = \delta_1\langle\Lambda(1)|\Lambda(b)\rangle = \delta_1\varphi(b).$$

Thus $(\mathrm{id} \otimes \varphi)(V) = p_1$ and hence

$$\rho(\varphi) = (\mathrm{id} \otimes \varphi)(V^*) = (\mathrm{id} \otimes \varphi)(V)^* = p_1^* = p_1.$$

In particular, $p_1 \in \rho(L^1(\mathcal{G})) \subseteq \widehat{\mathcal{G}}^{\mathrm{c}}$. We conclude that the operator

$$\begin{aligned}\langle\!\langle \xi \,|\, \eta \rangle\!\rangle &= \gamma_{\mathcal{E}}(\xi)^*(1_{\mathcal{K}(\mathcal{E})} \otimes p_1)\gamma_{\mathcal{E}}(\eta) \\ &= \big((1_{\mathcal{K}(\mathcal{E})} \otimes p_1)\gamma_{\mathcal{E}}(\xi)\big)^*\big((1_{\mathcal{K}(\mathcal{E})} \otimes p_1)\gamma_{\mathcal{E}}(\eta)\big)\end{aligned}$$

belongs to $(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})^*(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \subseteq B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ by Proposition 5.1.5. Here we are using that compact quantum groups are regular, so that $\gamma_{\mathcal{E}}$ is automatically continuous (see Remark 2.6.9(3)). Therefore any subset of $\mathcal{E}$ is relatively continuous.

The equation $|\xi\rangle\!\rangle = (1 \otimes \delta_1^*)\gamma_{\mathcal{E}}(\xi)$ yields

$$\mathcal{F}_{\mathcal{E}} = \overline{\operatorname{span}}(1 \otimes \delta_1^*)\gamma_{\mathcal{E}}(\mathcal{E})(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) = (1 \otimes \delta_1^*)\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.$$

Hence

$$\mathrm{Fix}(\mathcal{E}) = \overline{\operatorname{span}}(1 \otimes \delta_1^*)(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})^*(1 \otimes \delta_1) = (1 \otimes \delta_1^*)\mathcal{K}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(1 \otimes \delta_1),$$

which is therefore Morita equivalent to

$$\mathcal{I}_{\mathcal{E}} = \overline{\operatorname{span}}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})^*(1 \otimes p_1)(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}).$$

Moreover, by Proposition 2.6.10, the linear span of $L^1(\mathcal{G}) * \mathcal{E}$ is dense in $\mathcal{E}$. Combining this with Propositions 4.3.13 and 4.3.14 (and using that $\mathcal{G}$ is unimodular, so that $L^1_0(\mathcal{G}) = L^1(\mathcal{G})$), we get that

$$\begin{aligned}\overline{|\mathcal{E}\rangle\!\rangle} &= \overline{\operatorname{span}}\big((1 \otimes \delta_1^*)\gamma_{\mathcal{E}}(\mathcal{E})\big(1 \otimes \rho(L^1(\mathcal{G}))\big)\big)\\ &= \overline{\operatorname{span}}\big((1 \otimes \delta_1^*)\gamma_{\mathcal{E}}(\mathcal{E})(1 \otimes \widehat{\mathcal{G}}^{\mathrm{c}})\big)\\ &= (1 \otimes \delta_1^*)(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) = \mathcal{F}_{\mathcal{E}}.\end{aligned}$$

In particular,

$$\mathcal{I}_{\mathcal{E}} = \overline{\operatorname{span}}\langle\!\langle \mathcal{E} \,|\, \mathcal{E} \rangle\!\rangle = \overline{\operatorname{span}}\, \gamma_{\mathcal{E}}(\mathcal{E})^*(1 \otimes p_1)\gamma_{\mathcal{E}}(\mathcal{E}),$$

and (using the equality $\omega_{\delta_1,\delta_1} = \varphi$)

$$\begin{aligned}\operatorname{Fix}(\mathcal{E}) &= \overline{\operatorname{span}}\, |\mathcal{E}\rangle\!\rangle\langle\!\langle \mathcal{E}|\\ &= \overline{\operatorname{span}}(1 \otimes \delta_1^*)\gamma_{\mathcal{K}(\mathcal{E})}(\mathcal{K}(\mathcal{E}))(1 \otimes \delta_1)\\ &= \overline{\operatorname{span}}(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(\gamma_{\mathcal{K}(\mathcal{E})}(\mathcal{K}(\mathcal{E})))\\ &= \{x \in \mathcal{K}(\mathcal{E}) : \gamma_{\mathcal{K}(\mathcal{E})}(x) = x \otimes 1_{\mathcal{G}}\},\end{aligned}$$

where the last equality is proved in the following way: since $\gamma_{\mathcal{K}(\mathcal{E})}(\mathcal{K}(\mathcal{E}))$ is contained in $\tilde{\mathcal{M}}\big(\mathcal{K}(\mathcal{E}) \otimes \mathcal{G}\big)$ (which is equal to $\mathcal{K}(\mathcal{E}) \otimes \mathcal{G}$ because $\mathcal{G}$ is unital), and since $\varphi \in \mathcal{G}^*$, we have

$$(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(\gamma_{\mathcal{K}(\mathcal{E})}(\mathcal{K}(\mathcal{E}))) \subseteq \{x \in \mathcal{K}(\mathcal{E}) : \gamma_{\mathcal{K}(\mathcal{E})}(x) = x \otimes 1_{\mathcal{G}}\}.$$

Conversely, if $\gamma_{\mathcal{K}(\mathcal{E})}(x) = x \otimes 1_{\mathcal{G}}$, then $(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(\gamma_{\mathcal{K}(\mathcal{E})}(x)) = x$, and therefore

$$\operatorname{Fix}(\mathcal{E}) = \{x \in \mathcal{K}(\mathcal{E}) : \gamma_{\mathcal{K}(\mathcal{E})}(x) = x \otimes 1_{\mathcal{G}}\}. \qquad \square$$

Remark 5.2.13. In the case of a $\mathcal{G}$-C^*-algebra A with $\mathcal{G}$ compact, the Morita equivalence between $\operatorname{Fix}(A)$ and the ideal $\mathcal{I}_A$ in $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ was proved by Ng in [54]. He also defined an interesting condition on the coaction: γ_A is called *effective* if the linear span of $\gamma_A(A)(A \otimes 1)$ is dense in $A \otimes \mathcal{G}$. This condition implies that $\mathcal{R} = A$ is saturated, that is, $\mathcal{I}_A$ is equal to $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ ([54, Lemma 2.6]). Thus, in this case, $\operatorname{Fix}(A)$ is Morita equivalent to $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. Note that comultiplications are effective and hence any dual coaction is effective. Observe that this result applied to the comultiplication Δ of $\mathcal{G}$ and combined with Proposition 5.2.6 yields what we already know: any compact quantum group is regular.

The following result provides a canonical way to associate relatively continuous subspaces of $\mathcal{E}$ to relatively continuous subspaces of $\mathcal{K}(\mathcal{E})$. It also provides a formula for the corresponding Hilbert modules over the reduced crossed product (and therefore also for the generalized fixed point algebras).

Proposition 5.2.14. *Let $\mathcal{G}$ be a locally compact quantum group and let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module with a continuous coaction of $\mathcal{G}$.*

(i) *Suppose that there is a left action $\pi : A \to \mathcal{L}(\mathcal{E})$ of a $\mathcal{G}$-C^*-algebra A turning $\mathcal{E}$ into a $\mathcal{G}$-equivariant right-Hilbert A,B-bimodule. This means that π is a $\mathcal{G}$-equivariant nondegenerate $*$-homomorphism. We will use the module notation for the left action: $a \cdot \xi := \pi(a)\xi$ for all $a \in A$ and $\xi \in \mathcal{E}$.*

If $\mathcal{R}$ is a relatively continuous subset of A, then $\mathcal{R} \cdot \mathcal{E}$ is a relatively continuous subset of $\mathcal{E}$ and

$$\mathcal{F}(\mathcal{E}, \mathcal{R}\cdot\mathcal{E}) = \overline{\operatorname{span}}\big(\mathcal{F}(A,\mathcal{R})\cdot \mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}\big) \cong \mathcal{F}(A,\mathcal{R}) \otimes_{A\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}} (\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}),$$

where for $x \in \mathcal{F}(A,\mathcal{R}) \subseteq \mathcal{L}(A\otimes H, A)$ and $y \in \mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}} \subseteq \mathcal{L}(B\otimes H, \mathcal{E}\otimes H)$ we are using the notation $x\cdot y := (\pi\otimes \mathrm{id}_{H^*})(x)y$.[7]

In particular, if $\mathcal{R}$ is a relatively continuous subspace of $\mathcal{K}(\mathcal{E})$, then $\mathcal{R}(\mathcal{E})$ is a relatively continuous subspace of $\mathcal{E}$ and

$$\mathcal{F}\big(\mathcal{E},\mathcal{R}(\mathcal{E})\big) = \overline{\operatorname{span}}\big(\mathcal{F}(\mathcal{K}(\mathcal{E}),\mathcal{R})\circ(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\big) \cong \mathcal{F}\big(\mathcal{K}(\mathcal{E}),\mathcal{R}\big)\otimes_{\mathcal{K}(\mathcal{E})\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}}(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}).$$

(ii) *If $\mathcal{R}$ is a relatively continuous subset of $\mathcal{E}$, then $|\mathcal{R}\rangle\langle\mathcal{E}|$ is a relatively continuous subset of $\mathcal{K}(\mathcal{E})$ and*

$$\mathcal{F}\big(\mathcal{K}(\mathcal{E}), |\mathcal{R}\rangle\langle\mathcal{E}|\big) = \overline{\operatorname{span}}\big(\mathcal{F}(\mathcal{E},\mathcal{R})\circ(\mathcal{E}^*\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\big) \cong \mathcal{F}(\mathcal{E},\mathcal{R})\otimes_{B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}}(\mathcal{E}^*\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}).$$

Proof. (i) It follows from Proposition 4.1.10(iv) that $\mathcal{R}\cdot\mathcal{E}\subseteq\mathcal{E}_{\mathrm{si}}$ and, for all $a\in\mathcal{R}$ and $\xi\in\mathcal{E}$, we have

$$|a\cdot\xi\rangle\!\rangle = (\pi\otimes\mathrm{id}_{H^*})(|a\rangle\!\rangle)\gamma_{\mathcal{E}}(\xi) = |a\rangle\!\rangle\cdot\gamma_{\mathcal{E}}(\xi).$$

Thus, for all $a,b\in\mathcal{R}$ and $\xi,\eta\in\mathcal{E}$ we get

$$\langle\!\langle a\cdot\xi\,|\,b\cdot\eta\rangle\!\rangle = \gamma_{\mathcal{E}}(\xi)^*(\pi\otimes\mathrm{id}_{\mathcal{K}})(\langle\!\langle a\,|\,b\rangle\!\rangle)\gamma_{\mathcal{E}}(\eta) = \gamma_{\mathcal{E}}(\xi)^*(\pi\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})(\langle\!\langle a\,|\,b\rangle\!\rangle)\gamma_{\mathcal{E}}(\eta),$$

where $\mathcal{K}:=\mathcal{K}(H)$. Since $\mathcal{R}$ is relatively continuous, we have $\langle\!\langle a\,|\,b\rangle\!\rangle\in A\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$. Thus to prove that $\mathcal{R}\cdot\mathcal{E}$ is relatively continuous it is enough to prove that

$$\gamma_{\mathcal{E}}(\mathcal{E})^*(\pi\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})(A\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\gamma_{\mathcal{E}}(\mathcal{E})\subseteq B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}.$$

If $c\in A$, $\hat{x}\in\widehat{\mathcal{G}}^{\mathrm{c}}$ and $\xi,\eta\in\mathcal{E}$ then

$$\begin{aligned}\gamma_{\mathcal{E}}(\xi)^*(\pi\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})((1\otimes\hat{x})\gamma_A(c))\gamma_{\mathcal{E}}(\eta) &= \gamma_{\mathcal{E}}(\xi)^*((1\otimes\hat{x})\gamma_{\mathcal{K}(\mathcal{E})}(\pi(c))\gamma_{\mathcal{E}}(\eta)\\ &= \gamma_{\mathcal{E}}(\xi)^*(1\otimes\hat{x})\gamma_{\mathcal{E}}(\pi(c)\eta)\subseteq(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})^*(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\subseteq B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}.\end{aligned}$$

Hence $\mathcal{R}\cdot\mathcal{E}$ is relatively continuous. We compute

$$\begin{aligned}\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}} &= (A\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\cdot(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\\ &= (\pi\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})(A\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})\\ &= (\pi\otimes\mathrm{id}_{\mathcal{K}})(A\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}})(\mathcal{E}\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}),\end{aligned}$$

[7]Recall that $\pi\otimes\mathrm{id}_{H^*}:\mathcal{L}(A\otimes H,A)\to\mathcal{L}(\mathcal{E}\otimes H,\mathcal{E})$ and therefore the composition $(\pi\otimes\mathrm{id}_{H^*})(x)y$ makes sense. Note that $\mathcal{L}(A\otimes H,A)\cong\mathcal{M}(A\otimes H^*)$ and $\mathcal{L}(\mathcal{E}\otimes H,\mathcal{E})\cong\mathcal{M}(\mathcal{K}(\mathcal{E})\otimes H^*)$.

and hence

$$\begin{aligned}
\mathcal{F}(\mathcal{E}, \mathcal{R}\cdot\mathcal{E}) &= \overline{\operatorname{span}}\,|\mathcal{R}\cdot\mathcal{E}\rangle\!\rangle(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \\
&= \overline{\operatorname{span}}(|\mathcal{R}\rangle\!\rangle\cdot\gamma_{\mathcal{E}}(\mathcal{E}))(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \\
&= \overline{\operatorname{span}}(\pi\otimes \mathrm{id}_{H^*})(|\mathcal{R}\rangle\!\rangle)\gamma_{\mathcal{E}}(\mathcal{E})(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \\
&= \overline{\operatorname{span}}(\pi\otimes \mathrm{id}_{H^*})(|\mathcal{R}\rangle\!\rangle)(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \\
&= \overline{\operatorname{span}}(\pi\otimes \mathrm{id}_{H^*})(|\mathcal{R}\rangle\!\rangle)(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\cdot(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \\
&= \overline{\operatorname{span}}(\pi\otimes \mathrm{id}_{H^*})(|\mathcal{R}\rangle\!\rangle)(\pi\otimes\mathrm{id}_{\mathcal{K}})(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \\
&= \overline{\operatorname{span}}(\pi\otimes \mathrm{id}_{H^*})\big(|\mathcal{R}\rangle\!\rangle(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big)(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \\
&= \overline{\operatorname{span}}\,\mathcal{F}(A,\mathcal{R})\cdot(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}).
\end{aligned}$$

Finally, it is easy to see that the map $x\otimes y \mapsto x\cdot y$, where $x \in \mathcal{F}(A,\mathcal{R})$ and $y \in \mathcal{E}\rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, induces an isomorphism

$$\mathcal{F}(A,\mathcal{R}) \otimes_{A\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}} (\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \cong \overline{\operatorname{span}}\,\mathcal{F}(A,\mathcal{R})\cdot(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}).$$

(ii) By Proposition 4.1.10(v), we have $|\mathcal{R}\rangle\langle\mathcal{E}| \subseteq \mathcal{K}(\mathcal{E})_{\mathrm{si}}$ and, for all $\xi\in\mathcal{R}$, $\eta\in\mathcal{E}$,

$$\big||\xi\rangle\langle\eta|\big\rangle\!\!\big\rangle = |\xi\rangle\!\rangle\gamma_{\mathcal{E}}(\eta)^*.$$

Thus, if $\xi_1,\xi_2\in\mathcal{R}$ and $\eta_1,\eta_2\in\mathcal{E}$ we get

$$\big\langle\!\!\big\langle|\xi_1\rangle\langle\eta_1|\big||\xi_2\rangle\langle\eta_2|\big\rangle\!\!\big\rangle = \gamma_{\mathcal{E}}(\eta_1)\langle\!\langle\xi_1\,|\,\xi_2\rangle\!\rangle\gamma_{\mathcal{E}}(\eta_2)^* \in (\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})^* \subseteq \mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.$$

Thus $|\mathcal{R}\rangle\langle\mathcal{E}|$ is relatively continuous and because $\mathcal{E}^*\rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = (B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})$ we conclude that

$$\begin{aligned}
\mathcal{F}(\mathcal{K}(\mathcal{E}), |\mathcal{R}\rangle\langle\mathcal{E}|) &= \overline{\operatorname{span}}\big(\big||\mathcal{R}\rangle\langle\mathcal{E}|\big\rangle\!\!\big\rangle(\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big) \\
&= \overline{\operatorname{span}}\big(|\mathcal{R}\rangle\!\rangle\gamma_{\mathcal{E}}(\mathcal{E})^*(\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big) \\
&= \overline{\operatorname{span}}(|\mathcal{R}\rangle\!\rangle(\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})) \\
&= \overline{\operatorname{span}}\big(|\mathcal{R}\rangle\!\rangle(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})(\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big) \\
&= \overline{\operatorname{span}}\,\mathcal{F}(\mathcal{E},\mathcal{R})(\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}).
\end{aligned}$$

Finally, it is easy to see that the map $z\otimes w\mapsto z\circ w$, where $z\in\mathcal{F}(\mathcal{E},\mathcal{R})$ and $w\in\mathcal{E}^*\rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, induces an isomorphism

$$\mathcal{F}(\mathcal{E},\mathcal{R}) \otimes_{B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}} (\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \cong \overline{\operatorname{span}}\,\mathcal{F}(\mathcal{E},\mathcal{R})(\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}). \qquad \square$$

In the group case, it is a basic observation that $C^*_{\mathrm{r}}(G,A)$ appears as a generalized fixed point algebra of $A\otimes\mathcal{K}\big(L^2(G)\big)$, where G is a locally compact group and A is a G-C^*-algebra. Using the result above we can now prove the following generalization:

Proposition 5.2.15. *Let $\mathcal{G}$ be a regular locally compact quantum group. Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module with an injective coaction of $\mathcal{G}$ and consider the $\mathcal{G}$-C^*-algebra $A \otimes \mathcal{K}$, where $A := \mathcal{K}(\mathcal{E})$ and $\mathcal{K} := \mathcal{K}\big(L^2(\mathcal{G})\big)$. Then there is a dense, relatively continuous subspace $\mathcal{R} \subseteq A \otimes \mathcal{K}$ such that*

$$\mathcal{F}(A \otimes \mathcal{K}, \mathcal{R}) \cong (A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \otimes L^2(\mathcal{G})^*, \quad \mathrm{Fix}(A \otimes \mathcal{K}, \mathcal{R}) \cong A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}},$$

$$\textit{and} \quad \mathcal{I}(A \otimes \mathcal{K}, \mathcal{R}) \cong (A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \otimes \mathcal{K} \cong (A \otimes \mathcal{K}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.$$

Hence $A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ appears as a generalized fixed point algebra of $A \otimes \mathcal{K}$.

Proof. Note that $\gamma_{\mathcal{E}}$ is injective if and only if γ_A is injective. Thus (A, γ_A) is a reduced coaction of $\mathcal{G}$. Since $\mathcal{G}$ is regular we have (Proposition 2.7.16)

$$A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G} \cong A \otimes \mathcal{K}.$$

Hence $(A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \cong (A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \otimes \mathcal{K}$. By Proposition 5.2.10, there is a dense, relatively continuous subset $\mathcal{R}_0 \subseteq \mathcal{E} \otimes L^2(\mathcal{G})$ such that

$$\mathcal{F}(\mathcal{E} \otimes L^2(\mathcal{G}), \mathcal{R}_0) = \mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \cong \mathcal{E} \otimes_B (B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}).$$

By Proposition 5.2.14(ii), $\mathcal{R} := \mathrm{span}(|\mathcal{R}_0\rangle\langle\mathcal{E}|)$ is a dense, relatively continuous subspace of $\mathcal{K}(\mathcal{E} \otimes L^2(\mathcal{G})) \cong A \otimes \mathcal{K}$ and

$$\mathcal{F}(A \otimes \mathcal{K}, \mathcal{R}) \cong \mathcal{F}(\mathcal{E} \otimes L^2(\mathcal{G}), \mathcal{R}_0) \otimes_{B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}} (\mathcal{E} \otimes L^2(\mathcal{G}))^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}.$$

Now note that

$$\begin{aligned}
(\mathcal{E} \otimes L^2(\mathcal{G}))^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} &\cong (\mathcal{E}^* \otimes L^2(\mathcal{G})^*) \otimes_{A \otimes \mathcal{K}} (A \otimes \mathcal{K}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \\
&\cong (\mathcal{E}^* \otimes L^2(\mathcal{G})^*) \otimes_{A \otimes \mathcal{K}} (A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \otimes \mathcal{K} \\
&\cong \big(\mathcal{E}^* \otimes_A (A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big) \otimes \big(L^2(\mathcal{G})^* \otimes_{\mathcal{K}} \mathcal{K}\big) \\
&\cong (\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \otimes L^2(\mathcal{G})^*.
\end{aligned}$$

Thus

$$\mathcal{F}(A \otimes \mathcal{K}, \mathcal{R}) \cong (\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \otimes_{B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}} (\mathcal{E}^* \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \otimes L^2(\mathcal{G})^* \cong (A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \otimes L^2(\mathcal{G})^*.$$

Therefore,

$$\mathrm{Fix}(A \otimes \mathcal{K}, \mathcal{R}) \cong A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$$

and

$$\mathcal{I}(A \otimes \mathcal{K}, \mathcal{R}) \cong (A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}) \otimes \mathcal{K} \cong (A \otimes \mathcal{K}) \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}. \qquad \square$$

In the situation above, we have $A \otimes \mathcal{K} \cong A \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}$. Thus $A \otimes \mathcal{K}$ is a dual coaction and therefore the following result generalizes the proposition above.

Proposition 5.2.16. *Let $\mathcal{G}$ be a regular locally compact quantum group and suppose that $\mathcal{E}$ is a Hilbert $B,\mathcal{G}$-module, where B is a reduced $\mathcal{G}$-C^*-algebra. Consider the dual coaction of $\widehat{\mathcal{G}}^{\mathrm{c}}$ on $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ as in Remark 5.1.6(2). Then there is a dense, relatively continuous subspace $\mathcal{R}$ of $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ such that*

$$\mathcal{F}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, \mathcal{R}) \cong L^2(\mathcal{G})^* \otimes \mathcal{E}, \quad \mathrm{Fix}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, \mathcal{R}) \cong \mathcal{K}(\mathcal{E}),$$
$$\text{and} \quad \mathcal{I}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, \mathcal{R}) \cong I \otimes \mathcal{K} \subseteq B \otimes \mathcal{K} \cong B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G},$$

where $I := \overline{\mathrm{span}}\langle \mathcal{E} \,|\, \mathcal{E}\rangle_B \subseteq B$ and $\mathcal{K} := \mathcal{K}\big(L^2(\mathcal{G})\big)$. In particular, if $\mathcal{E}$ is full, then $\mathcal{R}$ is saturated.

Proof. Let $A := \widehat{\mathcal{G}}^{\mathrm{c}}$, where $\widehat{\mathcal{G}}^{\mathrm{c}}$ is regarded as a $\widehat{\mathcal{G}}^{\mathrm{c}}$-$C^*$-algebra (with the comultiplication as coaction). Since $\mathcal{G}$ is regular, Proposition 5.2.6 implies that $\mathcal{R}_0 := A_{\mathrm{si}}$ is a dense, relatively continuous subspace of A and

$$\mathcal{F}(A, \mathcal{R}_0) = (1_{\mathcal{G}} \otimes L^2(\mathcal{G})^*)W \cong L^2(\mathcal{G})^*.$$

Consider the canonical $\widehat{\mathcal{G}}^{\mathrm{c}}$-equivariant nondegenerate $*$-homomorphism

$$\pi : A \to \mathcal{L}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}), \quad x \mapsto \pi(x) := 1 \otimes x.$$

By Proposition 5.2.14(i), $\mathcal{R} := \mathrm{span}\big(\pi(\mathcal{R}_0)(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})\big)$ is a (dense) relatively continuous subspace of $\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ and we have

$$\mathcal{F}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, \mathcal{R}) \cong \mathcal{F}(A, \mathcal{R}_0) \otimes_{A \rtimes_{\mathrm{r}} \mathcal{G}} (\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}).$$

Since B is reduced and $\mathcal{G}$ is regular, we have

$$\begin{aligned}\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G} &\cong (\mathcal{E} \otimes_B (B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})) \otimes_{B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}} (B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G}) \\ &\cong \mathcal{E} \otimes_B (B \otimes \mathcal{K}) \cong \mathcal{E} \otimes \mathcal{K}.\end{aligned}$$

Since $\mathcal{G}$ is regular, we also have $A \rtimes_{\mathrm{r}} \mathcal{G} = \widehat{\mathcal{G}}^{\mathrm{c}} \rtimes_{\mathrm{r}} \mathcal{G} \cong \mathcal{K}$. Thus

$$\mathcal{F}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, \mathcal{R}) \cong L^2(\mathcal{G})^* \otimes_{\mathcal{K}} (\mathcal{E} \otimes \mathcal{K}) \cong L^2(\mathcal{G})^* \otimes \mathcal{E}.$$

And therefore

$$\mathrm{Fix}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, \mathcal{R}) \cong \mathcal{K}(\mathcal{E}) \quad \text{and} \quad \mathcal{I}(\mathcal{E} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}, \mathcal{R}) \cong I \otimes \mathcal{K}. \qquad \square$$

5.3 Completions of relatively continuous subsets

In general, for a given Hilbert $B,\mathcal{G}$-module $\mathcal{E}$ there may be several relatively continuous subspaces $\mathcal{R} \subseteq \mathcal{E}$ yielding the same Hilbert module $\mathcal{F} = \mathcal{F}(\mathcal{E}, \mathcal{R})$. In this section we impose more conditions on $\mathcal{R}$ to minimize this choice.

In this section we shall use the Banach algebra $L^1_0(\mathcal{G}) \subseteq L^1(\mathcal{G})$ defined in Section 4.3.

Definition 5.3.1. Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. A subspace $\mathcal{R}\subseteq\mathcal{E}_{\mathrm{si}}$ is called *complete* if $\mathcal{R}$ is $\|\cdot\|_{\mathrm{si}}$-closed, $L^1_0(\mathcal{G})$-invariant and also B-invariant, that is, if $\omega*\xi$ and $\xi\cdot b$ belong to $\mathcal{R}$ for all $\xi\in\mathcal{R}$, $\omega\in L^1_0(\mathcal{G})$ and $b\in B$. Here $*$ denotes the left action of $L^1(\mathcal{G})$ on $\mathcal{E}$ induced by the coaction of $\mathcal{E}$ (see Equation (2.18)) and $\cdot$ denotes the right B-action.

The *completion* of a subset $\mathcal{R}\subseteq\mathcal{E}_{\mathrm{si}}$, denoted by $\mathcal{R}_{\mathrm{c}}$, is the smallest complete subspace of $\mathcal{E}_{\mathrm{si}}$ containing $\mathcal{R}$.

Note that $\mathcal{E}_{\mathrm{si}}$ is complete by Propositions 4.1.10(ii), 4.1.11 and 4.3.8, and hence $\mathcal{R}$ is complete if and only if $\mathcal{R}$ is an $L^1_0(\mathcal{G}),B$-invariant closed subspace of $\mathcal{E}_{\mathrm{si}}$. Since the intersection of complete subspaces is clearly complete, the completion of a subset $\mathcal{R}\subseteq\mathcal{E}_{\mathrm{si}}$ always exists and is the intersection of all complete subspaces of $\mathcal{E}_{\mathrm{si}}$ containing $\mathcal{R}$.

If $\mathcal{R}\subseteq\mathcal{E}_{\mathrm{si}}$ is an $L^1_0(\mathcal{G}),B$-invariant subspace, then so is the si-closure $\overline{\mathcal{R}}^{\mathrm{si}}$ by Propositions 4.1.11 and 4.3.8, and therefore $\mathcal{R}_{\mathrm{c}}=\overline{\mathcal{R}}^{\mathrm{si}}$. In general, we can describe the completion of a subset $\mathcal{R}\subseteq\mathcal{E}_{\mathrm{si}}$ as the si-closure of the smallest $L^1_0(\mathcal{G}),B$-invariant subspace of $\mathcal{E}_{\mathrm{si}}$ containing $\mathcal{R}$. Let us describe this in more detail. We define $\mathcal{R}_0:=\operatorname{span}\mathcal{R}$ and also $\mathcal{R}_n$, $n\in\mathbb{N}$, recursively by

$$\mathcal{R}_1:=\operatorname{span}\big(\mathcal{R}_0\cup\mathcal{R}_0\cdot B\big),\quad \mathcal{R}_2:=\operatorname{span}\big(\mathcal{R}_1\cup L^1_0(\mathcal{G})*\mathcal{R}_1\big),\quad \mathcal{R}_3:=\operatorname{span}\big(\mathcal{R}_2\cup\mathcal{R}_2\cdot B\big),\quad\ldots$$

Then it is easy to see that

$$\mathcal{R}_\infty:=\operatorname{span}\bigcup_{n=0}^{\infty}\mathcal{R}_n$$

is the smallest $L^1_0(\mathcal{G}),B$-invariant subspace of $\mathcal{E}_{\mathrm{si}}$ containing $\mathcal{R}$, and therefore the completion of $\mathcal{R}$ is $\mathcal{R}_{\mathrm{c}}=\overline{\mathcal{R}}^{\mathrm{si}}_\infty$.

Proposition 5.3.2. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. If $\mathcal{R}\subseteq\mathcal{E}$ is relatively continuous, then so are $\mathcal{R}\cdot B$, $L^1_0(\mathcal{G})*\mathcal{R}$ and $\overline{\mathcal{R}}^{\mathrm{si}}$, and we have*

$$\mathcal{F}(\mathcal{E},\mathcal{R})=\mathcal{F}(\mathcal{E},\overline{\mathcal{R}}^{\mathrm{si}})=\mathcal{F}(\mathcal{E},\mathcal{R}\cdot B)=\mathcal{F}(\mathcal{E},L^1_0(\mathcal{G})*\mathcal{R}).$$

Moreover, the completion $\mathcal{R}_{\mathrm{c}}$ of $\mathcal{R}$ is also relatively continuous, and we have

$$\mathcal{F}(\mathcal{E},\mathcal{R})=\mathcal{F}(\mathcal{E},\mathcal{R}_{\mathrm{c}}).$$

Proof. Let $\xi\in\mathcal{R}$, $b\in B$ and $\omega\in L^1_0(\mathcal{G})$. By Propositions 4.1.10(ii) and 4.3.8, we have the formulas $|\xi\cdot b\rangle\!\rangle=|\xi\rangle\!\rangle\gamma_B(b)$ and $|\omega*\xi\rangle\!\rangle=|\xi\rangle\!\rangle(1_B\otimes\rho_\omega)$. Since $\gamma_B(b)$ and $(1_B\otimes\rho_\omega)$ are multipliers of $B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$ (remember that $\rho_\omega\in\widehat{\mathcal{G}}^{\mathrm{c}}$; see Proposition 4.3.13), it follows that $\mathcal{R}\cdot B$ and $L^1_0(\mathcal{G})*\mathcal{R}$ are relatively continuous. And from the definition of $\|\cdot\|_{\mathrm{si}}$ it also follows that $\overline{\mathcal{R}}^{\mathrm{si}}$ is relatively continuous.

Let $A:=B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$. By definition of $\|\cdot\|_{\mathrm{si}}$ and because $\mathcal{R}\subseteq\overline{\mathcal{R}}^{\mathrm{si}}$, we get

$$\mathcal{F}(\mathcal{E},\mathcal{R})\subseteq\mathcal{F}(\mathcal{E},\overline{\mathcal{R}}^{\mathrm{si}})=\overline{\operatorname{span}}(|\overline{\mathcal{R}}^{\mathrm{si}}\rangle\!\rangle\circ A)\subseteq\overline{\operatorname{span}}(|\mathcal{R}\rangle\!\rangle\circ A)=\mathcal{F}(\mathcal{E},\mathcal{R}).$$

Thus $\mathcal{F}(\mathcal{E},\mathcal{R})=\mathcal{F}(\mathcal{E},\overline{\mathcal{R}}^{\mathrm{si}})$. By Proposition 4.1.10(ii) and because the linear span of $\gamma_B(B)A$ is dense in A we get

$$\mathcal{F}(\mathcal{E},\mathcal{R}\cdot B)=\overline{\operatorname{span}}(|\mathcal{R}\rangle\!\rangle\gamma_B(B)A)=\mathcal{F}(\mathcal{E},\mathcal{R}).$$

Analogously, by Proposition 4.3.8, and because the linear span of $(1 \otimes \rho(L^1_0(\mathcal{G}))A$ is dense in A (by Proposition 4.3.13), we get that $\mathcal{F}(\mathcal{E}, L^1_0(\mathcal{G}) * \mathcal{R}) = \mathcal{F}(\mathcal{E}, \mathcal{R})$.

In particular, it follows that $\mathcal{R}_n$ (defined above) is relatively continuous and $\mathcal{F}(\mathcal{E}, \mathcal{R}_n) = \mathcal{F}(\mathcal{E}, \mathcal{R})$ for all $n \in \mathbb{N}$. In the same way one proves that $\mathcal{R}_\infty$ is relatively continuous. We conclude that $\mathcal{R}_c = \overline{\mathcal{R}_\infty}^{\mathrm{si}}$ is relatively continuous and

$$\mathcal{F}(\mathcal{E}; \mathcal{R}_{\mathrm{c}}) = \mathcal{F}(\mathcal{E}, \mathcal{R}_\infty) = \overline{\mathrm{span}}\left(\bigcup_{n=0}^{\infty} \mathcal{F}(\mathcal{E}, \mathcal{R}_n)\right) = \mathcal{F}(\mathcal{E}, \mathcal{R}). \qquad \square$$

Remark 5.3.3. Let $\mathcal{E}$ be a Hilbert $B, \mathcal{G}$-module and let $\mathcal{R}, \mathcal{R}' \subseteq \mathcal{E}$ be relatively continuous subsets. In general, it is not true that the union $\mathcal{R} \cup \mathcal{R}'$ is relatively continuous, even if $\mathcal{R}' = u(\mathcal{R})$, where $u \in \mathcal{L}^{\mathcal{G}}(\mathcal{E})$ is some equivariant unitary. This happens already in the case of (Abelian) groups (see [19, 48]). We are going to consider some examples later in Example 6.11.1.

Given a complete subspace $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$, Propositions 4.1.10(ii) and 4.3.8 imply that $\overline{|\mathcal{R}\rangle\!\rangle}$ is already a (concrete) A-module, where $A := B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. In other words, we have $\overline{|\mathcal{R}\rangle\!\rangle} \circ A \subseteq \overline{|\mathcal{R}\rangle\!\rangle}$. Therefore, if $\mathcal{R}$ is also relatively continuous, then it follows from Equation (5.4) that

$$\mathcal{F}(\mathcal{E}, \mathcal{R}) = \overline{\mathrm{span}}(|\mathcal{R}\rangle\!\rangle \circ A) \subseteq \overline{|\mathcal{R}\rangle\!\rangle} \subseteq \mathcal{F}(\mathcal{E}, \mathcal{R}),$$

that is, $\mathcal{F}(\mathcal{E}, \mathcal{R}) = \overline{|\mathcal{R}\rangle\!\rangle}$ for any complete, relatively continuous subspace. Combining this with Proposition 5.3.2 we get

$$\mathcal{F}(\mathcal{E}, \mathcal{R}) = \overline{|\mathcal{R}_{\mathrm{c}}\rangle\!\rangle}, \tag{5.11}$$

for any relatively continuous subset $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$.

Corollary 5.3.4. *For any relatively continuous subset $\mathcal{R}$ of a Hilbert $B, \mathcal{G}$-module $\mathcal{E}$, we have*

$$\mathrm{Fix}(\mathcal{E}, \mathcal{R}) = \overline{\mathrm{span}}(|\mathcal{R}_{\mathrm{c}}\rangle\!\rangle\langle\!\langle \mathcal{R}_{\mathrm{c}}|) \quad \textit{and} \qquad \mathcal{I}(\mathcal{E}, \mathcal{R}) = \overline{\mathrm{span}}(\langle\!\langle \mathcal{R}_{\mathrm{c}} \,|\, \mathcal{R}_{\mathrm{c}}\rangle\!\rangle).$$

Since the bra-ket operators are $\mathcal{G}$-equivariant we see (again) that $\mathrm{Fix}(\mathcal{E}, \mathcal{R})$ is a C^*-subalgebra of $\mathcal{L}^{\mathcal{G}}(\mathcal{E}) = \mathcal{M}_1\big(\mathcal{K}(\mathcal{E})\big)$. Proposition 4.1.10(i) yields the equality

$$\mathrm{Fix}(\mathcal{E}, \mathcal{R}) = \overline{\mathrm{span}}\{(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(\gamma_{\mathcal{K}(\mathcal{E})}(|\xi\rangle\langle\eta|)) : \xi, \eta \in \mathcal{R}_{\mathrm{c}}\}. \tag{5.12}$$

The following result gives a useful criterion to show that a subspace is complete or to calculate its completion.

Proposition 5.3.5. *Let $\mathcal{E}$ be a Hilbert $B, \mathcal{G}$-module, let $\mathcal{R}$ be a subspace of $\mathcal{E}_{\mathrm{si}}$ and suppose that $D_0 \subseteq L^1_0(\mathcal{G})$ and $B_0 \subseteq B$ are dense subsets.*

(i) *$\mathcal{R}$ is complete if and only if it is* si*-closed, $D_0 * \mathcal{R} \subseteq \mathcal{R}$ and $\mathcal{R} \cdot B_0 \subseteq \mathcal{R}$.*

(ii) *If $D_0 * \mathcal{R} \subseteq \overline{\mathcal{R}}^{\mathrm{si}}$ and $\mathcal{R} \cdot B_0 \subseteq \overline{\mathcal{R}}^{\mathrm{si}}$, then the completion of $\mathcal{R}$ is equal to $\overline{\mathcal{R}}^{\mathrm{si}}$.*

Proof. By Propositions 4.1.11 and 4.3.8 the left $L^1_0(\mathcal{G})$-action and the right B-action on $\mathcal{E}_{\mathrm{si}}$ are continuous with respect to $\|\cdot\|_{\mathrm{si}}$. The assertions now follow easily. $\square$

At this point, the following question naturally appears. Let $\mathcal{R}, \mathcal{R}' \subseteq \mathcal{E}$ be complete, relatively continuous subspaces and suppose that $\mathcal{F}(\mathcal{E}, \mathcal{R}) = \mathcal{F}(\mathcal{E}, \mathcal{R}')$. Does it follow that $\mathcal{R} = \mathcal{R}'$? For locally compact groups, that is, for $\mathcal{G} = \mathcal{C}_0(G)$, this is in fact true ([48, Theorem 6.1]). Unfortunately, this is not the case for general locally compact quantum groups. Problems appear for non-co-amenable locally compact quantum groups $\mathcal{G}$. In these cases, coactions are not necessarily injective. Take any non-injective coaction $(\mathcal{E}, \gamma_{\mathcal{E}})$ of a locally compact quantum group $\mathcal{G}$. Note that any $\xi \in \ker(\gamma_{\mathcal{E}})$ is square-integrable with $|\xi\rangle\!\rangle = 0$. Thus $\mathcal{R} := \{0\}$ and $\mathcal{R}' := \ker(\gamma_{\mathcal{E}})$ are different complete, relatively continuous subspaces with $\mathcal{F}(\mathcal{E}, \mathcal{R}) = \mathcal{F}(\mathcal{E}, \mathcal{R}') = \{0\}$. In order to circumvent this problem we need an extra condition.

Definition 5.3.6. Let $\mathcal{E}$ be a Hilbert $B, \mathcal{G}$-module. We say that a complete subspace $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$ is *slice-complete*, or shortly, *s-complete* if for all $\xi \in \mathcal{E}_{\mathrm{si}}$, with $\langle\!\langle \xi \,|\, \xi \rangle\!\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, one has

$$\omega * \xi \in \mathcal{R} \text{ for all } \omega \in L^1_0(\mathcal{G}) \Longrightarrow \xi \in \mathcal{R}.$$

The *s-completion* of a subset $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$, denoted by $\mathcal{R}_{\mathrm{sc}}$, is the smallest s-complete subspace of $\mathcal{E}_{\mathrm{si}}$ containing $\mathcal{R}$.

Note that, by definition, $\mathcal{E}_{\mathrm{si}}$ is s-complete, and intersections of s-complete subspaces are again s-complete. Thus the s-completion of a subset $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$ always exists: it is the intersection of all s-complete subspaces of $\mathcal{E}_{\mathrm{si}}$ containing $\mathcal{R}$.

Note also that any s-complete subspace contains $\ker(\gamma_{\mathcal{E}})$ because $\omega * \xi = 0$ for all $\omega \in L^1_0(\mathcal{G})$ and $\xi \in \ker(\gamma_{\mathcal{E}})$. Thus, if $\gamma_{\mathcal{E}}$ is not injective, the trivial subspace $\mathcal{R} = \{0\}$ is complete (and relatively continuous), but not s-complete. The converse is also true:

Proposition 5.3.7. *Let $\mathcal{E}$ be a Hilbert $B, \mathcal{G}$-module. Then the s-completion of $\{0\}$ is $\ker(\gamma_{\mathcal{E}})$. In particular, $\{0\}$ is s-complete if and only if $\gamma_{\mathcal{E}}$ is injective.*

Proof. It suffices to show that $\mathcal{R}_0 := \ker(\gamma_{\mathcal{E}})$ is s-complete. Of course, $\mathcal{R}_0$ is complete. Now suppose that $\xi \in \mathcal{E}$ and $\omega * \xi \in \mathcal{R}_0$ for all $\omega \in L^1_0(\mathcal{G})$, that is,

$$0 = \gamma_{\mathcal{E}}(\omega * \xi) = \gamma_{\mathcal{E}}((\mathrm{id}_{\mathcal{E}} \otimes \omega)(\gamma_{\mathcal{E}}(\xi))) = (\mathrm{id}_{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega)(\gamma_{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{G}})(\gamma_{\mathcal{E}}(\xi)).$$

Since $\omega \in L^1_0(\mathcal{G})$ is arbitrary, it follows that

$$0 = (\gamma_{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{G}})\big(\gamma_{\mathcal{E}}(\xi)\big) = (\mathrm{id}_{\mathcal{E}} \otimes \Delta)\big(\gamma_{\mathcal{E}}(\xi)\big).$$

And finally, because Δ is injective, we get $\gamma_{\mathcal{E}}(\xi) = 0$, that is, $\xi \in \ker(\gamma_{\mathcal{E}}) = \mathcal{R}_0$. Therefore $\mathcal{R}_0$ is s-complete. □

If one restricts to injective coactions, that is, reduced coactions, then it is not clear whether there exist examples of complete subspaces that are not s-complete. This is not clear even in the compact case. Suppose we have a compact quantum group $\mathcal{G}$ (in particular, $\mathcal{G}$ is unimodular and hence $L^1_0(\mathcal{G}) = L^1(\mathcal{G})$). In this case, we have $\mathcal{E}_{\mathrm{si}} = \mathcal{E}$ for any Hilbert $B, \mathcal{G}$-module, the si-norm is equivalent to the norm on $\mathcal{E}$ and any subset $\mathcal{R} \subseteq \mathcal{E}$ is automatically relatively continuous by Proposition 5.2.12. Thus a complete subspace

$\mathcal{R} \subseteq \mathcal{E}$, that is, a closed $L^1(\mathcal{G}), B$-invariant subspace of $\mathcal{E}$, is s-complete if and only if for all $\xi \in \mathcal{E}$, the condition $\omega * \xi \in \mathcal{R}$ for all $\omega \in L^1(\mathcal{G})$ implies $\xi \in \mathcal{R}$. Note that this has a relation with the "slice map property" for the triple $(\mathcal{G}, \mathcal{E}, \mathcal{R})$, which says the following:

$$\text{for all } x \in \mathcal{E} \otimes \mathcal{G}, \text{ if } (\mathrm{id}_{\mathcal{E}} \otimes \omega)(x) \in \mathcal{R} \text{ for all } \omega \in \mathcal{G}^*, \text{ then } x \in \mathcal{R} \otimes \mathcal{G}.^8$$

In fact, suppose that $\mathcal{R}$ is an $L^1(\mathcal{G}), B$-invariant closed subspace of $\mathcal{E}$ satisfying $\gamma_{\mathcal{E}}(\mathcal{R}) \subseteq \mathcal{R}\otimes\mathcal{G}$, and suppose that $\mathcal{R}$ is s-complete. This means that given $\xi \in \mathcal{E}$, if $(\mathrm{id}_{\mathcal{E}}\otimes\omega)(\gamma_{\mathcal{E}}(\xi)) = \omega * \xi \in \mathcal{R}$ for all $\omega \in L^1(\mathcal{G})$, then $\xi \in \mathcal{R}$. Since $L^1(\mathcal{G}) \subseteq \mathcal{G}^*$ and $\gamma_{\mathcal{E}}(\mathcal{R}) \subseteq \mathcal{R} \otimes \mathcal{G}$, this implies the slice map property above for elements $x \in \gamma_{\mathcal{E}}(\mathcal{R})$. The problem is that the slice map property is known to be false in general. But it is not clear to me whether some of these counterexamples fit into our setting.

By the way, the relation above between s-completeness and the slice map property was the reason to adopt the terminology "slice-complete".

Remark 5.3.8. Note that every complete subspace $\mathcal{R} \subseteq \mathcal{E}$ satisfies the following property: for all $\xi \in \mathcal{E}_{\mathrm{si}}$ with $\langle\!\langle \xi \,|\, \xi \rangle\!\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, if $\xi \cdot b \in \mathcal{R}$ for all $b \in B$, then $\xi \in \mathcal{R}$. In fact, let (e_i) be an approximate unit for B. Then $\xi \cdot e_i \to \xi$ and $\gamma_B(e_i) \to 1$ strictly in $\mathcal{M}(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})$. Now the condition $\langle\!\langle \xi \,|\, \xi \rangle\!\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ means that $\mathcal{R}' := \{\xi\}$ is relatively continuous. Thus $\mathcal{F} := \mathcal{F}(\mathcal{E}, \mathcal{R}')$ is a (concrete) Hilbert A-module, where $A := B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. Thus, by Cohen's Factorization Theorem, for any $x \in \mathcal{F}$, the map $\mathcal{M}(A) \ni a \mapsto x \cdot a \in \mathcal{F}$ is continuous for the strict topology on $\mathcal{M}(A)$ and the norm topology on $\mathcal{F}$. Equation (5.4) says that $|\xi\rangle\!\rangle \in \mathcal{F}$. Thus

$$|\xi \cdot e_i\rangle\!\rangle = |\xi\rangle\!\rangle \circ \gamma_B(e_i) \to |\xi\rangle\!\rangle.$$

Hence $\xi \cdot e_i \to \xi$ in the si-norm and therefore $\xi \in \mathcal{R}$.

Note that one important point above was the use of a (bounded) approximate unit for B. In order to follow the same idea above and try to prove the same property for the left $L^1_0(\mathcal{G})$-action, that is, to prove that every complete subspace is automatically s-complete, one needs a bounded approximate unit for $L^1_0(\mathcal{G})$, that is, one needs co-amenability of $\mathcal{G}$. This is the content of the next result.

Proposition 5.3.9. *Let $\mathcal{E}$ be a Hilbert $B, \mathcal{G}$-module and suppose that $\mathcal{G}$ is co-amenable. Then every complete subspace $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$ is automatically s-complete.*

Proof. Let (ω_i) be a bounded approximate unit for $L^1_0(\mathcal{G})$ (Proposition 4.3.17). Then $\rho_{\omega_i} \to 1$ strictly in $\mathcal{M}(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})$. Using Proposition 4.3.14, one can now follow the same idea as in Remark 5.3.8. □

Proposition 5.3.9 applies to actions of locally compact groups, that is, to coactions of $\mathcal{G} = \mathcal{C}_0(G)$, where G is a locally compact group, because $\mathcal{C}_0(G)$ is always co-amenable as a quantum group. On the other hand, it does not apply to coactions of groups, that is, to the dual $C^*_{\mathrm{r}}(G)$, unless G is amenable. Indeed, as already mentioned, the quantum group $C^*_{\mathrm{r}}(G)$ is co-amenable if and only if G is amenable (see comments before Definition 2.5.4).

[8] Since $\mathcal{R}$ is a closed B-invariant subspace of $\mathcal{E}$, that is, a Hilbert B-submodule of $\mathcal{E}$, it makes sense to write $\mathcal{R} \otimes \mathcal{G}$. This is the external tensor and it is identified with the closed linear span of $\mathcal{R} \odot \mathcal{G} \subseteq \mathcal{E} \otimes \mathcal{G}$.

Proposition 5.3.10. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module, and let $\mathcal{R}$ be a complete, relatively continuous subspace of $\mathcal{E}_{\mathrm{si}}$. Equipped with the* si*-norm, $\mathcal{R}$ is a nondegenerate Banach right B-module, that is, $\mathcal{R}\cdot B=\mathcal{R}$. Moreover, if $\mathcal{G}$ is co-amenable, then $\mathcal{R}$ is also a nondegenerate Banach left $L^1_0(\mathcal{G})$-module, that is, $L^1_0(\mathcal{G})*\mathcal{R}=\mathcal{R}$.*

Proof. We already know that $\mathcal{R}$ is a Banach left $L^1_0(\mathcal{G})$-module and also a Banach right B-module. We only have to prove the nondegeneracy of the actions. Now, if (e_j) and (ω_i) are bounded approximate units for B and $L^1_0(\mathcal{G})$, respectively, then, as we saw in Remark 5.3.8 and Proposition 5.3.9, we have $\xi\cdot e_j\to\xi$ and $\omega_i*\xi\to\xi$ with respect to the si-norm, for all $\xi\in\mathcal{R}$. Therefore, by Cohen's Factorization Theorem, $\mathcal{R}\cdot B=\mathcal{R}$ and $L^1_0(\mathcal{G})*\mathcal{R}=\mathcal{R}$. □

If $\mathcal{G}$ is not co-amenable, then the conclusion of Proposition 5.3.10 does not hold in general. A trivial example can be found in the case of non-injective coactions. In fact, if $(\mathcal{E},\gamma_{\mathcal{E}})$ is a non-injective coaction, then $\mathcal{R}:=\ker(\gamma_{\mathcal{E}})$ is relatively continuous and s-complete (Proposition 5.3.7), but $L^1(\mathcal{G})*\mathcal{R}=\{0\}$. Note that the same problem appears even if $\mathcal{G}$ is compact because compactness does not imply co-amenability.

Definition 5.3.11. Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. We say that a complete subspace $\mathcal{R}\subseteq\mathcal{E}_{\mathrm{si}}$ is *essential* if $\overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G})*\mathcal{R})=\mathcal{R}$. In this case we also say that $\mathcal{R}$ is *e-complete*.

With this new terminology, Proposition 5.3.10 says that every complete, relatively continuous subspace $\mathcal{R}\subseteq\mathcal{E}$ is essential provided $\mathcal{G}$ is co-amenable. And as we have seen above, $\ker(\gamma_{\mathcal{E}})$ is relatively continuous and s-complete, but it is e-complete if and only $\gamma_{\mathcal{E}}$ is injective. If one restricts to injective coactions, then it is not clear, whether there exist examples of non-essential, relatively continuous, complete subspaces.

Proposition 5.3.12. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module and let $\mathcal{F}\subseteq\mathcal{L}^{\mathcal{G}}(B\otimes L^2(\mathcal{G}),\mathcal{E})$ be a concrete Hilbert A-module, where $A:=B\rtimes_{\mathrm{r}}\widehat{\mathcal{G}}^{\mathrm{c}}$. Define*

$$\mathcal{R}_{\mathcal{F}}:=\{\xi\in\mathcal{E}_{\mathrm{si}}:|\xi\rangle\!\rangle\in\mathcal{F}\},$$

$$\mathcal{R}^0_{\mathcal{F}}:=\{x(K):x\in\mathcal{F},\,K\in B\odot\hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})\}.$$

Then $\mathcal{R}^0_{\mathcal{F}}\subseteq\mathcal{R}_{\mathcal{F}}$, both $\mathcal{R}^0_{\mathcal{F}}$ and $\mathcal{R}_{\mathcal{F}}$ are relatively continuous, $\mathcal{R}_{\mathcal{F}}$ is complete, and $|\mathcal{R}^0_{\mathcal{F}}\rangle\!\rangle$ and $|\mathcal{R}_{\mathcal{F}}\rangle\!\rangle$ are dense in $\mathcal{F}$. In particular, we have

$$\mathcal{F}(\mathcal{E},\mathcal{R}^0_{\mathcal{F}})=\mathcal{F}(\mathcal{E},\mathcal{R}_{\mathcal{F}})=\mathcal{F}.$$

Proof. Let $\mathcal{R}_0:=B\odot\hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})\subseteq B\otimes L^2(\mathcal{G})$. By Proposition 5.2.9, $\mathcal{R}_0$ is a relatively continuous subset of $B\otimes L^2(\mathcal{G})$ and $|\mathcal{R}_0\rangle\!\rangle$ is a dense subspace of A. Proposition 4.1.10(iii) implies that $\mathcal{R}^0_{\mathcal{F}}\subseteq\mathcal{E}_{\mathrm{si}}$ and

$$|\mathcal{R}^0_{\mathcal{F}}\rangle\!\rangle=\mathcal{F}\circ|\mathcal{R}_0\rangle\!\rangle\subseteq\mathcal{F}\circ A\subseteq\mathcal{F}.$$

Thus $\mathcal{R}^0_{\mathcal{F}}\subseteq\mathcal{R}_{\mathcal{F}}$. This implies that $|\mathcal{R}^0_{\mathcal{F}}\rangle\!\rangle\subseteq|\mathcal{R}_{\mathcal{F}}\rangle\!\rangle\subseteq\mathcal{F}$. The equation above also shows that $|\mathcal{R}^0_{\mathcal{F}}\rangle\!\rangle$ is dense in $\mathcal{F}\circ A$ which by Cohen's Factorization Theorem is equal to $\mathcal{F}$. Since

$\mathcal{F}^* \circ \mathcal{F} \subseteq A$, $\mathcal{R}_{\mathcal{F}}$ (and therefore $\mathcal{R}^0_{\mathcal{F}}$) is relatively continuous. Since $\mathcal{F}$ is a concrete A-module and $\mathcal{E}_{\mathrm{si}}$ is $L^1_0(\mathcal{G}), B$-invariant, it follows from Propositions 4.1.10(ii) and 4.3.8 that $\mathcal{R}_{\mathcal{F}}$ is $L^1_0(\mathcal{G}), B$-invariant as well. From the definition of $\|\cdot\|_{\mathrm{si}}$ and because $\mathcal{E}_{\mathrm{si}}$ is si-closed, it follows that $\mathcal{R}_{\mathcal{F}}$ is si-closed. Thus $\mathcal{R}_{\mathcal{F}}$ is complete. □

Remark 5.3.13. Proposition 5.3.12 is a quantum version of Meyer's result [48, Proposition 6.1] for classical groups. There is a small difference between our version and Meyer's version in [48], namely, the choice of $\mathcal{R}^0_{\mathcal{F}}$. For groups, that is, for $\mathcal{G} = \mathcal{C}_0(G)$, where G is some locally compact group, one can replace $\mathcal{R}^0_{\mathcal{F}}$ by the more canonical choice $\tilde{\mathcal{R}}^0_{\mathcal{F}} := \{x(K) : x \in \mathcal{F},\, K \in \mathcal{C}_c(G,B)\}$. This set satisfies the same properties of $\mathcal{R}^0_{\mathcal{F}}$ defined above (this is exactly [48, Proposition 6.1]). The point here is that $\mathcal{C}_c(G,B)$ is also a relatively continuous subset of $L^2(G,B)$ and $\mathcal{F}\big(L^2(G,B),\mathcal{C}_c(G,B)\big) = C^*_{\mathrm{r}}(G,B)$ (this is proved in [48]). For an arbitrary locally compact quantum group $\mathcal{G}$ we may take any relatively continuous subset $\mathcal{R}_0 \subseteq B \otimes L^2(\mathcal{G})$ satisfying $\mathcal{F}\big(B \otimes L^2(\mathcal{G}), \mathcal{R}_0\big) = B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ and define $\tilde{\mathcal{R}}^0_{\mathcal{F}} := \{x(K) : x \in \mathcal{F},\, K \in \mathcal{R}_0\}$. An argument analogous to that in the proof of Proposition 5.3.12 shows $\tilde{\mathcal{R}}^0_{\mathcal{F}} \subseteq \mathcal{R}_{\mathcal{F}}$ (so that $\tilde{\mathcal{R}}^0_{\mathcal{F}}$ is relatively continuous) and $\mathcal{F}(\mathcal{E}, \tilde{\mathcal{R}}^0_{\mathcal{F}}) = \mathcal{F}(\mathcal{E}, \mathcal{R}_{\mathcal{F}}) = \mathcal{F}$. We are going to see later that if $\tilde{\mathcal{R}}^0_{\mathcal{F}}$ is chosen in this way, then the s-completion of $\tilde{\mathcal{R}}^0_{\mathcal{F}}$ is equal to $\mathcal{R}_{\mathcal{F}}$. In this sense, all such choices are equivalent.

Proposition 5.3.14. *Let $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes L^2(\mathcal{G}), \mathcal{E})$ be a concrete Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-module, where $\mathcal{E}$ is some Hilbert $B, \mathcal{G}$-module. Then $\mathcal{R}_{\mathcal{F}}$ is s-complete.*

Proof. By Proposition 5.3.12, $\mathcal{R}_{\mathcal{F}}$ is complete. Suppose that $\xi \in \mathcal{E}_{\mathrm{si}}$ is such that $\langle\langle \xi \,|\, \xi \rangle\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ and $\omega * \xi \in \mathcal{R}_{\mathcal{F}}$ for all $\omega \in L^1_0(\mathcal{G})$. By Proposition 4.3.8, this means that

$$|\omega * \xi\rangle\rangle = |\xi\rangle\rangle(1_B \otimes \rho_\omega) \in \mathcal{F}$$

for all $\omega \in L^1_0(\mathcal{G})$. Since $\rho(L^1_0(\mathcal{G}))$ is dense in $\widehat{\mathcal{G}}^{\mathrm{c}}$, there is a bounded approximate unit (e_i) for $\widehat{\mathcal{G}}^{\mathrm{c}}$ of the form $e_i = \rho(\omega_i)$ with $\omega_i \in L^1_0(\mathcal{G})$ for all i. It follows that $(1_B \otimes e_i)$ converges strictly to 1 in $\mathcal{M}(B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}})$. Since $\langle\langle \xi \,|\, \xi \rangle\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, we get

$$|\omega_i * \xi\rangle\rangle = |\xi\rangle\rangle\big(1_B \otimes \rho(\omega_i)\big) \to |\xi\rangle\rangle.$$

Therefore $|\xi\rangle\rangle \in \mathcal{F}$, that is, $\xi \in \mathcal{R}_{\mathcal{F}}$ and hence $\mathcal{R}_{\mathcal{F}}$ is s-complete. □

Remark 5.3.15. Let $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes L^2(\mathcal{G}), \mathcal{E})$ be a concrete Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-modules as above. Suppose that $\xi \in \mathcal{E}_{\mathrm{si}}$ and $\omega * \xi \in \mathcal{R}_{\mathcal{F}}$ for all $\omega \in L^1_0(\mathcal{G})$. In general, this does *not* imply $\xi \in \mathcal{R}_{\mathcal{F}}$, even in the case of (Abelian) groups (see Example 6.11.1). Thus the condition $\langle\langle \xi \,|\, \xi \rangle\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is important.

5.4 Continuous square-integrability

Throughout this section, $\mathcal{G}$ denotes a locally compact quantum group and B denotes a $\mathcal{G}$-C^*-algebra. We are ready to give one of the main definitions of this work.

Definition 5.4.1. A *continuously square-integrable Hilbert $B,\mathcal{G}$-module* is a pair $(\mathcal{E},\mathcal{R})$ consisting of a Hilbert $B,\mathcal{G}$-module $\mathcal{E}$ and a dense, complete, relatively continuous subspace $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$. If, in addition, $\mathcal{R}$ is s-complete, then we say that $(\mathcal{E},\mathcal{R})$ is an *s-continuously square-integrable Hilbert $B,\mathcal{G}$-module.*

The *generalized fixed point algebra* associated to a continuously square-integrable Hilbert $B,\mathcal{G}$-module $(\mathcal{E},\mathcal{R})$ is the C^*-algebra $\mathrm{Fix}(\mathcal{E},\mathcal{R}) = \overline{\mathrm{span}}\,|\mathcal{R}\rangle\!\rangle\langle\!\langle\mathcal{R}|$.

By Equation (5.12), the generalized fixed point algebra $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ is the closed linear span of the "averages" $(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(x)$ where $x = |\xi\rangle\langle\eta|$ with $\xi,\eta \in \mathcal{R}$. Note that in the group case $\mathcal{G} = \mathcal{C}_0(G)$, the average $(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(x)$ is the same as integral $\int_G^{\mathrm{su}} \alpha_t(x)\,\mathrm{d}t$, where α is the corresponding action of G on $\mathcal{K}(\mathcal{E})$. In particular, $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ is contained in the big fixed point algebra $\mathcal{M}_1\big(\mathcal{K}(\mathcal{E})\big) = \{x \in \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big) : \gamma_{\mathcal{K}(\mathcal{E})}(x) = x \otimes 1\}$ and thus elements in $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ are fixed by the coaction of $\mathcal{K}(\mathcal{E})$. Proposition 5.2.4 tell us that $\mathrm{Fix}(\mathcal{E},\mathcal{R})$ is Morita equivalent to the ideal $\mathcal{I}(\mathcal{E},\mathcal{R}) = \overline{\mathrm{span}}\langle\!\langle\mathcal{R} \mid \mathcal{R}\rangle\!\rangle \subseteq B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, where $\mathcal{F}(\mathcal{E},\mathcal{R})$ is the canonical candidate for the imprimitivity Hilbert module.

In what follows, we are going to generalize [48, Theorem 6.1] to the setting of locally compact quantum groups. This result describes relatively continuous subspaces via concrete Hilbert modules. First we need some preliminary results.

Recall that σ denotes the modular group and $\mathcal{T}_\varphi$ denotes the Tomita $*$-algebra of the left Haar weight φ (see Equation (2.5)).

Lemma 5.4.2. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. Let $b \in \mathcal{T}_\varphi$, $\xi \in \mathcal{E}_{\mathrm{si}}$ and suppose that $a \in \mathcal{N}_\varphi$ is such that $\Lambda(a) \in \mathcal{D}(\delta^{\frac{1}{2}})$. Define $\omega := \omega_{\Lambda(b),\Lambda(a)} = a\varphi b^* \in L^1(\mathcal{G})$ and $x_\omega := a\sigma_{-\mathrm{i}}(b^*) \in \mathcal{N}_\varphi$. Then $\omega * \xi \in \mathcal{E}_{\mathrm{si}}$ and*

$$\|\omega * \xi\|_{\mathrm{si}} \leq c_\omega \||\xi\rangle\!\rangle\|,$$

where $c_\omega := \max\{\|\Lambda(x_\omega)\|, \|\rho(\omega)\|\}$.

Proof. Lemma 4.5.3 implies

$$\omega * \xi = |\xi\rangle\!\rangle\big(1_B \otimes \Lambda(x_\omega)\big).$$

Thus $\|\omega * \xi\| \leq \|\Lambda(x_\omega)\|\||\xi\rangle\!\rangle\|$. Proposition 4.3.8 says that $\omega * \xi \in \mathcal{E}_{\mathrm{si}}$ and

$$|\omega * \xi\rangle\!\rangle = |\xi\rangle\!\rangle(1_B \otimes \rho_\omega).$$

Hence $\||\omega * \xi\rangle\!\rangle\| \leq \|\rho_\omega\|\||\xi\rangle\!\rangle\|$. The desired result now follows. ☐

The following result (for the right Haar weight) appears in the proof of [73, Proposition 1.9.5].

Lemma 5.4.3. *Let $\mathcal{G}$ be a locally compact quantum group. Define*

$$C := \mathrm{span}\{e_n x : n \in \mathbb{N},\, x \in \mathcal{T}_\varphi\}.^9$$

Then $\Lambda(C)$ is a core for δ^z, where δ is the modular element of $\mathcal{G}$ and z is any complex number. In particular, $\mathcal{D}(\delta^z) \cap \Lambda(\mathcal{N}_\varphi)$ is a core for δ^z.

[9] See Equation (4.5) for the definition of e_n.

Proof. For any $x \in \mathcal{N}_\varphi$ we have $e_n x \in \mathcal{N}_\varphi$ and

$$\Lambda(e_n x) = e_n \Lambda(x).$$

Since $\delta^z e_n$ is bounded, it follows that $\Lambda(e_n x) \in \mathcal{D}(\delta^z)$ and

$$\delta^z \Lambda(e_n x) = \delta^z e_n \Lambda(x).$$

Thus $\Lambda(C) \subseteq \mathcal{D}(\delta^z)$. Now take any $v \in \mathcal{D}(\delta^z)$, and choose $(x_k) \subseteq \mathcal{T}_\varphi$ such that $v_k := \Lambda(x_k) \to v$. Define $c_{n,k} := e_n x_k \in C$. Then $\Lambda(c_{n,k}) \in \mathcal{D}(\delta^z)$ and (using that $e_n \to 1$ strictly and hence also strongly)

$$\Lambda(c_{n,k}) = e_n \Lambda(x_k) \to v, \text{ as } n, k \to \infty.$$

For each $n \in \mathbb{N}$, we have

$$\|\delta^z e_n v_k - \delta^z e_n v\| \to 0 \text{ as } k \to \infty,$$

because $\delta^z e_n$ is bounded. Thus we can find a sequence of natural numbers $k_1 < k_2 < \dots$ such that

$$\|\delta^z e_n v_{k_n} - \delta^z e_n v\| < 1/n, \text{ for each } n \in \mathbb{N}.$$

Define $\xi_n := e_n v_{k_n}$. Then $\xi_n \in \Lambda(C)$, $\xi_n \to v$ and (using that $\delta^z e_n = e_n \delta^z$)

$$\|\delta^z \xi_n - \delta^z v\| \leq \|\delta^z e_n v_{k_n} - \delta^z e_n v\| + \|\delta^z e_n v - \delta^z v\| \leq 1/n + \|e_n \delta^z v - \delta^z v\|,$$

which converges to 0, because $e_n \to 1$ strongly. Therefore $\Lambda(C)$ is a core for δ^z. □

We are ready to proof one of the main results of this thesis.

Theorem 5.4.4. *Let $\mathcal{G}$ be a locally compact quantum group, and let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module. Then the map $\mathcal{R} \mapsto \mathcal{F}(\mathcal{E},\mathcal{R})$ is a bijection between s-complete, relatively continuous subspaces of $\mathcal{E}$ and concrete Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-modules $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes L^2(\mathcal{G}), \mathcal{E})$. Its inverse is the map $\mathcal{F} \mapsto \mathcal{R}_{\mathcal{F}}$. A concrete Hilbert module $\mathcal{F}$ is essential if and only if $\mathcal{R}_{\mathcal{F}}$ is dense in $\mathcal{E}$.*

Proof. By Proposition 5.3.14, $\mathcal{R}_{\mathcal{F}}$ is relatively continuous and s-complete, so that the map $\mathcal{F} \mapsto \mathcal{R}_{\mathcal{F}}$ is well-defined. By Proposition 5.3.12 we have $\mathcal{F}(\mathcal{E},\mathcal{R}_{\mathcal{F}}) = \mathcal{F}$. It remains to show that $\mathcal{R} = \mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$ for every s-complete, relatively continuous subspace $\mathcal{R}$ of $\mathcal{E}$. By Equation (5.4), we have $\mathcal{R} \subseteq \mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$. Let $\xi \in \mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$. Then, by definition of $\mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$, we have $|\xi\rangle\!\rangle \in \mathcal{F}(\mathcal{E},\mathcal{R}) = \overline{|\mathcal{R}\rangle\!\rangle}$ (for the last equality we have used Equation (5.11) and the assumption that $\mathcal{R}$ is complete). Thus there is $\xi_n \in \mathcal{R}$ such that $|\xi_n\rangle\!\rangle \to |\xi\rangle\!\rangle$. Take any $a \in \mathcal{N}_\varphi$ and $b \in \mathcal{T}_\varphi$ such that $\Lambda(a) \in \mathcal{D}(\delta^{\frac{1}{2}})$ and define $\omega := a\varphi b^* = \omega_{u,v} \in L^1_0(\mathcal{G})$, where $u := \Lambda(b)$ and $v := \Lambda(a)$. By Lemma 5.4.2, we have $\|\omega * \eta\|_{\mathrm{si}} \leq c_\omega \||\eta\rangle\!\rangle\|$ for all $\eta \in \mathcal{E}_{\mathrm{si}}$, where c_ω is a constant depending only on ω. In particular,

$$\|\omega * \xi - \omega * \xi_n\|_{\mathrm{si}} \leq c_\omega \||\xi\rangle\!\rangle - |\xi_n\rangle\!\rangle\| \to 0.$$

Since $\mathcal{R}$ is complete, we get that $\omega * \xi \in \mathcal{R}$. Thus $\omega_{\Lambda(b),\Lambda(a)} * \xi \in \mathcal{R}$ for all $b \in \mathcal{T}_\varphi$ and $a \in \mathcal{N}_\varphi$ with $\Lambda(a) \in \mathcal{D}(\delta^{\frac{1}{2}})$. Now take any $u \in H$ and $v \in \mathcal{D}(\delta^{\frac{1}{2}})$. By Lemma 5.4.3, $\mathcal{D}(\delta^{\frac{1}{2}}) \cap \Lambda(\mathcal{N}_\varphi)$ is a core for $\delta^{\frac{1}{2}}$. So there is a sequence $(a_n) \subseteq \mathcal{N}_\varphi$ with $\Lambda(a_n) \in \mathcal{D}(\delta^{\frac{1}{2}})$ such that $\Lambda(a_n) \to v$ and $\delta^{\frac{1}{2}}(\Lambda(a_n)) \to \delta^{\frac{1}{2}}v$. And $\Lambda(\mathcal{T}_\varphi)$ is dense in H (see Lemma 2.4.11), there is a sequence $(b_n) \subseteq \mathcal{T}_\varphi$ such that $\Lambda(b_n) \to u$. It follows that $\omega_{\Lambda(b_n),\Lambda(a_n)} \to \omega_{u,v}$ in $L^1(\mathcal{G})$ and

$$\rho(\omega_{\Lambda(b_n),\Lambda(a_n)}) = (\mathrm{id} \otimes \omega_{\Lambda(b_n),\delta^{\frac{1}{2}}\Lambda(a_n)})(V^*) \to (\mathrm{id} \otimes \omega_{u,\delta^{\frac{1}{2}}v})(V^*) = \rho(\omega_{u,v}).$$

Proposition 4.3.8 implies that $\omega_{\Lambda(b_n),\Lambda(a_n)} * \xi \to \omega_{u,v} * \xi$ in the si-norm. Thus $\omega * \xi \in \mathcal{R}$ for all $\omega \in L^1_{00}(\mathcal{G})$ and hence also for all $\omega \in L^1_0(\mathcal{G})$ because $L^1_{00}(\mathcal{G})$ is dense in $L^1_0(\mathcal{G})$. Since $\mathcal{R}$ is s-complete we conclude that $\xi \in \mathcal{R}$. Therefore $\mathcal{R} = \mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$.

If $\mathcal{F}$ is essential, then by the definition of $\mathcal{R}^0_{\mathcal{F}}$ (see Proposition 5.3.12), the linear span of $\mathcal{R}^0_{\mathcal{F}}$ is dense in $\mathcal{E}$. Thus $\mathcal{R}_{\mathcal{F}} \supseteq \mathcal{R}^0_{\mathcal{F}}$ is dense in $\mathcal{E}$ as well. Conversely, if $\mathcal{R}_{\mathcal{F}}$ is dense, then $\mathcal{F}$ is essential by Proposition 5.2.3. □

Corollary 5.4.5. *Suppose $\mathcal{G}$ is a compact quantum group and $\mathcal{E}$ is a Hilbert $B,\mathcal{G}$-module. Then the map $\mathcal{R} \mapsto \mathcal{F}(\mathcal{E},\mathcal{R})$ is a bijection between $L^1(\mathcal{G}), B$-invariant closed subspaces of $\mathcal{E}$ satisfying*

$$\xi \in \mathcal{E} \text{ and } \omega * \xi \in \mathcal{R},\ \forall\, \omega \in L^1(\mathcal{G}) \Longrightarrow \xi \in \mathcal{R}, \tag{5.13}$$

and concrete Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-modules $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes L^2(\mathcal{G}), \mathcal{E})$. The inverse map is given by $\mathcal{F} \to \mathcal{R}_{\mathcal{F}}$.

Proof. Since $\mathcal{G}$ is compact, any subset of $\mathcal{E}$ is relatively continuous and the si-norm is equivalent to the norm of $\mathcal{E}$. Thus $\mathcal{R} \subseteq \mathcal{E}$ is complete if and only if it is an $L^1(\mathcal{G}), B$-invariant closed subspace of $\mathcal{E}$ (here we are using that $\mathcal{G}$ is unimodular so that $L^1(\mathcal{G}) = L^1_0(\mathcal{G})$). Such a subspace is s-complete if and only if it satisfies (5.13). Thus the assertion is a special case of Theorem 5.4.4. □

Corollary 5.4.6. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module, where $\mathcal{G}$ is a locally compact quantum group, and suppose that $\mathcal{R}$ is a relatively continuous subset of $\mathcal{E}$. Then the s-completion of $\mathcal{R}$ is equal to $\mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$. In particular, the s-completion of a relatively continuous subset is also relatively continuous.*

Proof. Let $\mathcal{R}_{\mathrm{sc}}$ be the s-completion of $\mathcal{R}$. By Proposition 5.3.14, $\mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$ is relatively continuous and s-complete and we have $\mathcal{R} \subseteq \mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$. Thus $\mathcal{R}_{\mathrm{sc}} \subseteq \mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$. In particular, $\mathcal{R}_{\mathrm{sc}}$ is relatively continuous. Now it is easy to see that the maps $\mathcal{R} \mapsto \mathcal{F}(\mathcal{E},\mathcal{R})$ and $\mathcal{F} \mapsto \mathcal{R}_{\mathcal{F}}$ preserve inclusion. Thus $\mathcal{R} \subseteq \mathcal{R}_{\mathrm{sc}}$ implies $\mathcal{F}(\mathcal{E},\mathcal{R}) \subseteq \mathcal{F}(\mathcal{E},\mathcal{R}_{\mathrm{sc}})$. Since $\mathcal{R}_{\mathrm{sc}}$ is relatively continuous and s-complete, Theorem 5.4.4 that $\mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})} \subseteq \mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R}_{\mathrm{sc}})} = \mathcal{R}_{\mathrm{sc}}$. □

Corollary 5.4.7. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module and suppose that $\mathcal{G}$ is co-amenable. Let $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$ be some relatively continuous subset. Then $\mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$ is the completion of $\mathcal{R}$. In particular, $\mathcal{R}$ is complete if and only if $\mathcal{R}$ is equal to $\mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$.*

Proof. This follows from Proposition 5.3.9 and Corollary 5.4.6. □

The result above implies, in particular, that our definition of completeness of relatively continuous subsets is equivalent to [48, Definition 6.2] in the case of groups (see [48, Proposition 6.3]).

Corollary 5.4.8. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module, and suppose that $\mathcal{G}$ is co-amenable. Then the map $\mathcal{R} \mapsto \mathcal{F}(\mathcal{E},\mathcal{R})$ is a bijection between complete, relatively continuous subspaces of $\mathcal{E}$ and concrete Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-modules $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes L^2(G), \mathcal{E})$. Its inverse is the map $\mathcal{F} \mapsto \mathcal{R}_{\mathcal{F}}$.*

The following result gives a description of the s-completion of a relatively continuous subset.

Proposition 5.4.9. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module, where $\mathcal{G}$ is a locally compact quantum group and let $\mathcal{R}$ be a relatively continuous subset of $\mathcal{E}$. Then the s-completion of $\mathcal{R}$ is given by*

$$\mathcal{R}_{\mathrm{sc}} = \{\xi \in \mathcal{E}_{\mathrm{si}} : \langle\!\langle \xi \,|\, \xi \rangle\!\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \text{ and } \omega * \xi \in \mathcal{R}_{\mathrm{c}} \text{ for all } \omega \in L^1_0(\mathcal{G})\},$$

where $\mathcal{R}_{\mathrm{c}}$ denotes the completion of $\mathcal{R}$.

Proof. Suppose that $\xi \in \mathcal{R}_{\mathrm{sc}} = \mathcal{R}_{\mathcal{F}(\mathcal{E},\mathcal{R})}$. By Equation (5.11), we have $\mathcal{F}(\mathcal{E},\mathcal{R}) = \overline{|\mathcal{R}_{\mathrm{c}}\rangle\!\rangle}$ and hence there is a sequence (ξ_n) in $\mathcal{R}_{\mathrm{c}}$ such that $|\xi_n\rangle\!\rangle \to |\xi\rangle\!\rangle$. As in the proof of Theorem 5.4.4, this implies that $\omega * \xi \in \mathcal{R}_{\mathrm{c}}$ for all $\omega \in L^1_0(\mathcal{G})$. Thus

$$\mathcal{R}_{\mathrm{sc}} \subseteq \{\xi \in \mathcal{E}_{\mathrm{si}} : \langle\!\langle \xi \,|\, \xi \rangle\!\rangle \in B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} \text{ and } \omega * \xi \in \mathcal{R}_{\mathrm{c}} \text{ for all } \omega \in L^1_0(\mathcal{G})\}.$$

And the other inclusion is trivial because $\mathcal{R}_{\mathrm{sc}}$ is s-complete and $\mathcal{R}_{\mathrm{c}} \subseteq \mathcal{R}_{\mathrm{sc}}$. □

Corollary 5.4.10. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module and let $\mathcal{R} \subseteq \mathcal{E}$ be a relatively continuous subset. Then*

$$\mathcal{F}(\mathcal{E},\mathcal{R}) = \mathcal{F}(\mathcal{E},\mathcal{R}_{\mathrm{sc}}).$$

Proof. The inclusion $\mathcal{R} \subseteq \mathcal{R}_{\mathrm{sc}}$ implies $\mathcal{F}(\mathcal{E},\mathcal{R}) \subseteq \mathcal{F}(\mathcal{E},\mathcal{R}_{\mathrm{sc}})$. On the other hand, if $\xi \in \mathcal{R}_{\mathrm{sc}}$, then by the description of $\mathcal{R}_{\mathrm{sc}}$ in Proposition 5.4.9, we have $\omega * \xi \in \mathcal{R}_{\mathrm{c}}$ for all $\omega \in L^1_0(\mathcal{G})$. Thus

$$|\omega * \xi\rangle\!\rangle = |\xi\rangle\!\rangle(1_B \otimes \rho_\omega) \in |\mathcal{R}_{\mathrm{c}}\rangle\!\rangle \subseteq \mathcal{F}(\mathcal{E},\mathcal{R}_{\mathrm{c}}) = \mathcal{F}(\mathcal{E},\mathcal{R})$$

for all $\omega \in L^1_{00}(\mathcal{G})$. Now taking a bounded approximate unit (e_i) for $\widehat{\mathcal{G}}^{\mathrm{c}}$ of the form $e_i = \rho(\omega_i)$, where $\omega_i \in L^1_0(\mathcal{G})$ for all i, it follows that $|\xi\rangle\!\rangle \in \mathcal{F}(\mathcal{E},\mathcal{R})$. Therefore

$$\mathcal{F}(\mathcal{E},\mathcal{R}_{\mathrm{sc}}) = \overline{|\mathcal{R}_{\mathrm{sc}}\rangle\!\rangle} \subseteq \mathcal{F}(\mathcal{E},\mathcal{R}).$$ □

Corollary 5.4.11. *Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module and suppose that $\mathcal{R} \subseteq \mathcal{E}$ is a complete, relatively continuous subspace of $\mathcal{E}$. Then we have*

$$\overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}) = \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}_{\mathrm{sc}}) \subseteq \mathcal{R} \subseteq \mathcal{R}_{\mathrm{sc}}.$$

In particular, if $\mathcal{R}_{\mathrm{sc}}$ is e-complete, then so is $\mathcal{R}$. And if $\mathcal{R}$ is e-complete, then

$$\overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}_{\mathrm{sc}}) = \mathcal{R}.$$

Proof. By Proposition 5.4.9, we have $L^1_0(\mathcal{G}) * \mathcal{R}_{\mathrm{sc}} \subseteq \mathcal{R}$. Using the fact that $L^1_0(\mathcal{G})$ is nondegenerate (Proposition 4.3.16) we get that

$$\overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}_{\mathrm{sc}}) = \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) \cdot L^1_0(\mathcal{G}) * \mathcal{R}_{\mathrm{sc}}) \subseteq \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}) \subseteq \mathcal{R}.$$

And since $\mathcal{R} \subseteq \mathcal{R}_{\mathrm{sc}}$, we also have $\overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}) \subseteq \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}_{\mathrm{sc}})$. □

Remark 5.4.12. Let $\mathcal{E}$ be a Hilbert $B,\mathcal{G}$-module and suppose that $\mathcal{R}$ is an e-complete, relatively continuous subspace of $\mathcal{E}$. Corollary 5.4.11 that $\overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}_{\mathrm{sc}})$ is again complete. Moreover, let $\mathcal{F} := \mathcal{F}(\mathcal{E},\mathcal{R})$ and suppose that $\mathcal{R}'$ is another complete, relatively continuous subspace of $\mathcal{E}$ such that $\mathcal{F}(\mathcal{E},\mathcal{R}') = \mathcal{F}$. By Corollary 5.4.6, the s-completions coincide, that is, $\mathcal{R}'_{\mathrm{sc}} = \mathcal{R}_{\mathrm{sc}}$. It follows from Proposition 5.4.9 that

$$\mathcal{R} = \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}_{\mathrm{sc}}) = \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}'_{\mathrm{sc}}) \subseteq \mathcal{R}'.$$

Thus $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{R}_{\mathrm{sc}}$. This shows that $\mathcal{R}$ is the smallest complete, relatively continuous subspace of $\mathcal{E}$ satisfying $\mathcal{F}(\mathcal{E},\mathcal{R}) = \mathcal{F}$. And as we already know, $\mathcal{R}_{\mathrm{sc}}$ is the biggest one with this property. Note also that

$$\mathcal{R} = \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}) \subseteq \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}') \subseteq \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}_{\mathrm{sc}}) = \mathcal{R}.$$

Thus $\mathcal{R} = \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}')$. In particular, $\overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}')$ is again complete.

In general, if $\mathcal{R}$ is complete but not e-complete, then this is not clear to me. So, I will leave the following question:

Question 5.4.13. Suppose that $\mathcal{R} \subseteq \mathcal{E}$ is a complete, relatively continuous subspace (not necessarily essential) and define $\tilde{\mathcal{R}} := \overline{\mathrm{span}}^{\mathrm{si}}(L^1_0(\mathcal{G}) * \mathcal{R}) \subseteq \mathcal{R}$. Is $\tilde{\mathcal{R}}$ complete?

Note that, by definition, $\tilde{\mathcal{R}}$ is si-closed and $L^1_0(\mathcal{G})$-invariant, but it is not clear, in general, whether it is also B-invariant. Of course, the problem here is that the left $L^1_0(\mathcal{G})$-action and the right B-action do not commute in general.

5.5 Functoriality and naturality

Throughout this section we fix a locally compact quantum group $\mathcal{G}$ and C^*-algebra B with a continuous coaction of $\mathcal{G}$, that is, a $\mathcal{G}$-C^*-algebra.

Definition 5.5.1. Let $(\mathcal{E}_1,\mathcal{R}_1)$ and $(\mathcal{E}_2,\mathcal{R}_2)$ be continuously square-integrable Hilbert $B,\mathcal{G}$-modules. An operator $T \in \mathcal{L}^{\mathcal{G}}(\mathcal{E}_1,\mathcal{E}_2)$ is called $\mathcal{R}$-*continuous* if $T(\mathcal{R}_1) \subseteq \mathcal{R}_2$ and $T^*(\mathcal{R}_2) \subseteq \mathcal{R}_1$.

Given a locally compact quantum group $\mathcal{G}$ and a $\mathcal{G}$-C^*-algebra B, the continuously square-integrable Hilbert $B,\mathcal{G}$-modules form a category with $\mathcal{R}$-continuous adjointable operators as morphisms. The s-continuously square-integrable Hilbert $B,\mathcal{G}$-modules form a full subcategory. By Proposition 5.3.9, these categories are identical if $\mathcal{G}$ is co-amenable.

In what follows, we analyze the functoriality of our constructions.

Proposition 5.5.2. *Let $\mathcal{G}$ be a locally compact quantum group and let B be a $\mathcal{G}$-C^*-algebra. The construction $(\mathcal{E},\mathcal{R}) \mapsto \mathcal{F}(\mathcal{E},\mathcal{R})$ is a functor from the category of continuously square-integrable Hilbert $B,\mathcal{G}$-modules to the category of Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-modules.*

Proof. Given an $\mathcal{R}$-continuous $\mathcal{G}$-equivariant operator $T : (\mathcal{E}_1,\mathcal{R}_1) \to (\mathcal{E}_2,\mathcal{R}_2)$, the associated adjointable operator $\tilde{T} : \mathcal{F}(\mathcal{E}_1,\mathcal{R}_1) \to \mathcal{F}(\mathcal{E}_2,\mathcal{R}_2)$ is given by $\tilde{T}(x) = T \circ x$ for all $x \in \mathcal{F}(\mathcal{E}_1,\mathcal{R}_1) \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E}_1)$. Here one uses that $|T(\xi)\rangle\rangle = T \circ |\xi\rangle\rangle$ for all $\xi \in \mathcal{E}_{\mathrm{si}}$ and the fact that $\mathcal{F}(\mathcal{E}_k,\mathcal{R}_k)$ is the closure of $|\mathcal{R}_k\rangle\rangle$, for $k = 1,2$. Since $T(\mathcal{R}_1) \subseteq \mathcal{R}_2$, this ensures that $\tilde{T}$ is a map $\mathcal{F}(\mathcal{E}_1,\mathcal{R}_1) \to \mathcal{F}(\mathcal{E}_2,\mathcal{R}_2)$. In the same way, since $T^*(\mathcal{R}_2) \subseteq \mathcal{R}_1$, the operator $\tilde{T}$ is, in fact, adjointable and its adjoint is given by $\tilde{T}^*(y) = T^* \circ y$ for all $y \in \mathcal{F}(\mathcal{E}_2,\mathcal{R}_2)$. ☐

Given an abstract Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-module $\mathcal{F}$, we can, by Theorem 5.1.2, identify it with the concrete Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-module $T(\mathcal{F}) \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E}_{\mathcal{F}})$, where $\mathcal{E}_{\mathcal{F}} := \mathcal{F} \otimes_A (B \otimes H)$, $A := B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$, and $T : \mathcal{F} \to \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E}_{\mathcal{F}})$ is the canonical representation (5.3). Recall that $T(x)f = x \otimes_A f$ for all $x \in \mathcal{F}$ and $f \in B \otimes H$. In this way we get an s-complete, relatively continuous subspace $\mathcal{R}_{\mathcal{F}} \subseteq \mathcal{E}_{\mathcal{F}}$ as in Proposition 5.3.12 by

$$\mathcal{R}_{\mathcal{F}} := \{\xi \in \mathcal{E}_{\mathcal{F}} : \xi \text{ is square-integrable and } |\xi\rangle\rangle \in T(\mathcal{F})\}.$$

Since $\mathcal{F}$ is essential, $\mathcal{R}_{\mathcal{F}}$ is dense $\mathcal{E}_{\mathcal{F}}$. In fact, note that by Theorem 5.4.4 and Proposition 5.3.12, $\mathcal{R}_{\mathcal{F}}$ is the s-completion of the linear span of $T(\mathcal{F})\big(B \odot \hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})\big)$ and this linear span equals $\mathcal{F} \odot_A \big(B \odot \hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})\big)$. Thus the pair $(\mathcal{E}_{\mathcal{F}},\mathcal{R}_{\mathcal{F}})$ is an s-continuously square-integrable Hilbert $B,\mathcal{G}$-module.

Lemma 5.5.3. *With notation as above we have $\mathcal{F}(\mathcal{E}_{\mathcal{F}},\mathcal{R}_{\mathcal{F}}) = T(\mathcal{F})$.*

Proof. Since $T(x) \in \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E})$ we have $x \otimes_A \zeta = T(x)\zeta \in \mathcal{E}_{\mathrm{si}}$ for all $x \in \mathcal{F}$ and $\zeta \in (B \otimes H)_{\mathrm{si}}$ and $|x \otimes_A \zeta\rangle\rangle = |T(x)\zeta\rangle\rangle = T(x) \circ |\zeta\rangle\rangle$. It follows that

$$\begin{aligned}\mathcal{F}(\mathcal{E}_{\mathcal{F}},\mathcal{R}_{\mathcal{F}}) &= \overline{\operatorname{span}}(|\mathcal{R}\rangle\rangle \circ A)\\ &= \overline{\operatorname{span}}(|\mathcal{F} \odot_A \mathcal{R}_0\rangle\rangle \circ A)\\ &= \overline{\operatorname{span}}(T(\mathcal{F}) \circ |\mathcal{R}_0\rangle\rangle \circ A)\\ &= \overline{\operatorname{span}}(T(\mathcal{F}) \circ A) = T(\mathcal{F}).\end{aligned}$$ ☐

Proposition 5.5.4. *The construction $\mathcal{F} \mapsto (\mathcal{E}_{\mathcal{F}},\mathcal{R}_{\mathcal{F}})$ is a functor from the category of Hilbert A-modules to the category of s-continuously square-integrable Hilbert $B,\mathcal{G}$-modules.*

Proof. As already observed in Section 5.1, the map $\mathcal{F} \mapsto \mathcal{E}_{\mathcal{F}}$ is functorial. To an adjointable operator $S : \mathcal{F}_1 \to \mathcal{F}_2$ we associate the $\mathcal{G}$-equivariant adjointable operator $S \otimes_A \mathrm{id} : \mathcal{E}_1 \to \mathcal{E}_2$, where $\mathcal{E}_k := \mathcal{F}_k \otimes_A (B \otimes H)$, $k = 1,2$. It remains to show that $S \otimes_A \mathrm{id}$ is $\mathcal{R}$-continuous, that is, $(S \otimes_A \mathrm{id})(\mathcal{R}_1) \subseteq \mathcal{R}_2$ and $(S \otimes_A \mathrm{id})^*(\mathcal{R}_2) \subseteq \mathcal{R}_1$, where $\mathcal{R}_k := \mathcal{R}_{\mathcal{F}_k}$, $k = 1,2$. Since $(S \otimes_A \mathrm{id})^* = S^* \otimes_A \mathrm{id}$, it is enough to show that $(S \otimes_A \mathrm{id})(\mathcal{R}_1) \subseteq \mathcal{R}_2$. Let $T_k : \mathcal{F}_k \to \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E}_k)$ be the canonical representation of $\mathcal{F}_k$, that is, $T_k(x)f = x \otimes_A f$ for all $x \in \mathcal{E}_k$ and $f \in B \otimes H$. Note that $(S \otimes_A \mathrm{id}) \circ T_1(x) = T_2\big(S(x)\big)$ for all $x \in \mathcal{F}_1$. Thus $(S \otimes_A \mathrm{id}) \circ T_1(\mathcal{F}_1) \subseteq T_2(\mathcal{F}_2)$. Combining this with the relation $|(S \otimes_A \mathrm{id})\xi\rangle\rangle = (S \otimes_A \mathrm{id}) \circ |\xi\rangle\rangle$, for every square-integrable element $\xi \in \mathcal{E}_1$ (see Proposition 4.1.10(iii)), the desired result follows. ☐

Corollary 5.5.5. *Let $\mathcal{G}$ be a locally compact quantum group and let B be a $\mathcal{G}$-C^*-algebra. Isomorphism classes of Hilbert modules over $A := B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ correspond bijectively to isomorphism classes of s-continuously square-integrable Hilbert $B, \mathcal{G}$-modules via the functors*

$$(\mathcal{E}, \mathcal{R}) \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R}) \quad \textit{and} \quad \mathcal{F} \mapsto (\mathcal{E}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}}), \tag{5.14}$$

where $\mathcal{E}_{\mathcal{F}} := \mathcal{F} \otimes_A (B \otimes H)$ and $\mathcal{R}_{\mathcal{F}}$ is the s-completion of $\mathcal{F} \odot_A \big(B \odot \hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})\big)$.

Proof. The proof combines Theorems 5.1.2 and 5.4.4. The maps in (5.14) are considered between isomorphism classes and are well-defined by Propositions 5.5.2 and 5.5.4. To prove that they are inverse to each other, let $(\mathcal{E}, \mathcal{R})$ be an s-continuously square-integrable Hilbert $B, \mathcal{G}$-module and define $\mathcal{F} := \mathcal{F}(\mathcal{E}, \mathcal{R})$. We have to prove that $(\mathcal{E}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}}) \cong (\mathcal{E}, \mathcal{R})$. Define $U : \mathcal{E}_{\mathcal{F}} \to \mathcal{E}$ by $U(x \otimes_A \zeta) := x(\zeta)$ for all $x \in \mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E})$ and $\zeta \in B \otimes H$. The unitary U appears in Theorem 5.4.4 and is $\mathcal{G}$-equivariant. It remains to show that $U(\mathcal{R}_{\mathcal{F}}) = \mathcal{R}$. Since $\mathcal{R}_{\mathcal{F}}, \mathcal{R} \subseteq \mathcal{E}$ are relatively continuous and s-complete, it is enough to show that $\mathcal{F}\big(\mathcal{E}, U(\mathcal{R}_{\mathcal{F}})\big) = \mathcal{F}(\mathcal{E}, \mathcal{R}) = \mathcal{F}$. Note that $U(\mathcal{R}_{\mathcal{F}})$ is the s-completion of $U(\mathcal{F} \odot_A \mathcal{R}_0) = \operatorname{span} \mathcal{F}(\mathcal{R}_0)$, where $\mathcal{R}_0 := B \odot \hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})$. Since $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E})$ and $\overline{|\mathcal{R}_0\rangle\!\rangle} = A$, we have $\mathcal{F}(\mathcal{E}, \mathcal{R}_{\mathcal{F}}) = \overline{\operatorname{span}}(\mathcal{F} \circ A) = \mathcal{F}$. Therefore $(\mathcal{E}, \mathcal{R}) \cong (\mathcal{E}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$. Now assume that $\mathcal{F}$ is a Hilbert A-module and define $(\mathcal{E}, \mathcal{R}) := (\mathcal{E}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$. We have to show that $\mathcal{F} \cong \mathcal{F}(\mathcal{E}, \mathcal{R})$. Let $T : \mathcal{F} \to \mathcal{L}^{\mathcal{G}}(B \otimes H, \mathcal{E}_{\mathcal{F}})$ be the canonical representation (5.3) of $\mathcal{F}$, that is, $T(x)\zeta := x \otimes_A \zeta$. We know, by Theorem 5.1.2, that $\mathcal{F} \cong T(\mathcal{F})$ (as Hilbert A-modules). Finally, by Lemma 5.5.3, $T(\mathcal{F}) = \mathcal{F}(\mathcal{E}, \mathcal{R})$. □

Finally, we prove that our constructions are natural and yield an equivalence between the respective categories.

Theorem 5.5.6. *Let $\mathcal{G}$ be a locally compact quantum group, and let B be a $\mathcal{G}$-C^*-algebra. Let $(\mathcal{E}, \mathcal{R})$ be an s-continuously square-integrable Hilbert $B, \mathcal{G}$-module, and let $\mathcal{F} := \mathcal{F}(\mathcal{E}, \mathcal{R})$. Then there is a canonical, injective, strictly continuous $*$-homomorphism $\phi : \mathcal{L}(\mathcal{F}) \to \mathcal{L}^{\mathcal{G}}(\mathcal{E})$, whose range is the space of $\mathcal{R}$-continuous operators. It maps $\mathcal{K}(\mathcal{F})$ isometrically onto* $\operatorname{Fix}(\mathcal{E}, \mathcal{R})$.

The categories of s-continuously square-integrable Hilbert $B, \mathcal{G}$-modules and Hilbert modules over $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ are equivalent via the functors $(\mathcal{E}, \mathcal{R}) \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R})$ and $\mathcal{F} \mapsto (\mathcal{E}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$.

The generalized fixed point algebra $\operatorname{Fix}(\mathcal{E}, \mathcal{R})$ *is the closed linear span of the operators $(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)(|\xi\rangle\langle\eta|) \in \mathcal{L}^{\mathcal{G}}(\mathcal{E})$, $\xi, \eta \in \mathcal{R}$, and it is Morita equivalent to the ideal $\mathcal{I}(\mathcal{E}, \mathcal{R}) = \overline{\operatorname{span}}\{\langle\!\langle \xi \,|\, \eta \rangle\!\rangle : \xi, \eta \in \mathcal{R}\}$ of $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$.*

Proof. Since $\mathcal{R}$ is s-complete, we have $\mathcal{R} = \mathcal{R}_{\mathcal{F}}$ and $\mathcal{F} = \overline{|\mathcal{R}\rangle\!\rangle}$ (by Corollary 5.4.6 and Equation (5.11)). These facts together with Proposition 4.1.10(iii) imply that the set $\mathcal{M}$ in Theorem 5.1.3 equals the set of $\mathcal{R}$-continuous operators. Therefore, the same ϕ of Theorem 5.1.3 yields the first statement. Combining this with Theorems 5.4.4 and 5.1.2 (see also Remark 5.1.4), we get the second statement. The last statement follows from Equation (5.12) and Proposition 5.2.4. □

Corollary 5.5.7. *Let $\mathcal{G}$ be a compact quantum group and suppose that B is a $\mathcal{G}$-C^*-algebra. Then the functor*

$$\mathcal{F} \mapsto \mathcal{F} \underset{B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}}{\otimes} \big(B \otimes L^2(\mathcal{G})\big)$$

is an equivalence between the categories of Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-modules and Hilbert $B, \mathcal{G}$-modules. In other words, any Hilbert $B, \mathcal{G}$-module appears in this way for a unique Hilbert module $\mathcal{F}$ over $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ and the map $\mathcal{L}(\mathcal{F}) \to \mathcal{L}^{\mathcal{G}}(\mathcal{E})$ is an isomorphism.

Given a Hilbert $B, \mathcal{G}$-module $\mathcal{E}$, the generalized fixed point algebra associated to $\mathcal{E}$ is the usual fixed point algebra:

$$\mathrm{Fix}(\mathcal{E}) = \{x \in \mathcal{K}(\mathcal{E}) : \gamma(x) = x \otimes 1\} \cong \mathcal{K}(\mathcal{F}_{\mathcal{E}}),$$

where $\mathcal{F}_{\mathcal{E}} = \mathcal{F}(\mathcal{E}, \mathcal{E})$ is the Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-module associated to $\mathcal{E}$.

Proof. If $\mathcal{G}$ is compact, then any Hilbert $B, \mathcal{G}$-module is continuously square-integrable and $\mathcal{R} = \mathcal{E}$ is the unique dense, complete, relatively continuous subspace. Therefore there is no difference between the categories of continuously (and hence also s-continuously) square-integrable Hilbert $B, \mathcal{G}$-modules and arbitrary Hilbert $B, \mathcal{G}$-modules. The assertions now follow from Theorem 5.5.6. □

In particular, for compact quantum groups every Hilbert $B, \mathcal{G}$-module is "proper" in the following sense:

Definition 5.5.8. We say that a Hilbert $B, \mathcal{G}$-module $\mathcal{E}$ is *$\mathcal{R}$-proper* if there is a unique dense, s-complete, relatively continuous subspace of $\mathcal{E}$.

By Theorem 5.4.4, $\mathcal{E}$ is $\mathcal{R}$-proper if and only if there is a unique concrete, essential Hilbert $B \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-module $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{G}}(B \otimes L^2(\mathcal{G}), \mathcal{E})$. Note that for an $\mathcal{R}$-proper Hilbert $B, \mathcal{G}$-module $\mathcal{E}$, the multiplier algebra of the generalized fixed point algebra is isomorphic to the big fixed point algebra $\mathcal{M}_1\big(\mathcal{K}(\mathcal{E})\big) \cong \mathcal{L}^{\mathcal{G}}(\mathcal{E})$. This follows from Remark 5.1.4(1).

Recall that in the group case $\mathcal{G} = \mathcal{C}_0(G)$, a G-C^*-algebra A is called *spectrally proper* if the canonical induced action of G on the primitive ideal space $\mathrm{Prim}(A)$ is proper (see [48, Definition 9.2]). This class includes all the proper G-C^*-algebras in the sense of Kasparov [35]. By Theorem 9.1 in [48], every Hilbert module over a spectrally proper G-C^*-algebra is $\mathcal{R}$-proper. In particular, a commutative G-C^*-algebra $\mathcal{C}_0(X)$ (where X is a locally compact G-space) is $\mathcal{R}$-proper if X is a proper G-space. Conversely, if $\mathcal{C}_0(X)$ is $\mathcal{R}$-proper, then it is, in particular, integrable and therefore, by Rieffel's Theorem 4.7 in [66], X is a proper G-space.

In the general quantum setting, unless $\mathcal{G}$ is compact, it is not easy to find non-trivial examples of $\mathcal{R}$-proper Hilbert modules. In this direction, we have the following result:

Proposition 5.5.9. *Let $\mathcal{G}$ be a locally compact quantum group and let $\mathcal{G}$ coact on itself by the comultiplication. Then $\mathcal{G}$ is an $\mathcal{R}$-proper $\mathcal{G}$-C^*-algebra if and only if $\mathcal{G}$ is semi-regular, that is, $\mathcal{K}\big(L^2(\mathcal{G})\big)$ is contained in $C := \overline{\mathrm{span}}(\mathcal{G}\widehat{\mathcal{G}}^{\mathrm{c}}) \cong \mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$. In this case, $\mathcal{R} = \mathcal{G}_{\mathrm{si}}$ is the unique dense, s-complete, relatively continuous subspace of $\mathcal{G}$. The Hilbert $\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$-module $\mathcal{F}(\mathcal{G}, \mathcal{R})$ is isomorphic to the dual $L^2(\mathcal{G})^*$ of $L^2(\mathcal{G})$ considered as a Hilbert C-module in the canonical way. In particular, $\mathrm{Fix}(\mathcal{G}, \mathcal{R}) \cong \mathbb{C}$ and $\mathcal{I}(\mathcal{G}, \mathcal{R}) \cong \mathcal{K}\big(L^2(\mathcal{G})\big)$. The quantum group $\mathcal{G}$ is regular if and only if $\mathcal{R}$ is saturated.*

Proof. By Proposition 3.2.12 we have $\mathcal{R} = \mathcal{G}_{\mathrm{si}} = \mathcal{N}_\varphi^*$ and

$$|\xi\rangle\!\rangle = \big(1_{\mathcal{G}} \otimes \Lambda(\xi^*)^*\big)W, \quad \langle\!\langle\xi| = W^*\big(1_{\mathcal{G}} \otimes \Lambda(\xi^*)\big), \quad \xi \in \mathcal{R}.$$

Since $\mathcal{G}$ is semi-regular, Proposition 5.2.6 says any subset of $\mathcal{G}_{\mathrm{si}}$ is relatively continuous. Now note that if $\mathcal{R}_0 \subseteq \mathcal{G}_{\mathrm{si}}$ is a complete subspace, then

$$\mathcal{F}(\mathcal{G}, \mathcal{R}_0) = \overline{|\mathcal{R}_0\rangle\!\rangle} = \overline{\big(1_{\mathcal{G}} \otimes \Lambda(\mathcal{R}_0^*)^*\big)W} = (1_{\mathcal{G}} \otimes H_0^*)W,$$

where $H_0 := \overline{\Lambda(\mathcal{R}_0^*)}$ (which is a closed subspace of $H = L^2(\mathcal{G})$). Equivalently,

$$\mathcal{F}(\mathcal{G}, \mathcal{R}_0)^* = \overline{\langle\!\langle\mathcal{R}_0|} = W^*(1_{\mathcal{G}} \otimes H_0).$$

In particular, $\mathcal{F}(\mathcal{G}, \mathcal{R})^* = W^*(1_{\mathcal{G}} \otimes H)$. Define the following linear map

$$T : \Lambda(\mathcal{N}_\varphi) \subseteq L^2(\mathcal{G}) \to \langle\!\langle\mathcal{R}| \subseteq \mathcal{F}(\mathcal{G}, \mathcal{R})^*, \quad T\big(\Lambda(\xi)\big) := W^*\big(1_{\mathcal{G}} \otimes \Lambda(\xi)\big).$$

Then (identifying $\mathbb{C} \cong \mathbb{C}1_{\mathcal{G}} \subseteq \mathcal{M}(\mathcal{G})$)

$$\begin{aligned}\big\langle T(\Lambda(\xi))|T(\Lambda(\eta))\big\rangle &= \big(W^*(1_{\mathcal{G}} \otimes \Lambda(\xi))\big)^* W^*\big(1_{\mathcal{G}} \otimes \Lambda(\eta)\big)\\ &= \big(1_{\mathcal{G}} \otimes \Lambda(\xi)^*\big)WW^*\big(1_{\mathcal{G}} \otimes \Lambda(\eta)\big)\\ &= 1_{\mathcal{G}}\varphi(\xi^*\eta) = \langle\Lambda(\xi)|\Lambda(\eta)\rangle_{L^2(\mathcal{G})}.\end{aligned}$$

It follows that T extends to an isomorphism $L^2(\mathcal{G}) \cong \mathcal{F}(\mathcal{G}, \mathcal{R})^*$ (as Hilbert spaces). Thus $\mathcal{F}(\mathcal{G}, \mathcal{R}) \cong L^2(\mathcal{G})^*$ as Hilbert modules over $\mathcal{K}\big(L^2(\mathcal{G})\big)$ and hence also as Hilbert modules over C.

Finally, suppose that $\mathcal{R}_0 \subseteq \mathcal{G}_{\mathrm{si}}$ is dense and s-complete. Then

$$\mathcal{I}(\mathcal{G}, \mathcal{R}_0) = \overline{\operatorname{span}}\big\{W^*(1_{\mathcal{G}} \otimes |\xi\rangle\langle\eta|)W : \xi, \eta \in \Lambda(\mathcal{R}_0^*)\big\} = W^*\big(1_{\mathcal{G}} \otimes \mathcal{K}(H_0)\big)W.$$

The subset $\mathcal{I}(\mathcal{G}, \mathcal{R}_0) \subseteq W^*\big(1_{\mathcal{G}} \otimes \mathcal{K}(H)\big)W \subseteq \mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ is an ideal of $\mathcal{G} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}}$ and hence also of $W^*\big(1_{\mathcal{G}} \otimes \mathcal{K}(H)\big)W$. It follows that $\mathcal{K}(H_0)$ is an ideal of $\mathcal{K}(H)$. Since $\mathcal{K}(H)$ is simple, we get $\mathcal{K}(H_0) = \mathcal{K}(H)$ (H_0 is not zero because $\mathcal{R}_0$ is dense in $\mathcal{G}$). Cohen's Factorization Theorem yields $H = \mathcal{K}(H)H = \mathcal{K}(H_0)H = H_0$. Hence

$$\mathcal{F}(\mathcal{G}, \mathcal{R}_0) = (1_{\mathcal{G}} \otimes H_0^*)W = (1_{\mathcal{G}} \otimes H^*)W = \mathcal{F}(\mathcal{G}, \mathcal{R}).$$

Therefore, $\mathcal{R}_0 = \mathcal{R}$ because both $\mathcal{R}_0$ and $\mathcal{R}$ are s-complete. The last assertion was already proved in Proposition 5.2.6. □

Remark 5.5.10. Examples of non-semi-regular quantum groups have been constructed in [7]. It has been observed there that for such examples the coaction of $\mathcal{G}$ on itself via the comultiplication is in some sense not "proper". We can now give this statement a precise meaning if we agree that "proper" means $\mathcal{R}$-proper.

Moreover, if we agree that a proper (that is, $\mathcal{R}$-proper) coaction is "free" if the corresponding dense, s-complete, relatively continuous subspace is, in addition, saturated, then we can also say that the comultiplication of a locally compact quantum group is proper and free if and only if it is regular.

Chapter 6

Coactions of groups

In this final chapter we study group coactions, that is, coactions of the locally compact quantum group $\mathcal{G} = C^*_\mathrm{r}(G)$, where G is a locally compact group. We are going to see that in this case, continuously square-integrable coactions of $\mathcal{G}$ are closely related to Fell bundles over G.

6.1 The Haar weight of $C^*_\mathrm{r}(G)$

In this section we collect some facts about the Haar weight of the locally compact quantum group $\mathcal{G} = C^*_\mathrm{r}(G)$ to be used later in the study of continuously square-integrable coactions of groups.

Let G be a locally compact group and consider the locally compact quantum group $\mathcal{G} = C^*_\mathrm{r}(G) \subseteq \mathcal{L}\big(L^2(G)\big)$, that is, the dual of $\mathcal{C}_0(G)$. The comultiplication Δ in this case is characterized by $\Delta(\lambda_s) = \lambda_s \otimes \lambda_s$, where λ is the left regular representation of G on $L^2(G)$. The Haar weight φ on $\mathcal{G}$, which is left and right invariant, is (up to a positive scalar) the restriction of the *Plancherel weight* $\tilde{\varphi}$ on the von Neumann algebra $\mathcal{L}(G) := C^*_\mathrm{r}(G)''$. In what follows we recall the definition and some basic facts about the weight $\tilde{\varphi}$ (see [58, Section 7.2] or [69, Section VII.3] for a detailed construction).

A function $\xi \in L^2(G)$ is called *left bounded* if the map $\mathcal{C}_c(G) \ni f \mapsto \xi * f \in L^2(G)$ extends to a bounded operator on $L^2(G)$. In this case, we denote this operator by $\lambda(\xi)$. Note that $\lambda(\xi)$ belongs to $\mathcal{L}(G)$ for every left bounded function ξ. The Plancherel weight $\tilde{\varphi} : \mathcal{L}(G)^+ \to [0, \infty]$ is defined by the formula

$$\tilde{\varphi}(x) = \begin{cases} \|\xi\|_2^2 & \text{if } x^{\frac{1}{2}} = \lambda(\xi) \text{ for some left bounded function } \xi \in L^2(G), \\ \infty & \text{otherwise.} \end{cases}$$

From the definition above it follows that (see notation in Section 2.4)

$$\mathcal{N}_{\tilde{\varphi}} = \big\{\lambda(\xi) : \xi \in L^2(G) \text{ is left bounded}\big\}$$

and (by polarization) $\tilde{\varphi}\big(\lambda(\xi)^*\lambda(\eta)\big) = \langle \xi \,|\, \eta \rangle$ whenever $\xi, \eta \in L^2(G)$ are left bounded.

There is a canonical GNS-construction $(L^2(G), \iota, \tilde{\Lambda})$ for $\tilde{\varphi}$, where $\tilde{\Lambda}\big(\lambda(\xi)\big) = \xi$ for every left bounded function $\xi \in L^2(G)$ and ι denotes the inclusion $C^*_\mathrm{r}(G) \hookrightarrow \mathcal{L}\big(L^2(G)\big)$. We always use this GNS-construction. Since φ is, by definition, the restriction of $\tilde{\varphi}$ to $C^*_{\mathrm{r}}(G)^+$, we have

$$\mathcal{N}_\varphi = \big\{\lambda(\xi) : \xi \in L^2(G) \text{ is left bounded and } \lambda(\xi) \in C^*_{\mathrm{r}}(G)\big\}.$$

and a canonical GNS-construction $(L^2(G), \iota, \Lambda)$, where Λ is the restriction of $\tilde{\Lambda}$ to $\mathcal{N}_\varphi$. We also get that

$$\bar{\mathcal{N}_\varphi} = \big\{\lambda(\xi) : \xi \in L^2(G) \text{ is left bounded and } \lambda(\xi) \in \mathcal{M}\big(C^*_{\mathrm{r}}(G)\big)\big\}. \tag{6.1}$$

It is easy to see that

$$(\xi^* * \eta)(t) = \langle \xi | V_t \eta \rangle \quad \text{for all } \xi, \eta \in L^2(G) \text{ and } t \in G, \tag{6.2}$$

where $V_t(\eta)(s) := \eta(st)$. In particular, the function $\xi^* * \eta$ is continuous and we have $(\xi^* * \eta)(e) = \langle \xi | \eta \rangle$. Thus, if $\xi, \eta \in L^2(G)$ are left bounded, the operator $\lambda(\xi^* * \eta) = \lambda(\xi)^* \lambda(\eta)$ belongs to $\mathcal{M}_{\tilde{\varphi}}$ and

$$\tilde{\varphi}\big(\lambda(\xi^* * \eta)\big) = \langle \xi \,|\, \eta \rangle = (\xi^* * \eta)(e).$$

Thus

$$\mathcal{M}_{\tilde{\varphi}} = \lambda\big(\mathcal{C}_e(G)\big),$$

where $\mathcal{C}_e(G) := \operatorname{span}\{\xi^* * \eta : \xi, \eta \in L^2(G) \text{ left bounded}\} \subseteq \mathcal{C}(G)$, and $\tilde{\varphi}$ is given on functions of $\mathcal{C}_e(G)$ by evaluation at $e \in G$. Since φ is the restriction of $\tilde{\varphi}$ to $C^*_{\mathrm{r}}(G)$, we have $\bar{\mathcal{M}_\varphi} \subseteq \mathcal{M}_{\tilde{\varphi}}$ and the same formula holds for φ. Note that Equation (6.2) implies $\delta_G(t)^{\frac{1}{2}}(\xi^* * \eta)(t) = \langle \xi \mid \rho_t \eta \rangle$, where $\rho_t = \delta_G(t)^{\frac{1}{2}} V_t$ is the right regular representation. It follows that $\delta_G^{\frac{1}{2}} \cdot (\xi^* * \eta) \in A(G)$, the Fourier algebra. In particular, $\delta_G^{\frac{1}{2}} \cdot \mathcal{C}_e(G) \subseteq A(G)$. This inclusion has dense image in $A(G)$ because $\mathcal{C}_e(G)$ contains all the functions in $\mathcal{C}_c(G)^2 = \operatorname{span}\big(\mathcal{C}_c(G) * \mathcal{C}_c(G)\big)$ (which is dense in $A(G)$).

Remark 6.1.1. Although the formula $\tilde{\varphi}\big(\lambda(f)\big) = f(e)$ makes sense for all $f \in \mathcal{C}_c(G)$, it is not true in general that $\lambda\big(\mathcal{C}_c(G)\big) \subseteq \mathcal{M}_{\tilde{\varphi}}$, that is, it is not true that $\mathcal{C}_c(G) \subseteq \mathcal{C}_e(G)$ (of course, we always have $\mathcal{C}_c(G) * \mathcal{C}_c(G) \subseteq \mathcal{C}_e(G)$). In fact, suppose that G is compact. Then $L^2(G) \subseteq L^1(G)$, so that any function $\xi \in L^2(G)$ is left bounded. Since G is unimodular the map $\xi \mapsto \xi^*$ is an anti-unitary operator on $L^2(G)$, so that $L^2(G)^* = L^2(G)$. Hence $\mathcal{C}_e(G) = \operatorname{span}\big(L^2(G) * L^2(G)\big)$. As already noted, we always have $\mathcal{C}_e(G) \subseteq \mathcal{C}(G)$. If, in addition, $\mathcal{C}(G) = \mathcal{C}_c(G) \subseteq \mathcal{C}_e(G)$, then we get $\operatorname{span}\big(L^2(G) * L^2(G)\big) = \mathcal{C}(G)$. But this is true only if G is finite (see [29, 34.16, 34.40 and 37.4]). Therefore, we have just seen that for every compact infinite group G, $\lambda\big(\mathcal{C}_c(G)\big)$ is not contained in $\mathcal{M}_{\tilde{\varphi}}$. However, we can prove a partial result:

Proposition 6.1.2. *Let G be a locally compact group and $f \in \mathcal{C}_c(G)$. If $\lambda(f) \geq 0$ as an operator on $L^2(G)$, then there exists a left bounded function $\xi \in L^2(G)$ such that $\lambda(f)^{\frac{1}{2}} = \lambda(\xi)$ and $f = \xi^* * \xi$. In particular, $\lambda(f) \in \mathcal{M}^+_\varphi$ and*

$$\varphi\big(\lambda(f)\big) = \|\xi\|_2^2 = (\xi^* * \xi)(e) = f(e).$$

Proof. The proof is just a suitable modification of [58, 7.2.4]. Let $(f_u)_{u\in U}$ be the usual approximate unit for $L^1(G)$, where U is the set of all compact neighborhoods of e and each f_u is a positive-valued continuous function with $\int_G f_u(t)\,\mathrm{d}t = 1$ and $\operatorname{supp}(f_u) \subseteq u$. Since $f \in \mathcal{C}_c(G)$, the nets $(f_u * f)$ and $(f * f_u)$ converge uniformly to f. The same is true for $(f_u^* * f)$ and $(f * f_u^*)$. Define $\xi_u := \lambda(f)^{\frac{1}{2}}(f_u) \in L^2(G)$. We have

$$\begin{aligned}\|\xi_u - \xi_v\|_2^2 &= \langle f_u - f_v \,|\, \lambda(f)(f_u - f_v)\rangle \\ &= \big((f_u - f_v)^* * f * (f_u - f_v)\big)(e) \to 0.\end{aligned}$$

Thus $\{\xi_u\}_u$ is a Cauchy net in $L^2(G)$. Let $\xi \in L^2(G)$ be the limit of this net. For each $g \in \mathcal{C}_c(G)$ let $\rho(g)$ be the operator in $\mathcal{L}\big(L^2(G)\big)$ given by $\rho(g)(\eta) = \eta * g$. It is clear that $\rho(g)$ is in the commutant $C^*_{\mathrm{r}}(G)'$ of $C^*_{\mathrm{r}}(G)$. Hence

$$\xi * g = \lim_u \rho(g)\lambda(f)^{\frac{1}{2}} f_u = \lim_u \lambda(f)^{\frac{1}{2}}\rho(g) f_u = \lambda(f)^{\frac{1}{2}}(g).$$

This means that ξ is left bounded and $\lambda(\xi) = \lambda(f)^{\frac{1}{2}}$. Finally, note that $V_t \in C^*_{\mathrm{r}}(G)'$ for all $t \in G$ and therefore

$$\begin{aligned}(\xi^* * \xi)(t) &= \langle \xi \,|\, V_t(\xi)\rangle \\ &= \lim_u \langle \lambda(f)^{\frac{1}{2}} f_u \,|\, V_t \lambda(f)^{\frac{1}{2}} f_u\rangle \\ &= \lim_u \langle f * f_u \,|\, V_t f_u\rangle \\ &= \lim_u (f_u^* * f * f_u)(t) = f(t).\end{aligned}$$

□

Finally, let us we remark that the modular group $\{\sigma_x\}_{x\in\mathbb{R}}$ of $\tilde{\varphi}$ is given by

$$\sigma_x(a) = \nabla^{ix} a \nabla^{-ix}, \quad a \in \mathcal{L}(G),\ x \in \mathbb{R}, \tag{6.3}$$

where ∇ is the modular operator. It is given by $(\nabla\xi)(t) = \delta_G(t)\xi(t)$, $\xi \in L^2(G)$, $t \in G$. The domain of ∇ is

$$\mathcal{D}(\nabla) = \left\{\xi \in L^2(G) : \int_G |\xi(t)|^2 \delta_G(t)^2\,\mathrm{d}t < \infty\right\}.$$

Equation (6.3) yields $\sigma_x(\lambda_t) = \delta_G(t)^{\mathrm{i}x}\lambda_t$ for all $t \in G$ and $x \in \mathbb{R}$. This implies that λ_t is analytic with respect to σ and

$$\sigma_z(\lambda_t) = \delta_G(t)^{\mathrm{i}z}\lambda_t \quad \text{for all } z \in \mathbb{C},\ t \in G. \tag{6.4}$$

6.2 Non-Abelian Fourier analysis

First we define a special family of densely defined linear functionals on $\mathcal{L}(G)$.

Definition 6.2.1. For each $t \in G$, we define the functional

$$\tilde{\varphi}_t \colon \mathcal{M}_{\tilde{\varphi}} \to \mathbb{C}, \quad \tilde{\varphi}_t(x) := \tilde{\varphi}(\lambda_{t^{-1}} x), \quad x \in \mathcal{M}_{\tilde{\varphi}}.$$

We shall also denote by φ_t the restriction of $\tilde{\varphi}_t$ to $\bar{\mathcal{M}}_\varphi \subseteq \mathcal{M}_{\tilde{\varphi}}$. The following result shows that the functionals $\tilde{\varphi}_t$ are well-defined.

Lemma 6.2.2. *For each $t \in G$, we have $\mathcal{N}_{\tilde{\varphi}}\lambda_t = \mathcal{N}_{\tilde{\varphi}}$ and $\lambda_t\mathcal{M}_{\tilde{\varphi}} = \mathcal{M}_{\tilde{\varphi}}\lambda_t = \mathcal{M}_{\tilde{\varphi}}$.*[1]

Proof. As already observed, λ_t is analytic with respect to the modular group σ of $\tilde{\varphi}$ (see Equation (6.4)). In particular, $\lambda_t \in \mathcal{D}(\sigma_{\frac{\mathrm{i}}{2}})$. It follows that $\mathcal{N}_{\tilde{\varphi}}\lambda_t \subseteq \mathcal{N}_{\tilde{\varphi}}$ (see, for example, [73, Theorem 4.6.2]). Since $t \in G$ is arbitrary, we get $\mathcal{N}_{\tilde{\varphi}} = \mathcal{N}_{\tilde{\varphi}}\lambda_t^{-1}\lambda_t \subseteq \mathcal{N}_{\tilde{\varphi}}\lambda_t$. From $\mathcal{M}_{\tilde{\varphi}} = \operatorname{span}\mathcal{N}_{\tilde{\varphi}}^*\mathcal{N}_{\tilde{\varphi}}$ we obtain $\lambda_t\mathcal{M}_{\tilde{\varphi}} = \mathcal{M}_{\tilde{\varphi}}\lambda_t = \mathcal{M}_{\tilde{\varphi}}$. □

Note that, since φ is the restriction of $\tilde{\varphi}$, it also follows from the lemma above that for all $t \in G$:

$$\mathcal{N}_\varphi\lambda_t = \mathcal{N}_\varphi, \quad \lambda_t\mathcal{M}_\varphi = \mathcal{M}_\varphi\lambda_t = \mathcal{M}_\varphi \tag{6.5}$$
$$\bar{\mathcal{N}}_\varphi\lambda_t = \bar{\mathcal{N}}_\varphi, \quad \lambda_t\bar{\mathcal{M}}_\varphi = \bar{\mathcal{M}}_\varphi\lambda_t = \bar{\mathcal{M}}_\varphi. \tag{6.6}$$

Definition 6.2.3. Given $x \in \mathcal{M}_{\tilde{\varphi}}$, we define the (*inverse*) *Fourier transform* of x to be the function $\check{x} : G \to \mathbb{C}$,

$$\check{x}(t) := \tilde{\varphi}_t(x) = \tilde{\varphi}(\lambda_t^{-1}x), \quad t \in G.$$

If G is Abelian, then under the isomorphism $\mathcal{L}(G) \cong L^\infty(\widehat{G})$, the Plancherel weight on $\mathcal{L}(G)$ corresponds to the usual Haar integral on $L^\infty(\widehat{G})$. In this picture, $\mathcal{M}_{\tilde{\varphi}}$ corresponds to $L^\infty(\widehat{G}) \cap L^1(\widehat{G})$ and $\check{x}$ corresponds to the inverse Fourier transform of the corresponding function in $L^\infty(\widehat{G}) \cap L^1(\widehat{G})$.

Proposition 6.2.4. *Let G be a locally compact group. Then the following properties hold*:

(i) *We have $\check{x} \in \mathcal{C}_e(G)$ for all $x \in \mathcal{M}_{\tilde{\varphi}}$. In particular, $\check{x}$ is a continuous function.*

(ii) *We have $\lambda(f)\check{} = f$ for all $f \in \mathcal{C}_e(G)$.*

(iii) *If we equip $\mathcal{C}_e(G)$ with convolution and the involution $f^*(t) := \delta_G(t^{-1})\overline{f(t^{-1})}$, then $\mathcal{C}_e(G)$ is a $*$-algebra and the map*

$$\mathcal{M}_{\tilde{\varphi}} \ni x \mapsto \check{x} \in \mathcal{C}_e(G)$$

is an isomorphism of $*$-algebras. The inverse is given by the map $f \mapsto \lambda(f)$. In particular, we have

$$(xy)\check{} = \check{x} * \check{y}, \quad \textit{and} \quad (x^*)\check{} = \check{x}^* \quad \textit{for all } x, y \in \mathcal{M}_{\tilde{\varphi}}.$$

(iv) *Suppose that $x \in \mathcal{M}_{\tilde{\varphi}}$ and that the function $t \mapsto \check{x}(t)\lambda_t \in \mathcal{L}(G)$ is weakly integrable. Then*

$$\int_G^{\mathrm{w}} \check{x}(t)\lambda_t \,\mathrm{d}t = x,$$

where the superscript "w" *above stands for weak integral. In particular, we get*

$$\lambda(\check{x}) = \int_G^{\mathrm{w}} \check{x}(t)\lambda_t \,\mathrm{d}t.$$

[1] Note that the equality $\lambda_t\mathcal{N}_{\tilde{\varphi}} = \mathcal{N}_{\tilde{\varphi}}$ is automatically satisfied because $\mathcal{N}_{\tilde{\varphi}}$ is a left ideal of $\mathcal{L}(G)$.

Proof. We already know that $\mathcal{M}_{\tilde{\varphi}} = \lambda\big(\mathcal{C}_e(G)\big)$. Let $x = \lambda(f)$ with $f \in \mathcal{C}_e(G)$. Then

$$\lambda_t^{-1} x = \lambda_t^{-1}\lambda(f) = \lambda\big(\mathrm{ev}_t(f)\big),$$

where ev_t denotes the evaluation functional at $t \in G$. Hence

$$\check{x}(t) = \tilde{\varphi}\big(\lambda(\mathrm{ev}_t(f))\big) = \mathrm{ev}_t(f)(e) = f(t).$$

Thus $\check{x} = f$. This proves (i) and (ii). If $f, g, \xi, \eta \in L^2(G)$ are left bounded, then

$$(f^* * g) * (\xi^* * \eta) = (\lambda(g)^* f)^* * (\lambda(\xi)^* \eta).$$

Note that, given $x \in \mathcal{L}(G)$ and $\zeta \in L^2(G)$ left bounded, $x\zeta \in L^2(G)$ is left bounded and $\lambda(x\zeta) = x\lambda(\zeta)$. It follows that $(f^* * g) * (\xi^* * \eta) \in \mathcal{C}_e(G)$. This shows that $\mathcal{C}_e(G)$ is an algebra with convolution. Note also that $(f^* * g)^* = g^* * f \in \mathcal{C}_e(G)$, and therefore $\mathcal{C}_e(G)$ is a $*$-algebra. It is easy to see that the map $\mathcal{M}_{\tilde{\varphi}} \ni x \mapsto \check{x} \in \mathcal{C}_e(G)$ preserves the $*$-algebra structures. For example, to prove that $(xy)\check{} = \check{x} * \check{y}$, take $f, g \in \mathcal{C}_e(G)$ such that $x = \lambda(f)$ and $y = \lambda(g)$. Then $(xy)\check{} = \big(\lambda(f * g)\big)\check{} = f * g = \check{x}\check{y}$. Item (ii) and the fact that any $x \in \mathcal{M}_{\tilde{\varphi}}$ has the form $x = \lambda(f)$ show that the map $x \mapsto \check{x}$ has $f \mapsto \lambda(f)$ as its inverse. Finally, we prove (iv). Take $\xi, \eta \in \mathcal{C}_c(G)$. Then

$$\begin{aligned}
\left\langle \xi \,\middle|\, \left(\int_G^{\mathrm{w}} \check{x}(t)\lambda_t \,\mathrm{d}t\right) \eta \right\rangle &= \int_G \check{x}(t) \langle \xi \,|\, \lambda_t(\eta) \rangle \,\mathrm{d}t \\
&= \int_G \int_G \check{x}(t) \overline{\xi(s)} \eta(t^{-1}s) \,\mathrm{d}t \,\mathrm{d}s \\
&= \int_G \overline{\xi(s)} (\check{x} * \eta)(s) \,\mathrm{d}s \\
&= \langle \xi \,|\, \lambda(\check{x})\eta \rangle = \langle \xi \,|\, x\eta \rangle.
\end{aligned}$$

□

6.3 Fell bundles

For full details concerning Fell bundles, we refer to [23]. In this section, we point out some important facts to be used later.

Definition 6.3.1. Let G be a locally compact group. A *Fell bundle* over G is a continuous Banach bundle $\mathcal{B} = \{\mathcal{B}_t\}_{t \in G}$ over G equipped with a continuous multiplication

$$\mathcal{B} \times \mathcal{B} \to \mathcal{B}, \quad (b, c) \mapsto b \cdot c$$

and a continuous involution

$$\mathcal{B} \mapsto \mathcal{B}, \quad b \mapsto b^*,$$

satisfying

(i) $\mathcal{B}_t \cdot \mathcal{B}_s \subseteq \mathcal{B}_{ts}$ and $\mathcal{B}_t^* = \mathcal{B}_{t^{-1}}$ for all $t, s \in G$;

(ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $(a \cdot b)^* = b^* \cdot a^*$ and $(a^*)^* = a$ for all $a, b, c \in \mathcal{B}$;

(iii) $\|b \cdot c\| \leq \|b\|\|c\|$ and $\|b^*\| = \|b\|$ for all $b, c \in \mathcal{B}$;

(iv) $\|b^*b\|_{\mathcal{B}_e} = \|b\|^2_{\mathcal{B}_t}$ whenever $b \in \mathcal{B}_t$ (e denotes the unit of G); and

(v) $b^*b \geq 0$ (in $\mathcal{B}_e$) for all $b \in \mathcal{B}$.

Let $\mathcal{C}_c(\mathcal{B})$ be the space of compactly supported continuous sections of $\mathcal{B}$. For each $\xi, \eta \in \mathcal{C}_c(\mathcal{B})$, define

$$(\xi * \eta)(t) := \int_G \xi(s)\eta(s^{-1}x)\,\mathrm{d}s$$

and

$$\xi^*(t) := \delta_G(t)^{-1}\xi(x^{-1})^*,$$

where δ_G denotes the modular function of G. Then $\xi * \eta \in \mathcal{C}_c(\mathcal{B})$ and $\xi^* \in \mathcal{C}_c(\mathcal{B})$, so that $\mathcal{C}_c(\mathcal{B})$ becomes a $*$-algebra. By definition, $L^1(\mathcal{B})$ is the completion of $\mathcal{C}_c(\mathcal{B})$ with respect to the L^1-norm:

$$\|\xi\|_1 := \int_G \|\xi(t)\|\,\mathrm{d}t,$$

and the *cross-sectional C^*-algebra* $C^*(\mathcal{B})$ is the enveloping C^*-algebra of $L^1(\mathcal{B})$.

Let $L^2(\mathcal{B})$ be the Hilbert $\mathcal{B}_e$-module defined as the completion of $\mathcal{C}_c(\mathcal{B})$ with respect to the $\mathcal{B}_e$-inner product:

$$\langle \xi | \eta \rangle_{\mathcal{B}_e} := \int_\Gamma \xi(t)^*\eta(t)dx,$$

and the right $\mathcal{B}_e$-action:

$$(\xi \cdot b)(t) := \xi(t) \cdot b.$$

The *left regular representation* of $\mathcal{B}$ is the map

$$\lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to \mathcal{L}\big(L^2(\mathcal{B})\big)$$

given by $\lambda_{\mathcal{B}}(\xi)\eta = \xi * \eta$ for all $\xi, \eta \in \mathcal{C}_c(\mathcal{B})$. The *reduced cross-sectional C^*-algebra* of $\mathcal{B}$ is, by definition, $C^*_{\mathrm{r}}(\mathcal{B}) := \lambda_{\mathcal{B}}\big(C^*(\mathcal{B})\big)$.

A Fell bundle $\mathcal{B}$ over G is called *amenable* if $\lambda_{\mathcal{B}}$ is an isomorphism. For example, if G is amenable, then all the Fell bundles over G are amenable (see [21, Theorem 3.9]).

Example 6.3.2. (1) Let (B, G, α) be a C^*-dynamical system. Then the trivial Banach bundle $\mathcal{B} := B \times G$ with operations

$$(b, t) \cdot (c, s) := (b\alpha_t(c), ts), \quad (b, t)^* := (\alpha_{t^{-1}}(b^*), t^{-1}),$$

is a Fell bundle over G, called the *semidirect product* of (B, G, α) and denoted by $\mathcal{B} = B \times_\alpha G$. Moreover,

$$C^*(\mathcal{B}) \cong C^*(G, B) \quad \text{and} \quad C^*_{\mathrm{r}}(\mathcal{B}) \cong C^*_{\mathrm{r}}(G, B).$$

In particular, if $B = \mathbb{C}$ with the trivial action of G, then

$$C^*(\mathcal{B}) \cong C^*(G) \quad \text{and} \quad C^*_{\mathrm{r}}(\mathcal{B}) \cong C^*_{\mathrm{r}}(G).$$

In this case, $\lambda_{\mathcal{B}}$ corresponds to the left regular representation λ_G of G.

(2) Let (B, G, α) be a C^*-dynamical system. Suppose that I is a closed two-sided ideal of B. For each $t \in G$, we define $I_t := I \cap \alpha_t(I)$ (which is also a closed two-sided ideal of B) and a map $\theta_t : I_{t^{-1}} \to I_t$ by $\theta_t(b) := \alpha_t(b)$. Then $\theta := \{I_t, \theta_t\}_{t\in G}$ is a *partial action* of G on I (see [20]). It is called the *restriction* of α to I.

Define

$$\mathcal{B} := \{(b,t) : b \in I_t\} \subseteq B \times G, \quad \mathcal{B}_t := I_t \times \{t\} \cong I_t$$

with operations

$$(b,t)\cdot(c,s) := (b\alpha_t(c), ts), \quad (b,t)^* := (\alpha_{t^{-1}}(b^*), t^{-1}).$$

Then $\mathcal{B}$ is a Fell bundle over G, called the *semidirect product* of the partial dynamical system (I, G, θ); it is denoted by $\mathcal{B} = I \times_\theta G$. We have

$$C^*(\mathcal{B}) \cong C^*(G, I, \theta) \quad \text{and} \quad C^*_{\mathrm{r}}(\mathcal{B}) \cong C^*_{\mathrm{r}}(G, I, \theta),$$

where $C^*(G, I, \theta)$ and $C^*_{\mathrm{r}}(G, I, \theta)$ denote the full and the reduced crossed product of the partial dynamical system (I, G, θ).

More generally, one can associate Fell bundles to *twisted partial actions*, and up to a regularity condition all Fell bundles are of this form (see [20] for details).

Definition 6.3.3. Let $\mathcal{E}$ be a Hilbert B-module, and let $\mathcal{B} = \{\mathcal{B}_t\}$ be a Fell bundle over a locally compact group G. A *representation* of $\mathcal{B}$ on $\mathcal{E}$ is a map $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{E})$ such that

(i) $\pi_{|\mathcal{B}_t} : \mathcal{B}_t \to \mathcal{L}(\mathcal{E})$ is linear for all $t \in G$;

(ii) $\pi(bc) = \pi(b)\pi(c)$ for all $b, c \in \mathcal{B}$;

(iii) $\pi(b)^* = \pi(b^*)$ for all $b \in \mathcal{B}$; and

(iv) for each $\xi \in \mathcal{E}$, the map $\mathcal{B} \ni b \to \pi(b)\xi \in \mathcal{E}$ is continuous.

A representation π of $\mathcal{B}$ on $\mathcal{E}$ is called *nondegenerate* if $\overline{\operatorname{span}}(\pi(\mathcal{B})\mathcal{E}) = \mathcal{E}$. It is called *isometric* if $\|\pi(b)\| = \|b\|$ for all $b \in \mathcal{B}$.

Every representation π of $\mathcal{B}$ on $\mathcal{E}$ corresponds to a unique representation (still denoted by π) of $C^*(\mathcal{B})$, called the *integrated form* of π, which is determined by

$$\pi(f)\xi = \int_G \pi\big(f(t)\big)\xi \,\mathrm{d}t \qquad \text{for all } f \in L^1(\mathcal{B}) \text{ and } \xi \in \mathcal{E}.$$

This induces a one-to-one correspondence between representations of $\mathcal{B}$ and representations of $C^*(\mathcal{B})$. This correspondence preserves nondegeneracy. Note that condition (iv) above implies that the function $G \ni t \mapsto \pi\big(f(t)\big) \in \mathcal{L}(\mathcal{E})$ is strongly continuous for all $f \in \mathcal{C}_c(\mathcal{B})$. Since it is also bounded, it is strictly continuous. Since $\mathcal{C}_c(\mathcal{B})$ is dense in $L^1(\mathcal{B})$, it follows that the function $G \ni t \mapsto \pi\big(f(t)\big) \in \mathcal{L}(\mathcal{E}) = \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$ is strictly measurable for all

$f \in L^1(\mathcal{B})$, that is, for all $x \in \mathcal{K}(\mathcal{E})$, the map $G \ni t \mapsto \pi\big(f(t)\big)x \in \mathcal{K}(\mathcal{E})$ is measurable. Moreover, we have

$$\int_G \big\|\pi\big(f(t)\big)x\big\| \,\mathrm{d}t \le \left(\int_G \|f(t)\| \,\mathrm{d}t\right) \|x\| < \infty.$$

It follows that the map $G \ni t \mapsto \pi\big(f(t)\big) \in \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big)$ is strictly integrable for all $f \in L^1(\mathcal{B})$. The formula above for the integrated form can now be rewritten in the form

$$\pi(f) = \int_G^{\mathrm{s}} \pi\big(f(t)\big) \,\mathrm{d}t \quad \text{ for all } f \in L^1(\mathcal{B}), \tag{6.7}$$

where the superscript "s" stands for strict integral.

Let $\mathcal{B}$ be a Fell bundle over G, and for each $t \in G$, let $\Phi_t : \mathcal{B}_t \to \mathcal{M}\big(C^*(\mathcal{B})\big)$ be the map defined by $\Phi_t(b)\xi|_s = b\xi(t^{-1}s)$ and $\xi\Phi_t(b)|_s = \delta_G(t^{-1})\xi(st^{-1})b$ for all $b \in \mathcal{B}_t$, $\xi \in \mathcal{C}_c(\mathcal{B})$ and $s \in G$. Let $\Psi_t : \mathcal{B}_t \to \mathcal{M}\big(C^*_{\mathrm{r}}(\mathcal{B})\big)$ be the composition $\Psi_t = \lambda_{\mathcal{B}} \circ \Phi_t$. Note that if we identify $\mathcal{M}\big(C^*_{\mathrm{r}}(\mathcal{B})\big) \subseteq \mathcal{L}\big(L^2(\mathcal{B})\big)$, then Ψ_t is given by $\Psi_t(b)\xi|_s = b\xi(t^{-1}s)$ for all $\xi \in \mathcal{C}_c(\mathcal{B}) \subseteq L^2(\mathcal{B})$.

Proposition 6.3.4. *Let $\mathcal{B}$ be a Fell bundle over G. With the notations above, we define $\Phi : \mathcal{B} \to \mathcal{M}\big(C^*(\mathcal{B})\big)$ and $\Psi : \mathcal{B} \to \mathcal{M}\big(C^*_{\mathrm{r}}(\mathcal{B})\big)$ by $\Phi(b) = \Phi_t(b)$ and $\Psi(b) = \Psi_t(b)$ for all $b \in \mathcal{B}_t$. Then Φ and Ψ are nondegenerate isometric representations of $\mathcal{B}$. The integrated forms are, respectively, the inclusion $C^*(\mathcal{B}) \hookrightarrow \mathcal{M}\big(C^*(\mathcal{B})\big)$ and the left regular representation $\lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to \mathcal{M}\big(C^*_{\mathrm{r}}(\mathcal{B})\big)$.*

Proof. It is easy to check the condition (i)–(iii) in Definition 6.3.3. To check (iv), it is enough to show that the map $b \mapsto \Phi(b)\xi$ is continuous from $\mathcal{B}$ to $\mathcal{C}_c(\mathcal{B})$ with respect to the inductive limit topology for all $\xi \in \mathcal{C}_c(\mathcal{B})$. Suppose that (b_i) is a net in $\mathcal{B}$ converging to some $b \in \mathcal{B}$. Take $t_i, t \in G$ such that $b_i \in \mathcal{B}_{t_i}$ and $b \in \mathcal{B}_t$. For each i, we define a function $f_i : G \to \mathbb{C}$ by

$$f_i(s) := \|(\Phi(b_i)\xi)(s) - (\Phi(b)\xi)(s)\| = \|b_i\xi(t_i^{-1}s) - b\xi(t^{-1}s)\|.$$

Note that f_i belongs to $\mathcal{C}_c(G)$. Since $t_i \to t$, we may assume that the net (t_i) is contained in a fixed compact subset of G. Thus the supports of the functions f_i are all contained in a fixed compact subset $K_0 \subseteq G$. Now for each i, there is $s_i \in K_0$ such that $x_i := \sup_{s \in G} f_i(s_i) = f_i(s_i)$. Passing to a subnet, if necessary, we may assume that (s_i) converges to some $s \in K_0$. It follows that $x_i \to 0$ and hence $\Phi(b_i)\xi \to \Phi(b)\xi$ in the inductive limit topology. Therefore, both Φ and Ψ are $*$-representations of $\mathcal{B}$. The integrated form of Φ is given by

$$\Phi(f)\xi|_s = \int_G \Phi\big(f(t)\big)\xi(s) \,\mathrm{d}t = \int_G f(t)\xi(t^{-1}s) \,\mathrm{d}t = f * \xi$$

for all $f, \xi \in \mathcal{C}_c(\mathcal{B})$, that is, Φ is the canonical inclusion of $C^*(\mathcal{B})$ into $\mathcal{M}\big(C^*(\mathcal{B})\big)$. Hence the integrated from of $\Psi = \lambda_{\mathcal{B}} \circ \Phi$ coincides with $\lambda_{\mathcal{B}}$. In particular, Φ and Ψ are nondegenerate. Suppose now that $b \in \mathcal{B}_e$ and $\Phi(b) = 0$. This means that $b\xi(t) = 0$ for every $\xi \in \mathcal{C}_c(\mathcal{B})$ and $t \in G$. In particular, $b\xi(e) = 0$ for all $\xi \in \mathcal{C}_c(\mathcal{B})$, which is equivalent to $bc = 0$ for all $c \in \mathcal{B}_e$. Thus $b = 0$ and therefore Φ, when restricted to $\mathcal{B}_e$, is injective and hence isometric. Hence, for all $b \in \mathcal{B}$:

$$\|\Phi(b)\|^2 = \|\Phi(b)^*\Phi(b)\| = \|\Phi(b^*b)\| = \|b^*b\| = \|b\|^2.$$

This shows that Φ is isometric. Similarly, one proves that Ψ is isometric. □

Remark 6.3.5. Let $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{E})$ be a nondegenerate representation of $\mathcal{B}$, and let us denote for a moment by $\tilde{\pi}$ its integrated form. Then we have $\tilde{\pi} \circ \Phi = \pi$. In fact, for every $b \in \mathcal{B}_t$, $f \in \mathcal{C}_c(\mathcal{B})$ and $\xi \in \mathcal{E}$ we have

$$\begin{aligned}\tilde{\pi}\big(\Phi(b)\big)\tilde{\pi}(f)\xi = \tilde{\pi}\big(\Phi(b)f\big)\xi &= \int_G \pi\big((\Phi(b)f)(s)\big)\xi \,\mathrm{d}s \\ &= \int_G \pi\big(b_t f(t^{-1}s)\big)\xi \,\mathrm{d}s = \pi(b_t)\int_G \pi\big(f(s)\big)\xi \,\mathrm{d}s = \pi(b)\tilde{\pi}(f)\xi.\end{aligned}$$

In particular, if $\tilde{\pi}$ is isometric, then so is π. Since, in general, $\lambda_{\mathcal{B}}$ is not injective, Proposition 6.3.4 shows that the converse is not true.

6.4 Square-integrability of dual coactions

In this section we prove that the dual coactions on the full and reduced cross-sectional C^*-algebras of a Fell bundle are square-integrable.

Let G be a locally compact group, let $\mathcal{B} = \{\mathcal{B}_t\}_{t\in G}$ be a Fell bundle over G and let $A := C^*(\mathcal{B})$ be the cross-sectional C^*-algebra of $\mathcal{B}$. We shall identify each $b_s \in \mathcal{B}_s$ with an element of $\mathcal{M}(A)$ via the map Φ in Proposition 6.3.4. Thus $b_s \in \mathcal{B}_s$ will be identified with the multiplier of $A = C^*(\mathcal{B})$ given by $(b_s \cdot \xi)(t) := b_s \cdot \xi(s^{-1}t)$ for all $\xi \in \mathcal{C}_c(\mathcal{B})$ and $s, t \in G$. With this identification, each section $\xi \in \mathcal{C}_c(\mathcal{B})$ can be seen as an element of $\mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$, the space of compactly supported strictly continuous functions from G to $\mathcal{M}(A)$. Since the integrated form of $\Phi : \mathcal{B} \to \mathcal{M}(A)$ coincides with the inclusion $A \subseteq \mathcal{M}(A)$, Equation (6.7) yields

$$\xi = \int_G^{\mathrm{s}} \Phi\big(\xi(s)\big)\ \mathrm{d}s = \int_G^{\mathrm{s}} \xi(s)\ \mathrm{d}s, \quad \xi \in \mathcal{C}_c(\mathcal{B}), \tag{6.8}$$

where the superscript "s" stands for strict integral. The same equation also holds for $\xi \in L^1(\mathcal{B})$, but we just need it for $\xi \in \mathcal{C}_c(\mathcal{B})$.

Let $\mathcal{G} = C^*_{\mathrm{r}}(G)$. There is a continuous coaction of $\mathcal{G}$, that is, a continuous coaction of G on A denoted by $\gamma_{\mathcal{B}} : A \to \mathcal{M}(A \otimes \mathcal{G})$ and characterized by $\gamma_{\mathcal{B}}(b_s) = b_s \otimes \lambda_s$ for $b_s \in \mathcal{B}_s$ (see [21] for details). It follows from Equation (6.8) that

$$\gamma_{\mathcal{B}}(\xi) = \int_G^{\mathrm{s}} \xi(s) \otimes \lambda_s \,\mathrm{d}s, \quad \xi \in \mathcal{C}_c(\mathcal{B}) \tag{6.9}$$

Recall that $A(G)$ denotes the Fourier algebra of G. Given $u \in A(G)$ and $\xi \in \mathcal{C}_c(\mathcal{B})$, we have

$$u * \xi = (\mathrm{id} \otimes u)\big(\gamma_{\mathcal{B}}(\xi)\big) = (\mathrm{id} \otimes u)\left(\int_G^{\mathrm{s}} \xi(s) \otimes \lambda_s \,\mathrm{d}s\right) = \int_G^{\mathrm{s}} \xi(s)u(s)\,\mathrm{d}s = u \cdot \xi. \tag{6.10}$$

That is, the Banach left action of $A(G)$ on A induced by the coaction $\gamma_{\mathcal{B}}$, when restricted to $\mathcal{C}_c(\mathcal{B})$, is given by pointwise multiplication.

Now let A be an arbitrary C^*-algebra. We consider in $\mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$ the operations

$$\xi * \eta(t) := \int_G^{\mathrm{s}} \xi(s)\eta(s^{-1}t)\,\mathrm{d}s, \quad \xi^*(t) := \delta_G(t^{-1})\xi(t^{-1})^*, \quad \xi, \eta \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big).$$

With these operations $\mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$ is a (locally convex) $*$-algebra as in [15, C.6]. Moreover, the map

$$\mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big) \ni \xi \mapsto \int_G^{\mathrm{s}} \xi(t) \otimes u_t \,\mathrm{d}t \in \mathcal{M}\big(A \otimes C^*(G)\big)$$

is an injective $*$-homomorphism which is continuous for the inductive limit topology on $\mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$ and the strict topology on $\mathcal{M}\big(A \otimes C^*(G)\big)$, where $t \mapsto u_t$ denotes the canonical inclusion of G into $\mathcal{M}\big(C^*(G)\big)$ (see [15, C.7]). We will view $\mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$ as a $*$-subalgebra of $\mathcal{M}\big(A \otimes C^*(G)\big)$. In this way, we can consider the $*$-homomorphism

$$\mathrm{id}_A \otimes \lambda : \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big) \subseteq \mathcal{M}\big(A \otimes C^*(G)\big) \to \mathcal{M}\big(A \otimes C^*_{\mathrm{r}}(G)\big)$$

given by the formula

$$(\mathrm{id}_A \otimes \lambda)(\xi) := \int_G^{\mathrm{s}} \xi(s) \otimes \lambda_s \,\mathrm{d}s, \quad \xi \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big). \tag{6.11}$$

Thus, if we have a Fell bundle $\mathcal{B}$ and set $A := C^*(\mathcal{B})$, then the dual coaction $\gamma_{\mathcal{B}}$ satisfies

$$\gamma_{\mathcal{B}}(\xi) = (\mathrm{id}_A \otimes \lambda)(\xi) \quad \text{for all } \xi \in \mathcal{C}_c(\mathcal{B}). \tag{6.12}$$

A special case of Fell bundle, which is very good to keep in mind, is the semidirect product bundle $\mathcal{B}$ associated to an action of G on a C^*-algebra B (see Example 6.3.2). The C^*-algebra $C^*(\mathcal{B})$ is identified in this way with the full crossed product $C^*(G, B)$ and the dual coaction $\gamma_{\mathcal{B}}$ is identified with the classical dual coaction of G.

We already know that every classical dual coaction is integrable (Corollary 3.3.5). We are going to generalize this result, proving that the dual coaction $\gamma_{\mathcal{B}}$ defined above is integrable for any Fell bundle $\mathcal{B}$. The argument used in Corollary 3.3.5 to prove that classical dual coactions are integrable does not apply to the coaction $\gamma_{\mathcal{B}}$. We are going to use the definition of integrability in order to prove that $\gamma_{\mathcal{B}}$ is integrable.

Recall that φ denotes the Haar weight of $\mathcal{G} = C^*_{\mathrm{r}}(G)$ described in Section 6.1.

Lemma 6.4.1. *Let A be a C^*-algebra. Let $\xi \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$ and $\theta \in A^*_+$. Consider the function $f \in \mathcal{C}_c(G)$ defined by $f(s) := \theta\big((\xi^* * \xi)(s)\big)$. Then $\lambda(f)$ is a positive element of $C^*_{\mathrm{r}}(G) \subseteq \mathcal{L}\big(L^2(G)\big)$. In particular, $\lambda(f) \in \mathcal{M}^+_\varphi$.*

Proof. Note that for all $\eta \in L^2(G)$,

$$\langle \eta, \lambda(f)\eta \rangle = \int_G \int_G \overline{\eta(t)} f(s)\eta(s^{-1}t)\,\mathrm{d}s\,\mathrm{d}t = \int_G \int_G \delta_G(s)^{-1} f(ts^{-1})\overline{\eta(t)}\eta(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Taking a GNS-construction for θ, we may assume that $A \subseteq \mathcal{L}(H)$ for some Hilbert space H and $\theta(a) = \langle v \,|\, av \rangle$ for some $v \in H$. Thus we have

$$\begin{aligned}\delta_G(s)^{-1} f(ts^{-1}) &= \delta_G(s)^{-1} \langle v \,|\, (\xi^* * \xi)(ts^{-1}) v \rangle \\ &= \delta_G(s)^{-1} \int_G \langle v \,|\, \xi(r)^* \xi(rts^{-1}) v \rangle \,\mathrm{d}r \\ &= \int_G \langle \delta_G(t)^{-1} \xi(rt^{-1}) v \,|\, \delta_G(s)^{-1} \xi(rs^{-1}) v \rangle \,\mathrm{d}r.\end{aligned}$$

Therefore

$$\langle \eta, \lambda(f) \eta \rangle = \int_G \left\langle \int_G \overline{\eta(t)} \delta_G(t)^{-1} \xi(rt^{-1}) v \,\mathrm{d}t \middle| \int_G \overline{\eta(s)} \delta_G(s)^{-1} \xi(rs^{-1}) v \,\mathrm{d}s \right\rangle \mathrm{d}r \geq 0.$$

The last assertion follows from Proposition 6.1.2. $\square$

Recall from Section 6.1 that we always use the GNS-construction $(L^2(G), \iota, \Lambda)$ for the Haar weight φ, where $\Lambda\big(\lambda(\xi)\big) = \xi$ for every left bounded function $\xi \in L^2(G)$. Here ι denotes the inclusion map $C^*_{\mathrm{r}}(G) \hookrightarrow \mathcal{L}\big(L^2(G)\big)$.

Proposition 6.4.2. *Let A be C^*-algebra and let $\mathcal{G} := C^*_{\mathrm{r}}(G)$. Consider the $*$-homomorphism $\mathrm{id}_A \otimes \lambda : \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big) \to \mathcal{M}(A \otimes \mathcal{G})$ given by Equation (6.11). Then*

(i) *for all $\xi \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$, the element $(\mathrm{id}_A \otimes \lambda)(\xi)$ belongs to $\bar{\mathcal{N}}_{\mathrm{id}_A \otimes \varphi}$ and*

$$(\mathrm{id}_A \otimes \Lambda)\big((\mathrm{id}_A \otimes \lambda)(\xi)\big) = T_\xi \in \mathcal{L}\big(A, A \otimes L^2(G)\big),$$

where T_ξ is element of $\mathcal{L}\big(A, A \otimes L^2(G)\big) \cong \mathcal{L}\big(A, L^2(G, A)\big)$ given by $T_\xi a|_t := \xi(t) a$ for all $a \in A$ and $t \in G$; and

(ii) *for all $\eta \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)^2 := \mathrm{span}\{f * g : f, g \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)\}$, the element $(\mathrm{id}_A \otimes \lambda)(\eta)$ belongs to $\bar{\mathcal{M}}_{\mathrm{id}_A \otimes \varphi}$ and*

$$(\mathrm{id}_A \otimes \varphi)\big((\mathrm{id}_A \otimes \lambda)(\eta)\big) = \eta(e).$$

Proof. We shall use Proposition 2.4.5. Let $\eta := \xi^* * \xi \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$, $x := (\mathrm{id}_A \otimes \lambda)(\eta) \in \mathcal{M}(A \otimes \mathcal{G})^+$ and $a := \eta(e) \in \mathcal{M}(A)$. Note that

$$a = (\xi^* * \xi)(e) = \int_G^{\mathrm{s}} \xi^*(s) \xi(s^{-1} e) \,\mathrm{d}s = \int_G^{\mathrm{s}} \xi(s)^* \xi(s) \,\mathrm{d}s,$$

so that $a \in \mathcal{M}(A)^+$. Let $\theta \in A^*_+$. We have

$$(\theta \otimes \mathrm{id}_{\mathcal{G}})(x) = (\theta \otimes \mathrm{id}_{\mathcal{G}}) \left(\int_G^{\mathrm{s}} \eta(s) \otimes \lambda_s \,\mathrm{d}s \right) = \int_G \theta\big(\eta(s)\big) \lambda_s \,\mathrm{d}s = \lambda(f),$$

where $f := [s \mapsto \theta(\eta(s))] \in \mathcal{C}_c(G)$. By Lemma 6.4.1, we have $(\theta \otimes \mathrm{id}_{\mathcal{G}})(x) \in \mathcal{M}^+_\varphi$ and

$$\varphi\big((\theta \otimes \mathrm{id}_{\mathcal{G}})(x)\big) = f(e) = \theta\big(\eta(e)\big) = \theta(a).$$

Now Proposition 2.4.5 yields $x = (\mathrm{id}_A \otimes \lambda)(\eta) \in \bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$. Thus $(\mathrm{id}_A \otimes \lambda)(\xi) \in \bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi}$ and

$$(\mathrm{id}_A \otimes \varphi)\big((\mathrm{id}_A \otimes \lambda)(\eta)\big) = \eta(e).$$

By polarization, we get $(\mathrm{id}_A \otimes \lambda)(\eta) \in \bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$ and

$$(\mathrm{id}_A \otimes \varphi)\big((\mathrm{id}_A \otimes \lambda)(\eta)\big) = \eta(e) \quad \text{for all } \eta \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)^2.$$

It is easy to see that the adjoint T_ξ^* of T_ξ is given by

$$T_\xi^*(f) = \int_G \xi(t)^* f(t)\,\mathrm{d}t, \quad f \in \mathcal{C}_c(G, A).$$

Thus we have, for all $f \in \mathcal{C}_c(G, A)$,

$$\begin{aligned}
(\mathrm{id}_A \otimes \Lambda)\big((\mathrm{id}_A \otimes \lambda)(\xi)\big)^*(f) &= (\mathrm{id}_A \otimes \Lambda)\big((\mathrm{id}_A \otimes \lambda)(\xi)\big)^*(\mathrm{id}_A \otimes \Lambda)\big((\mathrm{id}_A \otimes \lambda)(f)\big)\\
&= (\mathrm{id}_A \otimes \varphi)\big((\mathrm{id}_A \otimes \lambda)(\xi^* * f)\big)\\
&= (\xi^* * f)(e) = \int_G \xi^*(t) f(t^{-1}e)\,\mathrm{d}t\\
&= \int_G \xi(t)^* f(t)\,\mathrm{d}t = T_\xi^*(f). \qquad \square
\end{aligned}$$

In what follows we use the notations introduced in Definition 3.2.9.

Theorem 6.4.3. *Let $\mathcal{B}$ be a Fell bundle over G. Then the cross-sectional C^*-algebra $C^*(\mathcal{B})$ considered with the dual coaction is an integrable $\widehat{G}$-C^*-algebra. Moreover, each element $\xi \in \mathcal{C}_c(\mathcal{B})$ is square-integrable with*

$$\langle\!\langle \xi| = T_{\xi^*}.$$

Equivalently, each $\xi \in \mathcal{C}_c(\mathcal{B})^2 := \operatorname{span}\big\{\eta * \zeta : \eta, \zeta \in \mathcal{C}_c(\mathcal{B})\big\}$ is integrable with

$$E_1(\xi) = \xi(e). \tag{6.13}$$

Proof. Let $A := C^*(\mathcal{B})$. Since $\mathcal{C}_c(\mathcal{B}) \subseteq \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$, the assertions follow directly from Definition 3.2.1, Definition 3.2.9, Equation (6.12) and Proposition 6.4.2. $\square$

We now turn our attention to the reduced C^*-algebra $C^*_{\mathrm{r}}(\mathcal{B})$ of a Fell bundle $\mathcal{B}$. Recall that the left regular representation of $\mathcal{B}$ is defined by

$$\lambda_{\mathcal{B}} : \mathcal{B} \to \mathcal{L}\big(L^2(\mathcal{B})\big), \quad \lambda_{\mathcal{B}}(b_t)\xi(s) := b_t\xi(t^{-1}s) \quad \text{for all } b_t \in \mathcal{B}_t \text{ and } s, t \in G,$$

where $L^2(\mathcal{B})$ denotes the Hilbert $\mathcal{B}_e$-module completion of the pre-Hilbert $\mathcal{B}_e$-module $\mathcal{C}_c(\mathcal{B})$ with the obvious right action of $\mathcal{B}_e$ and the $\mathcal{B}_e$-inner product

$$\langle \xi, \eta\rangle_{\mathcal{B}_e} := \int_G \xi(t)^*\eta(t)\,\mathrm{d}t, \quad \xi, \eta \in \mathcal{C}_c(\mathcal{B}).$$

By definition, we have
$$C_{\mathrm{r}}^*(\mathcal{B}) := \lambda_{\mathcal{B}}\big(C^*(\mathcal{B})\big) \subseteq \mathcal{L}\big(L^2(\mathcal{B})\big).$$
There is a continuous coaction $\gamma_{\mathcal{B}}^{\mathrm{r}}$ of G on $C_{\mathrm{r}}^*(\mathcal{B})$, also called the *dual coaction*, such that $\lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C_{\mathrm{r}}^*(\mathcal{B})$ is equivariant (see [21] for details). Moreover, $\gamma_{\mathcal{B}}^{\mathrm{r}}$ is injective and satisfies
$$\gamma_{\mathcal{B}}^{\mathrm{r}}(x) = W_{\mathcal{B}}(x \otimes 1)W_{\mathcal{B}}^*,$$
where $W_{\mathcal{B}}$ is the unitary in $\mathcal{L}\left(L^2(\mathcal{B}) \otimes L^2(G)\right)$ defined by $W_{\mathcal{B}}\xi(s,t) = \xi(s, s^{-1}t)$ for all $\xi \in \mathcal{C}_c(\mathcal{B} \times G)$ and $s,t \in G$. Here we identify $L^2(\mathcal{B}) \otimes L^2(G) \cong L^2(\mathcal{B} \times G)$, where $\mathcal{B} \times G$ denotes the pull-back of $\mathcal{B}$ along the projection $G \times G \ni (s,t) \mapsto s \in G$.

The equivariance of $\lambda_{\mathcal{B}}$ yields
$$\gamma_{\mathcal{B}}^{\mathrm{r}}\big(\lambda_{\mathcal{B}}(b_t)\big) = (\lambda_{\mathcal{B}} \otimes \mathrm{id})\big(\gamma_{\mathcal{B}}(b_t)\big) = (\lambda_{\mathcal{B}} \otimes \mathrm{id})(b_t \otimes \lambda_t) = \lambda_{\mathcal{B}}(b_t) \otimes \lambda_t \quad \text{for all } b_t \in \mathcal{B}_t,$$
and hence
$$\gamma_{\mathcal{B}}^{\mathrm{r}}\big(\lambda_{\mathcal{B}}(\xi)\big) = \int_G^{\mathrm{s}} \lambda_{\mathcal{B}}\big(\xi(t)\big) \otimes \lambda_t \,\mathrm{d}t, \quad \xi \in \mathcal{C}_c(\mathcal{B}).$$
We can carry over our results from $C^*(\mathcal{B})$ to $C_{\mathrm{r}}^*(\mathcal{B})$:

Corollary 6.4.4. *Let $\mathcal{B}$ be a Fell bundle over G. Then $C_{\mathrm{r}}^*(\mathcal{B})$ considered with the dual coaction $\gamma_{\mathcal{B}}^{\mathrm{r}}$ is an integrable $\widehat{G}$-C^*-algebra. Moreover, for every $\xi \in \mathcal{C}_c(\mathcal{B})$, the element $\lambda_{\mathcal{B}}(\xi) \in \lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)$ is square-integrable and*
$$\langle\!\langle \lambda_{\mathcal{B}}(\xi)| = (\lambda_{\mathcal{B}} \otimes \mathrm{id})(T_{\xi^*}).$$
Equivalently, each element in $\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)^2$ is integrable and
$$E_1\big(\lambda_{\mathcal{B}}(\xi)\big) = \lambda_{\mathcal{B}}\big(\xi(e)\big) \quad \textit{for all } \xi \in \mathcal{C}_c(\mathcal{B})^2.$$

Proof. This follows from Theorem 6.4.3 and Proposition 3.3.1. □

6.5 Continuous square-integrability of dual coactions

Let G be a locally compact group and let $\mathcal{B}$ be a Fell bundle over G. Consider the $\widehat{G}$-C^*-algebra $A = C^*(\mathcal{B})$ with the dual coaction $\gamma_{\mathcal{B}}$. We already know that A is square-integrable. Moreover, from Theorem 6.4.3 we know that $\mathcal{C}_c(\mathcal{B})$ consists of square-integrable elements. In this section, we prove that $\mathcal{C}_c(\mathcal{B})$ is also relatively continuous, that is, we prove that $\langle\!\langle \mathcal{C}_c(\mathcal{B}) \,|\, \mathcal{C}_c(\mathcal{B})\rangle\!\rangle$ is contained in the crossed product $A \rtimes_{\mathrm{r}} G$.

Let $\xi, \eta \in \mathcal{C}_c(\mathcal{B})$. By Theorem 6.4.3, we have
$$\langle\!\langle \xi^* \,|\, \eta^* \rangle\!\rangle = T_\xi T_\eta^* \in \mathcal{L}\big(A \otimes L^2(G)\big) \cong \mathcal{L}\big(L^2(G, A)\big). \tag{6.14}$$
This operator is given by the formula
$$T_\xi T_\eta^* \zeta|_t = \int_G \xi(t)\eta(s)^*\zeta(s)\,\mathrm{d}s, \quad \zeta \in \mathcal{C}_c(G, A) \subseteq L^2(G, A).$$

Thus it is an integral operator

$$T_K\zeta|_t = \int_G K(t,s)\zeta(s)\,\mathrm{d}s,$$

with kernel $K = [\xi,\eta] \in \mathcal{C}_c\big(G\times G, \mathcal{M}^{\mathrm{s}}(A)\big)$ given by $[\xi,\eta](t,s) := \xi(t)\eta(s)^* \in \mathcal{B}_{ts^{-1}} \subseteq \mathcal{M}(A)$. Note that $(t,s)\mapsto \|K(t,s)\|$ is a scalar continuous function. Moreover, define the continuous function $\sigma : G\times G \to G$ by $\sigma(t,s) = ts^{-1}$. Let $\mathcal{A}$ be the pull-back of $\mathcal{B}$ along σ. Then $\mathcal{A}$ is a continuous Banach bundle over $G\times G$ whose fiber over (t,s) is naturally isomorphic to $\mathcal{B}_{ts^{-1}}$. In this way, the space $\mathcal{C}_c(\mathcal{A})$ of continuous compactly supported sections of $\mathcal{A}$ is naturally identified with the space of all continuous compactly supported functions $\zeta : G\times G\to \mathcal{B}$ satisfying $\zeta(t,s)\in\mathcal{B}_{ts^{-1}}$ for all $t,s\in G$. In particular, $[\xi,\eta]\in\mathcal{C}_c(\mathcal{A})$. If we identify $\mathcal{A}_{(t,s)} \cong \mathcal{B}_{ts^{-1}} \subseteq \mathcal{M}(A)$, then we may view $\mathcal{C}_c(\mathcal{A})$ as a subspace of $\mathcal{C}_c\big(G\times G,\mathcal{M}^{\mathrm{s}}(A)\big)$.

We want to prove that $\langle\langle \xi^* \,|\, \eta^*\rangle\rangle \in A\rtimes_{\mathrm{r}} G \subseteq \mathcal{L}\big(A\otimes L^2(G)\big)$ for all $\xi,\eta\in\mathcal{C}_c(\mathcal{B})$. Recall that

$$A\rtimes_{\mathrm{r}} G = \overline{\operatorname{span}}\{\gamma_A(a)(1_A\otimes M_f) : a\in A, f\in\mathcal{C}_0(G)\},$$

where $M:\mathcal{C}_0(G)\to\mathcal{L}\big(L^2(G)\big)$ is the multiplication representation, and we identify $\gamma_A(a)\in\mathcal{M}\big(A\otimes C^*_{\mathrm{r}}(G)\big)\subseteq\mathcal{M}\big(A\otimes\mathcal{K}\big(L^2(G)\big)\big)\cong\mathcal{L}\big(A\otimes L^2(G)\big)$. In this way, each $\gamma_A(a)$ defines a multiplier of $A\rtimes_{\mathrm{r}} G$. In particular, we have $\gamma_A(a)(1_A\otimes M_f)\gamma_A(b^*)\in A\rtimes_{\mathrm{r}} G$ for all $a,b\in A$ and $f\in\mathcal{C}_0(G)$. Now, the idea is to approximate $\langle\langle \xi^*\,|\,\eta^*\rangle\rangle$ by $\gamma_A(\xi)(1_A\otimes M_f)\gamma_A(\eta^*)$, for suitable functions $f\in\mathcal{C}_c(G)$. Note that

$$\gamma_A(\xi)(1_A\otimes M_f)\gamma_A(\eta^*) = \int_G^{\mathrm{s}}\int_G^{\mathrm{s}} \xi(t)\eta^*(s)\otimes\lambda_t M_f\lambda_s\,\mathrm{d}t\,\mathrm{d}s.$$

It is easy to see that $\lambda_t M_f = M_{\alpha_t(f)}\lambda_t$, where $\alpha_t(f)|_r := f(t^{-1}r)$. Thus, for all $a\in A$ and $y\in\mathcal{C}_c(G)$, we have

$$\begin{aligned}
\big(\gamma_A(\xi)(1_A\otimes M_f)\gamma_A(\eta^*)\big)(a\otimes y)|_r &= \int_G\int_G \big(\xi(t)\eta^*(s)a\otimes M(\alpha_t(f))\lambda_{ts}(y)\big)|_r\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_G\int_G \xi(t)\eta^*(s)af(t^{-1}r)y(s^{-1}t^{-1}r)\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_G \big([\xi,\eta]*f\big)(r,s)(a\otimes y)(s)\,\mathrm{d}s,
\end{aligned}$$

where

$$([\xi,\eta]*f)(r,s) := \int_G \xi(t)\eta(sr^{-1}t)^* f(t^{-1}r)\delta_G(r^{-1}t)\,\mathrm{d}t.$$

Note that $\xi(t)\eta(sr^{-1}t)^*f(t^{-1}r)\delta_G(r^{-1}t)\in\mathcal{B}_{rs^{-1}}$ for all $r\in G$. Thus the integral above is a Bochner integral with values in $\mathcal{B}_{rs^{-1}}$, which is viewed as a subspace of $\mathcal{M}(A)$. With this new notation, we therefore have

$$\gamma_A(\xi)(1_A\otimes M_f)\gamma_A(\eta^*) = T_{[\xi,\eta]*f}. \tag{6.15}$$

In other words, $\gamma_A(\xi)(1_A \otimes M_f)\gamma_A(\eta^*)$ is an integral operator with kernel $K = [\xi, \eta] * f$. Using [23, II.15.19] one proves that $[\xi, \eta] * f \in \mathcal{C}_c(\mathcal{A}) \subseteq \mathcal{C}_c\big(G \times G, \mathcal{M}^{\mathrm{s}}(A)\big)$. In particular, the function $(s,t) \mapsto \|([\xi, \eta] * f)(s,t)\|$ belongs to $\mathcal{C}_c(G \times G)$.

We may assume without loss of generality that $A \subseteq \mathcal{L}(H)$, for some Hilbert space H. Then $\mathcal{L}\big(A \otimes L^2(G)\big) \subseteq \mathcal{L}\big(H \otimes L^2(G)\big) \cong \mathcal{L}\big(L^2(G, H)\big)$, and an integral operator T_K with kernel $K \in \mathcal{C}_c(\mathcal{A}) \subseteq \mathcal{C}_c\big(G \times G, \mathcal{M}^{\mathrm{s}}(A)\big) \subseteq \mathcal{C}_c\big(G \times G, \mathcal{L}^{\mathrm{s}}(H)\big)$ takes the form

$$T_K\zeta|_t = \int_G K(t,s)\zeta(s)\,\mathrm{d}s, \quad \zeta \in \mathcal{C}_c(G, H).$$

The assumption $A \subseteq \mathcal{L}(H)$ allows us to calculate the norm of T_K more easily. In fact, by the Cauchy–Schwartz inequality, we have

$$\begin{aligned}\|T_K(\zeta)\|_2^2 &\leq \int_G \left(\int_G \|K(t,s)\|\|\zeta(s)\|\,\mathrm{d}s\right)^2 \mathrm{d}t \\ &\leq \int_G \left(\int_G \|K(t,s)\|^2\,\mathrm{d}s\right)\left(\int_G \|\zeta(s)\|^2\,\mathrm{d}s\right)\mathrm{d}t = \|K\|_2^2\|\zeta\|_2^2,\end{aligned}$$

where $\|K\|_2$ denotes the norm of the function $(t,s) \mapsto \|K(t,s)\|$ in $L^2(G \times G)$. Thus

$$\|T_K\| \leq \|K\|_2. \tag{6.16}$$

Before we proceed, let us consider a special case. Suppose that G is discrete. Let $\xi, \eta \in \mathcal{C}_c(\mathcal{B})$ and let δ_e be the delta Dirac function at $e \in G$. Then it is easy to see that the kernel function $[\xi, \eta] * \delta_e$ is equal to $[\xi, \eta]$. In particular, we see that in the case of a discrete group, $\mathcal{C}_c(\mathcal{B})$ is relatively continuous. In fact, this is something we already know from the general theory because in this case the quantum group $C^*_{\mathrm{r}}(G)$ is compact (see Proposition 5.2.12).

In general, if G is not discrete, we are going to approximate $[\xi, \eta]$ by elements of the form $[\xi, \eta] * f_i$, where (f_i) is some approximate unit of $L^1(G)$.

Lemma 6.5.1. *Let $\mathcal{B}$ be a Fell bundle over G, and let $\xi, \eta \in \mathcal{C}_c(\mathcal{B})$. Then*

$$\lim_{r \to e} \sup_{s,t \in G} \|\xi(sr)\eta(t) - \xi(s)\eta(rt)\| = 0$$

Proof. Define $f(s,r,t) := \|\xi(sr)\eta(t) - \xi(s)\eta(rt)\|$. Note that $f \in \mathcal{C}(G \times G \times G)$ because norm, product, and difference operations on a Fell Bundle are continuous. Fix some compact neighborhood U_0 of $e \in G$. Let g be the restriction of f to $G \times U_0 \times G$. Note that $g \in \mathcal{C}_c(G \times U_0 \times G)$. In fact, let $K := \mathrm{supp}(\xi)$ and $L := \mathrm{supp}(\eta)$. Then it is easy to see that $\mathrm{supp}(g) \subseteq KU_0^{-1} \times U_0 \times U_0^{-1}L$. Define the compact subset $K_0 := KU_0^{-1} \cup U_0^{-1}L$ and the function $h : U_0 \to \mathbb{R}$ by

$$h(r) := \sup_{s,t \in G} f(s,r,t).$$

We have to prove that h is continuous in e. Let (r_i) be a net converging to e with $r_i \in U_0$ for all i. Note that, for each i, there are $s_i, t_i \in K_0$ such that $h(r_i) = f(s_i, r_i, t_i)$. Since K_0 is compact, there are subnets (s_j') and (t_j') of (s_i) and (t_i), respectively, and $s_0, t_0 \in K_0$, such that $s_j' \to s_0$ and $t_j' \to t_0$. Therefore, $h(r_j') = f(s_j', r_j', t_j') \to f(s_0, e, t_0) = 0$. This yields the desired continuity of h. $\square$

Proposition 6.5.2. *Let $\mathcal{B}$ be a Fell bundle over G and let $\xi,\eta \in \mathcal{C}_c(\mathcal{B})$. Let (f_V) be the usual approximate unit of $L^1(G)$, consisting of positive-valued functions $f_V \in \mathcal{C}_c(G)$ with* $\mathrm{supp}(f_V) \subseteq V$ *and* $\int_G f_V(r)\,\mathrm{d}r = 1$ *for any compact neighborhood V of $e \in G$. Then*

$$[\xi,\eta] * f_V(s,t) \to [\xi,\eta](s,t) \quad \textit{uniformly in } s,t \in G.$$

Proof. We have

$$\begin{aligned}
[\xi,\eta] * f_V(s,t) &= \int_V \xi(r)\eta(ts^{-1}r)^* f_V(r^{-1}s)\delta_G(s^{-1}r)\,\mathrm{d}r \\
&= \int_V \xi(sr^{-1})\eta(tr^{-1})^* f_V(r)\delta_G(r)^{-2}\,\mathrm{d}r.
\end{aligned}$$

Thus, setting $c := \sup\limits_{s,r,t\in G}\big\|\xi(sr^{-1})\eta(tr^{-1})^*\big\| < \infty$,

$$\begin{aligned}
\big\|[\xi,\eta] * f_V(s,t) - [\xi,\eta](s,t)\big\| &\le \int_V \big\|\xi(sr^{-1})\eta(tr^{-1})^*\delta_G(r)^{-2} - \xi(s)\eta(t)^*\big\| f_V(r)\,\mathrm{d}r \\
&\le \int_V \big\|\xi(sr^{-1})\eta(tr^{-1})^*\big\|\big|\delta_G(r)^{-2} - 1\big| f_V(r)\,\mathrm{d}r \\
&\quad + \int_V \big\|\xi(sr^{-1})\eta(tr^{-1})^* - \xi(s)\eta(t)^*\big\| f_V(r)\,\mathrm{d}r \\
&\le c\int_V \big|\delta_G(r)^{-2} - 1\big| f_V(r)\,\mathrm{d}r \\
&\quad + \int_V \sup_{s,t\in G}\big\|\xi(sr^{-1})\eta(tr^{-1})^* - \xi(s)\eta(t)^*\big\| f_V(r)\,\mathrm{d}r.
\end{aligned}$$

Observe that

$$\begin{aligned}
\sup_{s,t\in G}\big\|\xi(sr^{-1})\eta(tr^{-1})^* - \xi(s)\eta(t)^*\big\| &= \sup_{s,t\in G}\big\|\xi(sr^{-1})\eta(t^{-1})^* - \xi(s)\eta(t^{-1}r)^*\big\| \\
&= \sup_{s,t\in G}\big\|\xi(sr^{-1})\tilde{\eta}(t) - \xi(s)\tilde{\eta}(r^{-1}t)\big\|,
\end{aligned}$$

where $\tilde{\eta}(g) := \eta(g^{-1})^*$ for all $g \in G$. The assertion now follows from Lemma 6.5.1. $\square$

Theorem 6.5.3. *Let $\mathcal{B}$ be a Fell bundle over G and consider the $\widehat{G}$-C^*-algebra $A := C^*(\mathcal{B})$ with the dual coaction $\gamma_{\mathcal{B}}$. Then $\mathcal{C}_c(\mathcal{B})$ is a relatively continuous subspace of A. In other words, we have*

$$\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in A \rtimes_{\mathrm{r}} G \quad \textit{for all } \xi,\eta \in \mathcal{C}_c(\mathcal{B}).$$

The pair $(C^*(\mathcal{B}), \overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}})$ is a continuously square-integrable $\widehat{G}$-C^*-algebra.

Proof. Since $\mathcal{C}_c(\mathcal{B})$ is self-adjoint, it is enough to prove that

$$\langle\!\langle \xi^* \,|\, \eta^* \rangle\!\rangle \in A \rtimes_{\mathrm{r}} G \quad \text{for all } \xi,\eta \in \mathcal{C}_c(\mathcal{B}).$$

We may assume that $A \subseteq \mathcal{L}(H)$ for some Hilbert space H. Equation (6.14) implies that $\langle\!\langle \xi^* \,|\, \eta^* \rangle\!\rangle \in \mathcal{L}\big(L^2(G,H)\big)$ is given by

$$\langle\!\langle \xi^* \,|\, \eta^* \rangle\!\rangle \zeta|_t = \int_G \xi(t)\eta(s)^*\zeta(s)\,\mathrm{d}s = \int_G [\xi,\eta](t,s)\zeta(s)\,\mathrm{d}s, \quad \zeta \in \mathcal{C}_c(G,H).$$

And by Equation (6.15), the operator $\gamma_A(\xi)(1_A \otimes M_{f_V})\gamma_A(\eta^*) \in A \rtimes_{\mathrm{r}} G \subseteq \mathcal{L}\big(L^2(G,H)\big)$ is given by

$$\gamma_A(\xi)(1_A \otimes M_{f_V})\gamma_A(\eta^*)\zeta|_t = \int_G ([\xi,\eta] * f_V)(t,s)\zeta(s)\,\mathrm{d}s, \quad \zeta \in \mathcal{C}_c(G,H).$$

By Proposition 6.5.2, $([\xi,\eta] * f_V)(s,t) \to [\xi,\eta](s,t)$ uniformly in $s,t \in G$. Moreover, it is easy to see that

$$\operatorname{supp}([\xi,\eta] * f_V) \subseteq \operatorname{supp}(\xi)\operatorname{supp}(f_V) \times \operatorname{supp}(\eta)\operatorname{supp}(f_V).$$

Thus $[\xi,\eta] * f_V \to [\xi,\eta]$ in the inductive limit topology. Using Equation (6.16), we conclude that

$$\gamma_A(\xi)(1_A \otimes M_{f_V})\gamma_A(\eta^*) \to \langle\!\langle \xi^* \,|\, \eta^* \rangle\!\rangle \text{ in } \mathcal{L}\big(L^2(G,H)\big).$$

Therefore $\mathcal{C}_c(\mathcal{B})$ is relatively continuous. Equation (6.10) says that $\mathcal{C}_c(\mathcal{B})$ is $A(G)$-invariant. Proposition 5.3.5(ii) implies that $\overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}}$ is the completion of $\mathcal{C}_c(\mathcal{B})$. The last assertion now follows. □

Corollary 6.5.4. *Let $\mathcal{B} = \{\mathcal{B}_t\}_{t\in G}$ be a Fell bundle over G and consider $A_{\mathrm{r}} := C^*_{\mathrm{r}}(\mathcal{B})$ with the dual coaction of G. Then $\mathcal{R}_{\mathrm{r}} := \overline{\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}}$ is a dense, complete, relatively continuous subspace of A_{r}, and we have*

$$\langle\!\langle \lambda_{\mathcal{B}}(\xi) \,|\, \lambda_{\mathcal{B}}(\eta) \rangle\!\rangle = \lambda_{\mathcal{B}}\left(\langle\!\langle \xi \,|\, \eta \rangle\!\rangle\right) \quad \textit{for all } \xi,\eta \in \mathcal{R}_{\mathrm{r}}.$$

The pair $(A_{\mathrm{r}}, \mathcal{R}_{\mathrm{r}})$ is a continuously square-integrable $\widehat{G}$-C^*-algebra.

Proof. By Corollary 3.3.3 and Theorem 6.5.3, $\mathcal{R}_{\mathrm{r}}$ is relatively continuous and the formula above holds. Since $\mathcal{C}_c(\mathcal{B})$ is $A(G)$-invariant and $\lambda_{\mathcal{B}}$ is equivariant, $\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)$ is also $A(G)$-invariant. Proposition 5.3.5(ii) implies that $\mathcal{R}_{\mathrm{r}}$ is complete. □

6.6 The Fourier transform

We are keeping the notations $\mathcal{G} = C^*_{\mathrm{r}}(G)$ and φ for its Haar weight. Recall that $\varphi_t : \bar{\mathcal{M}}_\varphi \to \mathbb{C}$ is the functional defined by $\varphi_t(x) = \varphi(\lambda_t^{-1}x)$. We can also define slice maps with these functionals. For a C^*-algebra A, we define

$$\mathrm{id}_A \otimes \varphi_t : \bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi} \to \mathcal{M}(A), \quad (\mathrm{id}_A \otimes \varphi_t)(x) := (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_t^{-1})x\big).$$

Since the elements λ_t are analytic with respect to the modular group of φ (see Equation (6.4)), Proposition 2.4.13(ii) yields the following generalization of (6.5):

$$(1_A \otimes \lambda_t)\bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi} = \bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi} \quad \text{and} \quad \bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi}(1_A \otimes \lambda_t) = \bar{\mathcal{N}}_{\mathrm{id}_A\otimes\varphi} \quad \text{for all } t \in G.$$

In particular, the functional $\mathrm{id}_A \otimes \varphi_t$ is well-defined on $\bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$.

Definition 6.6.1. Let $\gamma_A : A \to \mathcal{M}(A \otimes \mathcal{G})$ be a coaction of G. For $a \in A_{\mathrm{i}}$ we define the *t-Fourier coefficient* $E_t(a)$ by

$$E_t(a) := (\mathrm{id}_A \otimes \varphi_t)\big(\gamma_A(a)\big).$$

The map $t \mapsto E_t(a)$ from G to $\mathcal{M}(A)$ is called the *Fourier transform* of $a \in A_{\mathrm{i}}$.

If the group is Abelian and if we identify $\mathcal{G} \cong \mathcal{C}_0(\widehat{G})$ by means of the Fourier transform, then $(1_A \otimes \lambda_{t^{-1}})(f)(x) = \langle x \,|\, t\rangle f(x)$ for all $f \in \mathcal{C}_b\big(\widehat{G}, \mathcal{M}^{\mathrm{s}}(A)\big) \cong \mathcal{M}(A \otimes \mathcal{G})$ and $x \in \widehat{G}$, where $\langle x \,|\, t\rangle := x(t)$. So if γ_A corresponds to an action α of $\widehat{G}$ on A, then the t-Fourier coefficient coincides with the Fourier coefficient defined by Exel [18, 19]:

$$E_t(a) = \int_{\widehat{G}}^{\mathrm{su}} \langle x, t\rangle \alpha_x(a)\,\mathrm{d}x, \quad a \in A_{\mathrm{i}}.$$

If $\mathcal{E}$ is a Hilbert B-module with a coaction $\gamma_{\mathcal{E}}$ of $\mathcal{G}$ and $\xi, \eta \in \mathcal{E}_{\mathrm{si}}$, then we already know that $|\xi\rangle\langle\eta| \in \mathcal{K}(\mathcal{E})_{\mathrm{i}}$ (see Proposition 4.1.10). Using the induced coaction of $\mathcal{G}$ on $\mathcal{K}(\mathcal{E})$, we can apply Definition 6.6.1 to $A = \mathcal{K}(\mathcal{E})$ to get the t-Fourier coefficient

$$E_t(|\xi\rangle\langle\eta|) = (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi_t)\big(\gamma_{\mathcal{K}(\mathcal{E})}(|\xi\rangle\langle\eta|)\big) \in \mathcal{M}\big(\mathcal{K}(\mathcal{E})\big) \cong \mathcal{L}(\mathcal{E}).$$

Proposition 6.6.2. *Let A be a $\widehat{G}$-C^*-algebra and let $a \in A_{\mathrm{i}}$. Then the t-Fourier coefficient $E_t(a)$ belongs to the t-spectral subspace $\mathcal{M}_t(A)$ of $\mathcal{M}(A)$ defined by*

$$\mathcal{M}_t(A) := \{b \in \mathcal{M}(A) : \gamma_A(b) = b \otimes \lambda_t\}.$$

Proof. Since the comultiplication Δ of $\mathcal{G} = C^*_{\mathrm{r}}(G)$ satisfies $\Delta(\lambda_t) = \lambda_t \otimes \lambda_t$, the coaction identity $(\gamma_A \otimes \mathrm{id}_{\mathcal{G}}) \circ \gamma_A = (\mathrm{id}_A \otimes \Delta) \circ \gamma_A$ gives

$$(\gamma_A \otimes \mathrm{id}_{\mathcal{G}})\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big) = (1_A \otimes \lambda_t \otimes 1_{\mathcal{G}})(\mathrm{id}_A \otimes \Delta)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big).$$

Using Lemma 2.4.8(i) and Corollary 4.2.3, we get

$$\begin{aligned}
\gamma_A\big(E_t(a)\big) &= \gamma_A\Big((\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big)\Big)\\
&= (\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi)\Big((\gamma_A \otimes \mathrm{id}_{\mathcal{G}})\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big)\Big)\\
&= (\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi)\Big((1_A \otimes \lambda_t \otimes 1_{\mathcal{G}})(\mathrm{id}_A \otimes \Delta)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big)\Big)\\
&= (1_A \otimes \lambda_t)(\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \varphi)\Big((\mathrm{id}_A \otimes \Delta)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big)\Big)\\
&= (1_A \otimes \lambda_t)\Big((\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big) \otimes 1_{\mathcal{G}}\Big)\\
&= E_t(a) \otimes \lambda_t.
\end{aligned}$$

□

Let $e \in G$ be the unit element. Note that the e-spectral subspace $\mathcal{M}_e(A)$ is exactly the fixed point algebra:

$$\mathcal{M}_e(A) = \mathcal{M}_1(A) = \{b \in \mathcal{M}(A) : \gamma_A(b) = b \otimes 1\}.$$

The following result is a generalization of [19, Proposition 6.4].

Proposition 6.6.3. *Let A be a $\widehat{G}$-C^*-algebra and consider $a, b \in A_{\mathrm{i}}$, $s, t \in G$ and $m \in \mathcal{M}_s(A)$. Then*

(i) $E_t(a)^* = \delta_G(t)^{-1} E_{t^{-1}}(a^*)$,

(ii) $ma \in A_{\mathrm{i}}$ *and* $mE_t(a) = E_{st}(ma)$,

(iii) $am \in A_{\mathrm{i}}$ *and* $E_t(a)m = \delta_G(s) E_{ts}(am)$,

(iv) $E_t(a)E_s(b) = E_{ts}\big(E_t(a)b\big) = \delta_G(s) E_{ts}\big(aE_s(b)\big)$.

Proof. (i) Recall from Equation (6.4) that λ_t is an analytic element and $\sigma_z(\lambda_t) = \delta_G(t)^{\mathrm{i}z}\lambda_t$ for all $z \in \mathbb{C}$. Proposition 2.4.13(ii) yields

$$\begin{aligned} E_t(a)^* &= (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big)^* \\ &= (\mathrm{id}_A \otimes \varphi)\big(\gamma_A(a^*)(1_A \otimes \lambda_t)\big) \\ &= (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \sigma_i(\lambda_t))\gamma_A(a^*)\big) \\ &= \delta_G(t)^{-1} E_{t^{-1}}(a^*). \end{aligned}$$

(ii) Since $A_{\mathrm{i}} = \operatorname{span} A_{\mathrm{si}} A_{\mathrm{si}}^*$, we may assume that $a = bc^*$ with $b, c \in A_{\mathrm{si}}$. By definition of A_{si}, we have $\gamma_A(b) \in \bar{\mathcal{N}}^*_{\mathrm{id}_A \otimes \varphi}$. Thus, by Proposition 2.4.6(iv) and Proposition 2.4.13(ii), $\gamma_A(mb) = (m \otimes \lambda_t)\gamma_A(b) \in \bar{\mathcal{N}}^*_{\mathrm{id}_A \otimes \varphi}$, that is, $mb \in A_{\mathrm{si}}$. Therefore $ma = mbc^* \in A_{\mathrm{si}} A_{\mathrm{si}}^* \subseteq A_{\mathrm{i}}$. Moreover, since $\gamma_A(m) = m \otimes \lambda_s$, we conclude that

$$\begin{aligned} mE_t(a) &= m(\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_{t^{-1}})\gamma_A(a)\big) \\ &= (\mathrm{id}_A \otimes \varphi)\big((m \otimes \lambda_{t^{-1}})\gamma_A(a)\big) \\ &= (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_{t^{-1}}\lambda_{s^{-1}})\gamma_A(ma)\big) \\ &= E_{st}(ma). \end{aligned}$$

(iii) As in (ii) one proves that $am \in A_{\mathrm{i}}$. Using Proposition 2.4.13(ii) as well as the relations $\sigma_z(\lambda_s) = \delta_G(s)^{\mathrm{i}z}\lambda_s$ and $\gamma_A(m) = m \otimes \lambda_s$, we get

$$\begin{aligned} E_t(a)m &= (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_{t^{-1}})\gamma_A(a)\big)m \\ &= (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_{t^{-1}})\gamma_A(a)(m \otimes 1_{\mathcal{G}})\big) \\ &= (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_{t^{-1}})\gamma_A(am)(1_A \otimes \lambda_{s^{-1}})\big) \\ &= (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \sigma_i(\lambda_{s^{-1}})\lambda_{t^{-1}})\gamma_A(am)\big) \\ &= \delta_G(s)(\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_{s^{-1}}\lambda_{t^{-1}})\gamma_A(am)\big) \\ &= \delta_G(s) E_{ts}(am). \end{aligned}$$

(iv) Follows from (ii) and (iii).

□

Now we generalize the formula (6.13).

Theorem 6.6.4. *Let $\mathcal{B} = \{\mathcal{B}_t\}_{t\in G}$ be a Fell bundle and consider $A := C^*(\mathcal{B})$ with the dual coaction $\gamma_{\mathcal{B}}$ of G defined in* (6.9)*. If we view $\mathcal{B}_t$ as a subspace of $\mathcal{M}(A)$, then for all $\xi \in \mathcal{C}_c(\mathcal{B})^2$ and $t \in G$, we have*

$$E_t(\xi) = \xi(t).$$

Proof. For $t = e$ this is exactly (6.13). The idea now is to make a change of variables in order to prove the general case. The right change will be $\eta(s) := \lambda_{t^{-1}}(\xi)(s) = \xi(ts)$. The problem is that η is not a section of $\mathcal{B}$ because $\eta(s) \in \mathcal{B}_{ts}$. To solve this problem we consider each $\xi \in \mathcal{C}_c(\mathcal{B})$ as an element of $\mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$. By Proposition 6.4.2, $(\mathrm{id}_A \otimes \lambda)(\xi) \in \bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$ for every $\xi \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)^2$ and

$$(\mathrm{id}_A \otimes \varphi)\big((\mathrm{id}_A \otimes \lambda)(\xi)\big) = \xi(e).$$

Now for $\xi \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$, the change of variable $\eta := \lambda_{t^{-1}}(\xi)$ makes sense. And if $\xi \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)^2$, then $\eta \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)^2$ because $\lambda_{t^{-1}}(\xi_1 * \xi_2) = \lambda_{t^{-1}}(\xi_1) * \xi_2$ for all $\xi_1, \xi_2 \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)$. Thus $(\mathrm{id}_A \otimes \lambda)(\eta) \in \bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$ and

$$(\mathrm{id}_A \otimes \varphi)\big((\mathrm{id}_A \otimes \lambda)(\eta)\big) = \eta(e) = \xi(t).$$

Note that

$$\begin{aligned}(\mathrm{id}_A \otimes \lambda)(\eta) &= \int_G^{\mathrm{s}} \eta(s) \otimes \lambda_s \,\mathrm{d}s = \int_G^{\mathrm{s}} \xi(ts) \otimes \lambda_s \,\mathrm{d}s \\ &= \int_G^{\mathrm{s}} \xi(s) \otimes \lambda_{t^{-1}s} \,\mathrm{d}s = (1_A \otimes \lambda_{t^{-1}})(\mathrm{id}_A \otimes \lambda)(\xi).\end{aligned}$$

Therefore, for every $\xi \in \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)^2$,

$$(\mathrm{id}_A \otimes \varphi_t)\big((\mathrm{id}_A \otimes \lambda)(\xi)\big) = (\mathrm{id} \otimes \varphi)\big((\mathrm{id}_A \otimes \lambda)(\eta)\big) = \xi(t).$$

In particular, the equation above holds for $\xi \in \mathcal{C}_c(\mathcal{B})^2 \subseteq \mathcal{C}_c\big(G, \mathcal{M}^{\mathrm{s}}(A)\big)^2$. Equation (6.12) implies the desired result:

$$E_t(\xi) = (\mathrm{id}_A \otimes \varphi_t)\big(\gamma_{\mathcal{B}}(\xi)\big) = (\mathrm{id}_A \otimes \varphi_t)\big((\mathrm{id}_A \otimes \lambda)(\xi)\big) = \xi(t). \qquad \square$$

Remark 6.6.5. Let us illustrate how easy it is to prove Theorem 6.6.4 if G is discrete. Moreover, in this case the result is true for every $\xi \in L^1(\mathcal{B})$. In fact, in this case we have

$$\gamma_{\mathcal{B}}(\xi) = \sum_{s\in G} \xi(s) \otimes \lambda_s$$

for all $\xi \in L^1(\mathcal{B})$ and the functional $\varphi_t \in \mathcal{G}^*$ satisfies $\varphi_t(\lambda_s) = \delta_{t,s}$ for every $t, s \in G$, where $\delta_{t,s}$ denotes the Kronecker delta function. Therefore,

$$E_t(\xi) = (\mathrm{id} \otimes \varphi_t)\left(\sum_{s\in G} \xi(s) \otimes \lambda_s\right) = \sum_{s\in G} \xi(s)\varphi_t(\lambda_s) = \xi(t).$$

Theorem 6.6.4 is a generalization of [18, Theorem 5.5] for non-Abelian groups. Theorem 5.5 in [18] is proved using an appropriate Fourier inversion formula. Let us explain what we mean by Fourier inversion formula in this context. If G is Abelian, then the dual coaction $\gamma_{\mathcal{B}}$ corresponds to the action β of $\widehat{G}$ on $C^*(\mathcal{B})$ given by

$$\beta_x(\xi)(t) = \overline{\langle x\,|\,t\rangle}\xi(t) \quad \text{for all } \xi \in \mathcal{C}_c(\mathcal{B}),\ t \in G,\ x \in \widehat{G}.$$

It is easy to see that $\beta_x(b_t) = \overline{\langle x\,|\,t\rangle} b_t$ for all $b_t \in \mathcal{B}_t \subseteq \mathcal{M}\big(C^*(\mathcal{B})\big)$. Equation (6.8) yields

$$\beta_x(\xi) = \int_G^{\mathrm{s}} \overline{\langle x\,|\,s\rangle}\xi(s)\,\mathrm{d}s = \int_G^{\mathrm{su}} \overline{\langle x\,|\,s\rangle}\xi(s)\,\mathrm{d}s.$$

Thus we can think of $\beta_x(\xi)$ as a generalized Fourier transform of $\xi \in \mathcal{C}_c(\mathcal{B})$. In this way, Theorem 6.6.4 is the Fourier inversion formula

$$\int_{\widehat{G}}^{\mathrm{su}} \langle x, t\rangle \left(\int_G^{\mathrm{su}} \overline{\langle x\,|\,s\rangle}\xi(s)\,\mathrm{d}s\right)\mathrm{d}x = \xi(t) \quad \text{for } t \in G,\ \xi \in \mathcal{C}_c(\mathcal{B})^2.$$

Now we carry Theorem 6.6.4 over to $C^*_{\mathrm{r}}(\mathcal{B})$ using $\lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_{\mathrm{r}}(\mathcal{B})$. First, we need a preliminary result.

Proposition 6.6.6. *Let (A, γ_A) and (B, γ_B) be coactions of G. Suppose that $\pi : A \to B$ is an equivariant nondegenerate $*$-homomorphism. If $a \in A_{\mathrm{i}}$, then $\pi(a) \in B_{\mathrm{i}}$ and*

$$E_t\big(\pi(a)\big) = \pi\big(E_t(a)\big) \quad \textit{for all } t \in G.$$

Proof. By Corollary 3.3.2, $\pi(a) \in B_{\mathrm{i}}$ for all $a \in A_{\mathrm{i}}$. Moreover, the equivariance of π yields

$$\begin{aligned}
E_t\big(\pi(a)\big) &= (\mathrm{id}_B \otimes \varphi)\Big((1_B \otimes \lambda_t^{-1})\gamma_B\big(\pi(a)\big)\Big)\\
&= (\mathrm{id}_B \otimes \varphi)\Big((\pi \otimes \mathrm{id})\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big)\Big)\\
&= \pi\Big((\mathrm{id}_B \otimes \varphi)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big)\Big)\\
&= \pi\big(E_t(a)\big).
\end{aligned}$$

□

Corollary 6.6.7. *Let $\mathcal{B} = \{\mathcal{B}_t\}_{t\in G}$ be a Fell bundle over G and consider the dual coaction $\gamma^{\mathrm{r}}_{\mathcal{B}}$ of G on $C^*_{\mathrm{r}}(\mathcal{B})$. Then*

$$E_t\big(\lambda_{\mathcal{B}}(\xi)\big) = \lambda_{\mathcal{B}}\big(\xi(t)\big) \quad \textit{for all } \xi \in \mathcal{C}_c(\mathcal{B})^2.$$

Proof. This follows from Theorem 6.6.4 and Proposition 6.6.6. □

Recall from Section 6.1 that V_t denotes the operator in $\mathcal{L}\big(L^2(G)\big)$ given by $V_t(\xi)(s) = \xi(st)$ for all $s, t \in G$ and $\xi \in L^2(G)$. Note that $V_tV_s = V_{ts}$, $V_t^{-1} = V_{t^{-1}}$ and $V_t^* = \delta_G(t)^{-1}V_{t^{-1}}$ for all $t, s \in G$. Therefore, the map

$$V : G \to \mathcal{L}\big(L^2(G)\big), \quad t \mapsto V_t$$

is a (non-unitary) representation of G on $L^2(G)$, and we have $V_t = \delta_G(t)^{-\frac{1}{2}}\rho_t$, where ρ is the right regular (unitary) representation of G. Since ρ is strongly continuous, the same is true for V (see also [29, 20.4(ii)]). The same argument shows that V is also strictly continuous if we identify $\mathcal{L}\big(L^2(G)\big) = \mathcal{M}\big(\mathcal{K}(L^2(G))\big)$. Note that $\|V_t\| = \|V_t^*V_t\|^{\frac{1}{2}} = \delta_G(t)^{-\frac{1}{2}}$, so that V is not bounded for non-unimodular groups.

If B is an arbitrary C^*-algebra, we also write V for the representation of G on the Hilbert B-module $B \otimes L^2(G)$ given by

$$V : G \to \mathcal{L}\big(B \otimes L^2(G)\big), \quad V_t(\xi)(s) = \xi(st)$$

for all $\xi \in \mathcal{C}_c(G, B) \subseteq L^2(G, B) \cong B \otimes L^2(G)$.

Recall that $(L^2(G), \iota, \Lambda)$ denotes the canonical GNS-construction for the Haar weight φ, where $\Lambda(\lambda(\xi)) = \xi$ for every left bounded function $\xi \in L^2(G)$ (see Section 6.1). The next result is a generalization of Proposition 2.4.20(ii) for $\mathcal{G} = C^*_{\mathrm{r}}(G)$.

Proposition 6.6.8. *Let $\mathcal{E}$ be a Hilbert B-module. Then, for every $x, y \in \bar{\mathcal{N}}_{\mathrm{id}_{\mathcal{E}^*}\otimes\varphi}$ and $t \in G$, we have*

$$(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi_t)(x^*y) = (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x)^* V_t (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y).$$

Proof. Recall from Equation (6.4) that λ_t is an analytic element and $\sigma_z(\lambda_t) = \delta_G(t)^{\mathrm{i}z}\lambda_t$ for all $z \in \mathbb{C}$. Propositions 2.4.20 and 2.4.22 imply

$$\begin{aligned}
(\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi_t)(x^*y) &= (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)\big((1_{\mathcal{K}(\mathcal{E})} \otimes \lambda_t^{-1})x^*y\big) \\
&= (\mathrm{id}_{\mathcal{K}(\mathcal{E})} \otimes \varphi)\big((x(1_{\mathcal{K}(\mathcal{E})} \otimes \lambda_t))^*y\big) \\
&= (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)\big(x(1_{\mathcal{K}(\mathcal{E})} \otimes \lambda_t)\big)^*(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y) \\
&= (\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(x)^*\big(1_B \otimes J\sigma_{\frac{\mathrm{i}}{2}}(\lambda_t)J\big)(\mathrm{id}_{\mathcal{E}^*} \otimes \Lambda)(y),
\end{aligned}$$

where J denotes the modular conjugation of φ in the GNS-construction $(L^2(G), \iota, \Lambda)$. It remains to shows that $J\sigma_{\frac{\mathrm{i}}{2}}(\lambda_t)J = V_t$. The modular conjugation J is given by $J\xi(s) = \delta_G(s)^{-\frac{1}{2}}\overline{\xi(s^{-1})}$ for all $\xi \in L^2(G)$, $s \in G$. Therefore,

$$\begin{aligned}
(J\sigma_{\frac{\mathrm{i}}{2}}(\lambda_t)J\xi)(s) &= \delta_G(s)^{-\frac{1}{2}}\overline{(\sigma_{\frac{\mathrm{i}}{2}}(\lambda_t)J\xi)(s^{-1})} \\
&= \delta_G(s)^{-\frac{1}{2}}\delta_G(t)^{-\frac{1}{2}}\overline{(\lambda_t J\xi)(s^{-1})} \\
&= \delta_G(s)^{-\frac{1}{2}}\delta_G(t)^{-\frac{1}{2}}\overline{(J\xi)(t^{-1}s^{-1})} \\
&= \xi(st) = V_t\xi(s). \qquad \square
\end{aligned}$$

The next result is a generalization of Proposition 4.1.10(i) for $\mathcal{G} = C^*_{\mathrm{r}}(G)$. It can also be seen as a generalization of [19, Lemma 7.4].

Corollary 6.6.9. *Let $\mathcal{E}$ be a Hilbert B-module with a coaction $\gamma_{\mathcal{E}}$ of G. Then*

$$E_t\big(|\xi\rangle\langle\eta|\big) = |\xi\rangle\!\rangle V_t \langle\!\langle\eta| \quad \textit{for all } \xi, \eta \in \mathcal{E}_{\mathrm{si}} \textit{ and } t \in G.$$

Proof. Proposition 6.6.8 yields

$$\begin{aligned} E_t\big(|\xi\rangle\langle\eta|\big) &= (\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi_t)\big(\gamma_{\mathcal{K}(\mathcal{E})}(|\xi\rangle\langle\eta|)\big)\\ &= (\mathrm{id}_{\mathcal{K}(\mathcal{E})}\otimes\varphi_t)\big(\gamma_{\mathcal{E}}(\xi)\gamma_{\mathcal{E}}(\eta)^*\big)\\ &= (\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)\big(\gamma_{\mathcal{E}}(\xi)^*\big)^*V_t(\mathrm{id}_{\mathcal{E}^*}\otimes\Lambda)\big(\gamma_{\mathcal{E}}(\eta)^*\big)\\ &= |\xi\rangle\!\rangle V_t\langle\!\langle\eta|. \end{aligned}$$

□

Let A be a C^*-algebra with a coaction of G and let $a\in A_{\mathrm{i}}$. Corollary 6.6.9 yields several properties for the Fourier coefficients $E_t(a)$ which are not so easy to obtain directly from the definition. For example, since $E_t(a)$ is defined in terms of φ, which is not bounded unless G is discrete, it is not so clear that one can obtain any type of continuity or boundedness property of the Fourier transform

$$G\ni t\mapsto E_t(a)\in\mathcal{M}(A).$$

But since $\|V_t\|=\delta_G(t)^{-\frac{1}{2}}$, it follows directly from Corollary 6.6.9 that

$$\|E_t(a)\|\leq c\delta_G(t)^{-\frac{1}{2}}$$

for all $t\in G$, where c is a positive constant. In particular, the map $t\mapsto\delta_G(t)^{\frac{1}{2}}E_t(a)$ is bounded.

Concerning continuity of the Fourier transform, we prove the following result, which is a generalization of [18, Proposition 6.3] to non-Abelian groups.

Corollary 6.6.10. *Let A be a $\widehat{G}$-C^*-algebra and consider $a\in A_{\mathrm{i}}$. Then the Fourier transform $G\ni t\mapsto E_t(a)\in\mathcal{M}(A)$ is strictly continuous.*

Proof. Since $A_{\mathrm{i}}=\operatorname{span}A_{\mathrm{si}}A_{\mathrm{si}}^*$, we may assume $a=\xi\eta^*$, with $\xi,\eta\in A_{\mathrm{si}}$. Corollary 6.6.9 yields $E_t(a)=|\xi\rangle\!\rangle V_t\langle\!\langle\eta|$. Thus, for $t,t_0\in G$ and $b\in A$,

$$\big\|E_t(a)b-E_{t_0}(a)b\big\|\leq\big\||\xi\rangle\!\rangle\big\|\cdot\big\|V_t\big(\langle\!\langle\eta|b\big)-V_{t_0}\big(\langle\!\langle\eta|b\big)\big\|,$$

and

$$\begin{aligned} \big\|bE_t(a)-bE_{t_0}(a)\big\| &\leq \big\|b|\xi\rangle\!\rangle V_t-b|\xi\rangle\!\rangle V_{t_0}\big\|\cdot\big\|\langle\!\langle\eta|\big\|\\ &= \big\|V_t^*\big(\langle\!\langle\xi|b^*\big)-V_{t_0}^*\big(\langle\!\langle\xi|b^*\big)\big\|\cdot\big\|\langle\!\langle\eta|\big\|\\ &= \big\|\delta_G\big(t^{-1}\big)V_{t^{-1}}\big(\langle\!\langle\xi|b^*\big)-\delta_G\big(t_0^{-1}\big)V_{t_0^{-1}}\big(\langle\!\langle\xi|b^*\big)\big\|\cdot\big\|\langle\!\langle\eta|\big\|. \end{aligned}$$

The assertion now follows from the strong continuity of $t\mapsto V_t$ and the continuity of the modular function δ_G. □

For Abelian groups, [19, Proposition 6.3] contains a stronger result, namely, that the Fourier transform $G\ni t\mapsto E_t(a)\in\mathcal{M}(A)$ $(a\in A_{\mathrm{i}})$ is uniformly continuous in the strict topology of $\mathcal{M}(A)$. This means that

$$\lim_{s\to e}\sup_{t\in G}\|E_{st}(a)b-E_t(a)b\|=0$$

and

$$\lim_{s\to e}\sup_{t\in G}\|bE_{st}(a)-bE_t(a)\|=0$$

for every $b\in A$. This is a special feature of the Abelian case. First of all, if G is non-Abelian, we have two concepts of uniform continuity on G, namely, left and right uniform continuity (see [29, 4.12]). The two equations above say that the Fourier transform $G\ni t\mapsto E_t(a)\in\mathcal{M}(A)$ is right uniformly continuous in the strict topology of $\mathcal{M}(A)$. Left uniform continuity means that

$$\lim_{s\to e}\sup_{t\in G}\|E_{ts}(a)b-E_t(a)b\|=0$$

and

$$\lim_{s\to e}\sup_{t\in G}\|bE_{ts}(a)-bE_t(a)\|=0$$

for every $b\in A$.

The problem in general is that $G\ni t\mapsto V_t\in\mathcal{L}\big(L^2(G)\big)$ is neither left nor right uniformly continuous in the strong topology of $\mathcal{L}\big(L^2(G)\big)$. In fact, V is left uniformly continuous in the strong topology of $\mathcal{L}\big(L^2(G)\big)$ if and only if G is unimodular (see [29, 20.30(b)]). Moreover, if G is unimodular, then V is right uniformly continuous in the strong topology of $\mathcal{L}\big(L^2(G)\big)$ if and only if G has equivalent left and right uniform structures (see [29, 4.13] and [29, 20.30(c)]). Of course, if G is Abelian, then (G is unimodular and) the left and right uniform structures are equivalent (in fact equal).

Suppose that V is left uniformly continuous in the strict topology of $\mathcal{M}\big(\mathcal{K}(L^2(G))\big)\cong\mathcal{L}\big(L^2(G)\big)$. Since $L^2(G)=\mathcal{K}(L^2(G))\cdot L^2(G)$, strict convergence implies $*$-strong convergence.[2] From the discussion above, G is unimodular. Moreover, for every $\xi\in L^2(G)$ we get (using the unimodularity of G, so that $V_t^*=V_{t^{-1}}$)

$$0=\lim_{s\to e}\sup_{t\in G}\|V_{ts}^*(\xi)-V_t^*(\xi)\|=\lim_{s\to e}\sup_{t\in G}\|V_{s^{-1}t^{-1}}(\xi)-V_{t^{-1}}(\xi)\|=\lim_{s\to e}\sup_{t\in G}\|V_{st}(\xi)-V_t(\xi)\|.$$

This means that V is also right uniformly continuous in the strong topology of $\mathcal{L}\big(L^2(G)\big)$. Again, the discussion above implies that the left and right uniform structures of G are equivalent.

Similarly, if we suppose that V is right uniformly continuous in the strict topology of $\mathcal{L}\big(L^2(G)\big)$, then G is necessarily unimodular and has equivalent uniform structures.

Thus, in general, we cannot expect left or right uniform continuity of the Fourier transform $G\ni t\mapsto E_t(a)\in\mathcal{M}(A)$ in the strict topology. But we can at least prove the following partial result.

Proposition 6.6.11. *Assume that G is unimodular and has equivalent left and right uniform structures . Then the Fourier transform*

$$G\ni t\mapsto E_t(a)\in\mathcal{M}(A)$$

is (left and right) uniformly continuous in the strict topology of $\mathcal{M}(A)$.

[2] A net $\{T_i\}$ in $\mathcal{L}(\mathcal{E})$, where $\mathcal{E}$ is a Hilbert B-module, converges $*$-*strongly* to $T\in\mathcal{L}(\mathcal{E})$ if and only if $T_i\xi\to T\xi$ and $T_i^*\xi\to T^*\xi$ for all $\xi\in\mathcal{E}$.

Proof. We shall use the term "uniformly continuous" meaning both left and right uniformly continuous (which are equivalent in this case). The discussion above shows that V is uniformly continuous in the $*$-strong topology of $\mathcal{L}\big(L^2(G)\big)$. Hence the map

$$G \ni t \mapsto 1_A \otimes V_t \in \mathcal{L}\big(A \otimes L^2(G)\big)$$

is also uniformly continuous in the $*$-strong topology of $\mathcal{L}\big(A \otimes L^2(G)\big)$. As in the proof of Corollary 6.6.10, we estimate

$$\big\|E_{ts}(a)b - E_t(a)b\big\| \leq C_1\big\|(1_A \otimes V_{ts})(\zeta_1) - (1_A \otimes V_t)(\zeta_1)\big\|$$

and

$$\big\|bE_{ts}(a) - bE_t(a)\big\| \leq C_2\big\|(1_A \otimes V_{ts})^*(\zeta_2) - (1_A \otimes V_t)^*(\zeta_2)\big\|$$

for all $b \in A$, where C_1, C_2 are constants and ζ_1, ζ_2 are elements of $A \otimes L^2(G)$. The assertion now follows. □

Definition 6.6.12. Let B be a C^*-algebra and let $T \in \mathcal{L}\big(B \otimes L^2(G)\big)$. We say that T is *V-continuous* if the map

$$G \ni t \mapsto V_t T V_t^{-1} \in \mathcal{L}\big(B \otimes L^2(G)\big)$$

is continuous in the norm of $\mathcal{L}\big(B \otimes L^2(G)\big)$.

Proposition 6.6.13. *Let $\mathcal{E}$ be a Hilbert B-module with a coaction of G and let $\xi, \eta \in \mathcal{E}_{\mathrm{si}}$. Define $p := |\xi\rangle\langle\xi|$ and $q := |\eta\rangle\langle\eta|$. Then the following assertions are equivalent:*

(i) $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle \in \mathcal{L}\big(B \otimes L^2(G)\big)$ *is V-continuous;*

(ii) $\lim\limits_{r \to e} \sup\limits_{s,t \in K} \|E_{tr}(p)E_s(q) - E_t(p)E_{rs}(q)\| = 0$ *for every compact $K \subseteq G$;*

(iii) $\lim\limits_{t \to e} \|E_t(p)E_e(q) - E_e(p)E_t(q)\| = 0$ *and* $\lim\limits_{t \to e} \|E_t(p)E_{t^{-1}}(q) - E_e(p)E_e(q)\| = 0$.

Proof. The idea is basically the same as the one in [19, Theorem 7.5]. Corollary 6.6.9 and the relation $\|V_t\| = \delta_G(t)^{-\frac{1}{2}}$ yield

$$\begin{aligned}\big\|E_{rs}(p)E_t(q) - E_r(p)E_{st}(q)\big\| &= \big\||\xi\rangle\!\rangle V_{rs}\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t \langle\!\langle \eta| - |\xi\rangle\!\rangle V_r \langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_{st}\langle\!\langle \eta|\big\| \\ &\leq \big\||\xi\rangle\!\rangle\big\| \cdot \big\|V_r\big\| \cdot \big\|V_s\langle\!\langle \xi \,|\, \eta\rangle\!\rangle - \langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_s\big\| \cdot \big\|V_t\big\| \cdot \big\|\langle\!\langle \eta|\big\| \\ &\leq \delta_G(rst)^{-\frac{1}{2}}\big\||\xi\rangle\!\rangle\big\| \cdot \big\|V_s\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_s^{-1} - \langle\!\langle \xi \,|\, \eta\rangle\!\rangle\big\| \cdot \big\|\langle\!\langle \eta|\big\|.\end{aligned}$$

Hence (i) implies (ii). Suppose now that (ii) is true. Taking $K = \{e\}$, it follows that $\lim\limits_{t \to e} \|E_t(p)E_e(q) - E_e(p)E_t(q)\| = 0$. Moreover, taking K to be a compact neighborhood of e, and using that $t \mapsto t^{-1}$ is continuous in G, we also get $\lim\limits_{t \to e} \|E_t(p)E_{t^{-1}}(q) - E_e(p)E_e(q)\| = 0$.

Therefore, (ii) implies (iii). Finally, we show that (iii) implies (i). Using $V_t^* = \delta_G(t)^{-\frac{1}{2}} V_t^{-1}$, we get

$$\begin{aligned}
&\big(V_t\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t^{-1} - \langle\!\langle \xi \,|\, \eta\rangle\!\rangle\big)\big(V_t\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t^{-1} - \langle\!\langle \xi \,|\, \eta\rangle\!\rangle\big)^*\big(V_t\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t^{-1} - \langle\!\langle \xi \,|\, \eta\rangle\!\rangle\big)\\
&\qquad = V_t\langle\!\langle \xi \,|\, \eta\rangle\!\rangle\langle\!\langle \eta \,|\, \xi\rangle\!\rangle\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t^{-1} - V_t\langle\!\langle \xi \,|\, \eta\rangle\!\rangle\langle\!\langle \eta \,|\, \xi\rangle\!\rangle V_t^{-1}\langle\!\langle \xi \,|\, \eta\rangle\!\rangle\\
&\qquad\quad - V_t\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t^{-1}\langle\!\langle \eta \,|\, \xi\rangle\!\rangle V_t\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t^{-1} + V_t\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t^{-1}\langle\!\langle \eta \,|\, \xi\rangle\!\rangle\langle\!\langle \xi \,|\, \eta\rangle\!\rangle\\
&\qquad\quad - \langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t\langle\!\langle \eta \,|\, \xi\rangle\!\rangle\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t^{-1} + \langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t\langle\!\langle \eta \,|\, \xi\rangle\!\rangle V_t^{-1}\langle\!\langle \xi \,|\, \eta\rangle\!\rangle\\
&\qquad\quad + \langle\!\langle \xi \,|\, \eta\rangle\!\rangle\langle\!\langle \eta \,|\, \xi\rangle\!\rangle V_t\langle\!\langle \xi \,|\, \eta\rangle\!\rangle V_t^{-1} - \langle\!\langle \xi \,|\, \eta\rangle\!\rangle\langle\!\langle \eta \,|\, \xi\rangle\!\rangle\langle\!\langle \xi \,|\, \eta\rangle\!\rangle\\
&\qquad = V_t\langle\!\langle \xi|(E_e(q)E_e(p) - E_{t^{-1}}(q)E_t(p))|\eta\rangle\!\rangle V_t^{-1}\\
&\qquad\quad + V_t\langle\!\langle \xi|(E_{t^{-1}}(q)E_e(p) - E_e(q)E_{t^{-1}}(p))|\eta\rangle\!\rangle V_t^{-1}\\
&\qquad\quad + \langle\!\langle \xi|(E_e(q)E_t(p) - E_t(q)E_e(p))|\eta\rangle\!\rangle V_t^{-1}\\
&\qquad\quad + \langle\!\langle \xi|(E_t(q)E_{t^{-1}}(p) - E_e(q)E_e(p))|\eta\rangle\!\rangle.
\end{aligned}$$

Taking adjoints and using the fact that for an element a of a C^*-algebra, $\|aa^*a\| = \|a\|^3$, we conclude that (iii) implies (i). $\square$

Remark 6.6.14. If one uses the modified Fourier transform $\tilde{E}_t(a) := \delta_G(t)^{\frac{1}{2}} E_t(a)$, then one can replace the statement (ii) above by

(ii)' $\lim_{r\to e} \sup_{s,t\in G} \|\tilde{E}_{tr}(p)\tilde{E}_s(q) - \tilde{E}_t(p)\tilde{E}_{rs}(q)\| = 0.$

That is, we do not need to restrict to compact subsets of G. This can be seen from the proof above.

Remark 6.6.15. Let (A, γ_A) be a continuous coaction of G. Recall that the crossed product is defined by

$$A \rtimes_{\mathrm{r}} G = \operatorname{span}\big\{\gamma_A(a)(1_A \otimes M_f) : a \in A, f \in \mathcal{C}_0(G)\big\} \subseteq \mathcal{L}\big(A \otimes L^2(G)\big).$$

The dual action α of G on $A \rtimes_{\mathrm{r}} G$ is given by $\alpha_t(\gamma_A(a)(1_A \otimes M_f)) = \gamma_A(a)(1_A \otimes M_{t\cdot f})$, where $(t\cdot f)(s) := f(st)$. It is easy to see that

$$1_A \otimes M_{t\cdot f} = V_t(1_A \otimes M_f)V_t^{-1}.$$

Since $\gamma_A(a) \in A \otimes C^*_{\mathrm{r}}(G)$ and V_t is in the commutant of $A \otimes C^*_{\mathrm{r}}(G)$, it follows that

$$\alpha_t(T) = V_t T V_t^{-1} \quad \text{for all } T \in A \rtimes_{\mathrm{r}} G.$$

In particular, all elements of $A \rtimes_{\mathrm{r}} G$ are V-continuous. Therefore, if $\xi, \eta \in A_{\mathrm{si}}$, then

$$\xi \overset{rc}{\sim} \eta \quad \Longrightarrow \quad \langle\!\langle \xi \,|\, \eta\rangle\!\rangle \in \mathcal{L}\big(A \otimes L^2(G)\big) \text{ is } V\text{-continuous.}$$

Moreover, for G Abelian, Exel proved in [19] that the converse of the implication above also holds. In general, it is not clear to me whether this is true. Although we do not need this characterization of relative continuity, it would be interesting to see if it is also true in general. Note that if G is discrete, then every operator in $\mathcal{L}\big(L^2(G, A)\big)$ is V-continuous. Hence, in order to show that the converse implication above holds, one has to prove that $\xi \overset{rc}{\sim} \eta$ for all $\xi, \eta \in A_{\mathrm{si}}$. Since in this case $\mathcal{G} = C^*_{\mathrm{r}}(G)$ is a compact quantum group, this follows from Proposition 5.2.12.

6.7 The Fourier inversion theorem

Let G be a locally compact group, let $\mathcal{B}$ be a Fell bundle over G and let $a \in \mathcal{C}_c(\mathcal{B})^2$. Theorem 6.6.4 allows us to rewrite the formula

$$a = \int_G^{\mathrm{s}} a(t)\,\mathrm{d}t = \int_G^{\mathrm{su}} a(t)\,\mathrm{d}t$$

in the form

$$a = \int_G^{\mathrm{su}} E_t(a)\,\mathrm{d}t.$$

The last equation above makes sense for an arbitrary integrable element a of an arbitrary $\widehat{G}$-C^*-algebra A such that $t \mapsto E_t(\xi)$ is strictly-unconditionally integrable. It is the content of this section to investigate when the last equation above holds.

Recall that $B_{\mathrm{r}}(G) \cong C^*_{\mathrm{r}}(G)^*$ denotes the (reduced) Fourier–Stieltjes algebra of G.

Lemma 6.7.1. *Let (A, γ_A) be a $\widehat{G}$-C^*-algebra. If $a \in A_{\mathrm{i}}$ and $\omega \in B_{\mathrm{r}}(G)$, then $\omega * a \in A_{\mathrm{i}}$ and*

$$E_t(\omega * a) = E_t(a)\omega(t) \textit{ for all } t \in G.$$

Proof. We may assume that a and ω are positive and hence $\omega * a$ is positive as well. Note that

$$\begin{aligned}\gamma_A(\omega * a) &= \gamma_A\big((\mathrm{id}_A \otimes \omega)\gamma_A(a)\big)\\ &= (\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega)\big((\gamma_A \otimes \mathrm{id})\gamma_A(a)\big)\\ &= (\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega)\big((\mathrm{id} \otimes \Delta)\gamma_A(a)\big).\end{aligned}$$

Since $a \in A_{\mathrm{i}}^+$, we have $\gamma_A(a) \in \bar{\mathcal{M}}_{\mathrm{id}_A\otimes\varphi}$. Since φ is right invariant, Proposition 4.2.4(ii) says that $(\mathrm{id} \otimes \Delta)\big(\gamma_A(a)\big) \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi\otimes\mathrm{id}_{\mathcal{G}}}$. The calculation above implies $\gamma_A(\omega * a) \in \bar{\mathcal{M}}^+_{\mathrm{id}_A\otimes\varphi}$, that is, $\omega * a \in A_{\mathrm{i}}^+$, and

$$\begin{aligned}E_e(\omega * a) &= (\mathrm{id}_A \otimes \varphi)\big(\gamma_A(\omega * a)\big)\\ &= (\mathrm{id}_A \otimes \varphi)\Big((\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega)\big((\mathrm{id} \otimes \Delta)\gamma_A(a)\big)\Big)\\ &= (\mathrm{id}_A \otimes \omega)\Big((\mathrm{id}_A \otimes \varphi \otimes \mathrm{id}_{\mathcal{G}})\big((\mathrm{id} \otimes \Delta)\gamma_A(a)\big)\Big)\\ &= (\mathrm{id}_A \otimes \omega)\big((\mathrm{id}_A \otimes \varphi)(\gamma_A(a)) \otimes 1_{\mathcal{G}}\big)\\ &= (\mathrm{id}_A \otimes \varphi)\big(\gamma_A(a)\big)\omega(1_{\mathcal{G}}) = E_e(a)\omega(e).\end{aligned}$$

Using Proposition 4.2.4(ii) again, we get the desired result for an arbitrary $t \in G$:

$$\begin{aligned}
E_t(\omega * a) &= (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_t^{-1})\gamma_A(\omega * a)\big) \\
&= (\mathrm{id}_A \otimes \varphi)\Big((\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega)\big((1_A \otimes \lambda_t^{-1} \otimes 1_{\mathcal{G}})(\mathrm{id}_A \otimes \Delta)\gamma_A(a)\big)\Big) \\
&= (\mathrm{id}_A \otimes \varphi)\Big((\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{G}} \otimes \omega\lambda_t)(\mathrm{id}_A \otimes \Delta)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big)\Big) \\
&= (\mathrm{id}_A \otimes \omega\lambda_t)\Big((\mathrm{id}_A \otimes \varphi \otimes \mathrm{id}_{\mathcal{G}})\big((\mathrm{id}_A \otimes \Delta)((1_A \otimes \lambda_t^{-1})\gamma_A(a))\big)\Big) \\
&= (\mathrm{id}_A \otimes \omega\lambda_t)\Big((\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big) \otimes 1_{\mathcal{G}}\Big) \\
&= E_t(a)\omega(t).
\end{aligned}$$

□

Theorem 6.7.2. *Let (A, γ_A) be a $\widehat{G}$-C^*-algebra. Let $a \in A_{\mathrm{i}}$ and suppose that the Fourier transform $G \ni t \mapsto E_t(a) \in \mathcal{M}(A)$ is strictly-unconditionally integrable. Then we have*

$$\gamma_A(a) = \int_G^{\mathrm{su}} E_t(a) \otimes \lambda_t \,\mathrm{d}t.$$

If γ_A is faithful (for example, if G is amenable), then $\int_G^{\mathrm{su}} E_t(a)\,\mathrm{d}t = a$. In general, we have

$$\int_G^{\mathrm{su}} E_t(\omega * a)\,\mathrm{d}t = \omega * a \quad \textit{for all } \omega \in B_{\mathrm{r}}(G).$$

Proof. Since the function $t \mapsto E_t(a)$ is strictly-unconditionally integrable, the same is true for $t \mapsto E_t(a) \otimes \lambda_t = \gamma_A\big(E_t(a)\big)$, and

$$\gamma_A\left(\int_G^{\mathrm{su}} E_t(a)\,\mathrm{d}t\right) = \int_G^{\mathrm{su}} E_t(a) \otimes \lambda_t \,\mathrm{d}t.$$

Take any $\theta \in A^*$ and define $x := (\theta \otimes \mathrm{id})\big(\gamma_A(a)\big)$. Since $\gamma_A(a) \in \bar{\mathcal{M}}_{\mathrm{id}_A \otimes \varphi}$, we have $x \in \bar{\mathcal{M}}_\varphi$. Thus

$$\begin{aligned}
(\theta \otimes \mathrm{id})\left(\int_G^{\mathrm{su}} E_t(a) \otimes \lambda_t \,\mathrm{d}t\right) &= \int_G^{\mathrm{su}} \theta\big(E_t(a)\big)\lambda_t \,\mathrm{d}t \\
&= \int_G^{\mathrm{su}} \theta\Big((\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big)\Big)\lambda_t \,\mathrm{d}t \\
&= \int_G^{\mathrm{su}} \varphi\big(\lambda_t^{-1}(\theta \otimes \mathrm{id})\gamma_A(a)\big)\lambda_t \,\mathrm{d}t \\
&= \int_G^{\mathrm{su}} \varphi(\lambda_t^{-1}x)\lambda_t \,\mathrm{d}t \\
&= \int_G^{\mathrm{su}} \check{x}(t)\lambda_t \,\mathrm{d}t \\
&= x \quad \text{(see Proposition 6.2.4(iv))} \\
&= (\theta \otimes \mathrm{id})\big(\gamma_A(a)\big).
\end{aligned}$$

Since $\theta \in A^*$ is arbitrary, we conclude that

$$\gamma_A(a) = \int_G^{\mathrm{su}} E_t(a) \otimes \lambda_t \,\mathrm{d}t.$$

This implies

$$\gamma_A\left(\int_G^{\mathrm{su}} E_t(a)\,\mathrm{d}t\right) = \int_G^{\mathrm{su}} E_t(a) \otimes \lambda_t \,\mathrm{d}t = \gamma_A(a).$$

Therefore, if γ_A is faithful, then $\int_G^{\mathrm{su}} E_t(a)\,\mathrm{d}t = a$. Finally, if $\omega \in B_{\mathrm{r}}(G)$, then Lemma 6.7.1 yields

$$\begin{aligned}\omega * a &= (\mathrm{id}_A \otimes \omega)\bigl(\gamma_A(a)\bigr)\\ &= (\mathrm{id}_A \otimes \omega)\left(\int_G^{\mathrm{su}} E_t(a) \otimes \lambda_t \,\mathrm{d}t\right)\\ &= \int_G^{\mathrm{su}} E_t(a)\omega(t)\,\mathrm{d}t\\ &= \int_G^{\mathrm{su}} E_t(\omega * a)\,\mathrm{d}t. \qquad \square\end{aligned}$$

Remark 6.7.3. The injectivity of γ_A in Theorem 6.7.2 is really necessary. In fact, if $a \in \ker(\gamma_A)$, then $a \in A_{\mathrm{i}}$ and $E_t(a) = 0$ for all $t \in G$. Thus, if γ_A is not injective, and if $0 \neq a \in \ker(\gamma_A)$, then

$$\int_G^{\mathrm{su}} E_t(a)\,\mathrm{d}t = 0 \neq a.$$

Theorem 6.7.2 generalizes Proposition 6.6 in [19] to non-Abelian groups. Assume that G is Abelian. Then, under the usual identification $\mathcal{M}\bigl(A \otimes C^*_{\mathrm{r}}(G)\bigr) \cong \mathcal{C}_b\bigl(\widehat{G}, \mathcal{M}^{\mathrm{s}}(A)\bigr)$, the element $E_t(a) \otimes \lambda_t$ corresponds to the function $x \mapsto \overline{\langle x\,|\,t\rangle} E_t(a)$. Thus Theorem 6.7.2 says that

$$\int_G^{\mathrm{su}} \overline{\langle x\,|\,t\rangle} E_t(a)\,\mathrm{d}t = \alpha_x(a),$$

where α is the action of $\widehat{G}$ on A corresponding to the coaction γ_A. The Fourier coefficient $E_t(a)$ in this case is given by the integral $\int_{\widehat{G}}^{\mathrm{su}} \langle x\,|\,t\rangle \alpha_x(a)\,\mathrm{d}x$. Thus we can also rewrite the equation above in the form of a generalized Fourier inversion formula:

$$\int_G^{\mathrm{su}} \overline{\langle x\,|\,t\rangle} \left(\int_{\widehat{G}}^{\mathrm{su}} \langle x\,|\,t\rangle \alpha_x(a)\,\mathrm{d}x\right) \mathrm{d}t = \alpha_x(a).$$

6.8 Fell bundles from Hilbert modules over crossed products

In this section we associate Fell bundles to Hilbert modules over crossed products $B \rtimes_{\mathrm{r}} G$, where B is a $\widehat{G}$-C^*-algebra. By Theorem 5.1.2, it suffices to consider concrete Hilbert modules $\mathcal{F} \subseteq \mathcal{L}^{\widehat{G}}\bigl(B \otimes L^2(G), \mathcal{E}\bigr)$, where $\mathcal{E}$ is a Hilbert $B, \widehat{G}$-module.

Given a concrete Hilbert module $\mathcal{F} \subseteq \mathcal{L}^{\widehat{G}}\big(B \otimes L^2(G), \mathcal{E}\big)$ over $B \rtimes_{\mathrm{r}} G$, we define

$$\mathcal{B}_t(\mathcal{F}) := \overline{\mathrm{span}}(\mathcal{F} V_t \mathcal{F}^*) \subseteq \mathcal{L}(\mathcal{E}), \quad t \in G.$$

Recall that $V_t \in \mathcal{L}\big(B \otimes L^2(G)\big)$ is defined by the formula

$$V_t(f)(s) = f(st), \quad \text{for all } f \in \mathcal{C}_c(G, B) \text{ and } s \in G.$$

Note that

$$\mathcal{B}_e(\mathcal{F}) = \overline{\mathrm{span}}\, \mathcal{F}\mathcal{F}^* \cong \mathcal{K}(\mathcal{F}). \tag{6.17}$$

Throughout this section we denote the dual action of G on $B \rtimes_{\mathrm{r}} G$ by β. It is given by

$$\beta_t(x) = V_t x V_{t^{-1}}, \quad x \in B \rtimes_{\mathrm{r}} G.$$

Lemma 6.8.1. *We have*

$$\mathcal{B}_s(\mathcal{F})\mathcal{B}_t(\mathcal{F}) \subseteq \mathcal{B}_{st}(\mathcal{F}) \quad \textit{and} \quad \mathcal{B}_t(\mathcal{F})^* = \mathcal{B}_{t^{-1}}(\mathcal{F})$$

for all $s, t \in G$.

Proof. Since $\mathcal{F}$ is a concrete Hilbert $B \rtimes_{\mathrm{r}} G$-module, we have $\mathcal{F}^*\mathcal{F} \subseteq B \rtimes_{\mathrm{r}} G$. Thus

$$\begin{aligned} \mathcal{B}_s(\mathcal{F})\mathcal{B}_t(\mathcal{F}) &\subseteq \overline{\mathrm{span}}\big(\mathcal{F} V_s \mathcal{F}^* \mathcal{F} V_t \mathcal{F}^*\big) \\ &\subseteq \overline{\mathrm{span}}\big(\mathcal{F} \beta_s\left(\mathcal{F}^*\mathcal{F}\right) V_{st} \mathcal{F}^*\big) \\ &\subseteq \overline{\mathrm{span}}\big(\mathcal{F} \beta_s\left(B \rtimes_{\mathrm{r}} G\right) V_{st} \mathcal{F}^*\big) \\ &= \overline{\mathrm{span}}\big(\mathcal{F}(B \rtimes_{\mathrm{r}} G) V_{st} \mathcal{F}^*\big) \\ &= \overline{\mathrm{span}}\big(\mathcal{F} V_{st} \mathcal{F}^*\big) = \mathcal{B}_{st}(\mathcal{F}). \end{aligned}$$

The second equality follows from the identity $V_t^* = \delta_G(t)^{-1} V_{t^{-1}}$. □

Let $\mathcal{B}(\mathcal{F})$ be the disjoint union of the family of Banach spaces $\{\mathcal{B}_t(\mathcal{F})\}_{t \in G}$. By Lemma 6.8.1, $\mathcal{B}(\mathcal{F})$ forms a Fell bundle over G considered with the discrete topology. In order to turn $\mathcal{B}(\mathcal{F})$ into a Fell bundle over G with its own topology we have to define an appropriate topology on $\mathcal{B}(\mathcal{F})$.

Lemma 6.8.2. *There is a unique topology on $\mathcal{B}(\mathcal{F})$ making it into a continuous Banach bundle such that the sections $t \mapsto x V_t y^*$ are continuous for all $x, y \in \mathcal{F}$.*

Proof. Consider the space Γ of sections of $\mathcal{B}(\mathcal{F})$ spanned by the sections

$$t \mapsto x V_t y^*, \quad \text{for } x, y \in \mathcal{F}.$$

Given $x, y, z, w \in \mathcal{F}$, the function

$$G \ni t \mapsto \left(x V_t y^*\right)\left(z V_t w^*\right)^* = \delta_G(t)^{-1} x \beta_t(y^* w) z^* \in \mathcal{L}(\mathcal{E})$$

is (norm) continuous because $y^* w \in B \rtimes_{\mathrm{r}} G$. Therefore, the function $t \mapsto \|f(t)\|^2 = \|f(t)f(t)^*\|$ is continuous for all $f \in \Gamma$. By [23, II.13.18], there is a unique topology on $\mathcal{B}(\mathcal{F})$ making it into a continuous Banach bundle such that all the sections of Γ are continuous. □

Lemma 6.8.3. *The Banach bundle $\mathcal{B}(\mathcal{F})$ with the topology given in Lemma 6.8.2 and the multiplication and involution induced from $\mathcal{L}(\mathcal{E})$ is a Fell bundle over G.*

Proof. The only non-trivial axioms are the continuity of multiplication and involution. In order to prove the continuity of the multiplication, it is enough to show (by [23, VIII.2.4]) that given sections of the form

$$f(t) := xV_t y^*, \qquad g(t) := zV_t w^*,$$

where $x, y, z, w \in \mathcal{F}$, the map

$$G \times G \ni (s,t) \mapsto f(t)g(t) \in \mathcal{B}(\mathcal{F})$$

is continuous. To prove continuity at a given point $(s_0, t_0) \in G \times G$ we use [23, II.13.12]. Thus we show that there is a continuous section h of $\mathcal{B}(\mathcal{F})$ such that $h(s_0 t_0) = f(s_0)g(s_0)$ and

$$\|h(st) - f(s)g(t)\| \to 0 \quad \text{as } (s,t) \to (s_0, t_0).$$

Define $h(r) := x\beta_{s_0}(y^*z)V_r w^*$. Note that $x\beta_{s_0}(y^*z) \in \mathcal{F}(B \rtimes_{\mathrm{r}} G) = \mathcal{F}$ and hence h is a continuous section of $\mathcal{B}(\mathcal{F})$. Moreover, we have $h(s_0t_0) = f(s_0)g(t_0)$ and

$$\begin{aligned}\|h(st) - f(s)g(t)\| &= \|x\beta_{s_0}(y^*z)V_{st}w^* - xV_s y^* zV_t w^*\| \\ &\leq \delta_G(st)^{-\frac{1}{2}}\|x\| \cdot \|\beta_{s_0}(y^*z) - \beta_s(y^*z)\| \cdot \|w\| \to 0.\end{aligned}$$

To prove that the involution is continuous, it is enough to show that the map

$$G \ni t \mapsto f(t)^* \in \mathcal{B}$$

is continuous for every section of the form $f(t) = xV_t y^*$ with $x, y \in \mathcal{F}$. Since $f(t)^* = \delta_G(t)^{-1} yV_{t^{-1}}x^*$, this follows from the definition of the topology on $\mathcal{B}(\mathcal{F})$, the continuity of δ_G and the continuity of the inversion map on G. □

Proposition 6.8.4. *Suppose that $\mathcal{F}$ is a full Hilbert $B \rtimes_{\mathrm{r}} G$-module. Then $\mathcal{B}(\mathcal{F})$ is a saturated Fell bundle, that is,*

$$\overline{\operatorname{span}}\, \mathcal{B}_s(\mathcal{F})\mathcal{B}_t(\mathcal{F}) = \mathcal{B}_{st}(\mathcal{F}) \quad \textit{for all } s, t \in G.$$

Proof. Since $\mathcal{F}$ is full, we have $\overline{\operatorname{span}}\, \mathcal{F}^*\mathcal{F} = B \rtimes_{\mathrm{r}} G$. Thus

$$\begin{aligned}\overline{\operatorname{span}}\, \mathcal{B}_s(\mathcal{F})\mathcal{B}_t(\mathcal{F}) &= \overline{\operatorname{span}}\big(\mathcal{F}V_s\mathcal{F}^*\mathcal{F}V_t\mathcal{F}^*\big) \\ &= \overline{\operatorname{span}}\big(\mathcal{F}V_s\,(B \rtimes_{\mathrm{r}} G)\,V_t\mathcal{F}^*\big) \\ &= \overline{\operatorname{span}}\big(\mathcal{F}\beta_s(B \rtimes_{\mathrm{r}} G)V_{st}\mathcal{F}^*\big) \\ &= \overline{\operatorname{span}}\big(\mathcal{F}V_{st}\mathcal{F}^*\big) \\ &= \mathcal{B}_{st}(\mathcal{F}).\end{aligned}$$

□

Definition 6.8.5. Given a Hilbert $B,\widehat{G}$-module $\mathcal{E}$ and a relatively continuous subset $\mathcal{R} \subseteq \mathcal{E}$, we define

$$\mathcal{B}(\mathcal{E},\mathcal{R}) := \mathcal{B}\big(\mathcal{F}(\mathcal{E},\mathcal{R})\big).^3$$

Proposition 5.3.2, Equation (5.11) and Corollary 5.4.10 yield

$$\mathcal{B}(\mathcal{E},\mathcal{R}) = \mathcal{B}(\mathcal{E},\overline{\mathcal{R}}^{\mathrm{si}}) = \mathcal{B}(\mathcal{E},A(G)*\mathcal{R}) = \mathcal{B}(\mathcal{E},\mathcal{R}\cdot B) = \mathcal{B}(\mathcal{E},\mathcal{R}_{\mathrm{c}}) = \mathcal{B}(\mathcal{E},\mathcal{R}_{\mathrm{sc}}).$$

Thus, in general, there are many relatively continuous subspaces generating the same Fell bundle.

Proposition 6.8.6. *Let $\mathcal{F} \subseteq \mathcal{L}^{\widehat{G}}\big(B\otimes L^2(G),\mathcal{E}\big)$ be a concrete Hilbert $B,\widehat{G}$-module, and suppose that $\mathcal{F}_0$ is a dense subset of $\mathcal{F}$. Then*

$$\mathcal{B}_t(\mathcal{F}) = \overline{\operatorname{span}}\{xV_ty^* : x,y \in \mathcal{F}_0\} \quad \textit{for all } t \in G.$$

The topology is determined by the continuous sections $t \mapsto xV_ty^*$, $x,y \in \mathcal{F}_0$.

Proof. The description of the fibers follows from the definition of $\mathcal{B}(\mathcal{F})$. The last assertion follows from the observation that the topology of a continuous Banach bundle is determined by any pointwise dense subspace of continuous sections (see [23, II.13.18]). □

Corollary 6.8.7. *Let $\mathcal{E}$ be a Hilbert $B,\widehat{G}$-module and assume that $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$ is relatively continuous. Then*

$$\mathcal{B}_t(\mathcal{E},\mathcal{R}) = \overline{\operatorname{span}}\{|\xi\rangle\!\rangle V_t \langle\!\langle \eta| : \xi,\eta \in \mathcal{R}_0\} = \overline{\{E_t(a) : a \in \mathcal{W}_0\}},$$

where $\mathcal{W}_0 := \operatorname{span}|\mathcal{R}_0\rangle\langle\mathcal{R}_0| \subseteq \mathcal{K}(\mathcal{E})$ and $\mathcal{R}_0$ is any relatively continuous subset of $\mathcal{E}$ such that $|\mathcal{R}_0\rangle\!\rangle$ is dense in $\mathcal{F}(\mathcal{E},\mathcal{R})$. The topology of $\mathcal{B}(\mathcal{E},\mathcal{R})$ is determined by the continuous sections $t \mapsto E_t(a)$, $a \in \mathcal{W}_0$.

Proof. Corollary 6.6.9 yields $E_t(|\xi\rangle\langle\eta|) = |\xi\rangle\!\rangle V_t \langle\!\langle \eta|$ for all $\xi,\eta \in \mathcal{E}_{\mathrm{si}}$. The assertions now follow from Proposition 6.8.6. □

Corollary 6.8.8. *Let $(\mathcal{E},\mathcal{R})$ be a continuously square-integrable Hilbert $B,\widehat{G}$-module. Then the generalized fixed point algebra* $\operatorname{Fix}(\mathcal{E},\mathcal{R})$ *is equal to the unit fiber $\mathcal{B}_e(\mathcal{E},\mathcal{R})$ of $\mathcal{B}(\mathcal{E},\mathcal{R})$.*

Proof. This follows from Corollaries 5.3.4 and 6.8.7. □

6.9 Fell bundle structures

Definition 6.9.1. Let A be a $\widehat{G}$-C^*-algebra.

(i) A *full Fell bundle structure* for A is a pair $(\mathcal{B},\pi)$ consisting of a Fell bundle $\mathcal{B}$ over G and a $\widehat{G}$-equivariant $*$-isomorphism $\pi : C^*(\mathcal{B}) \to A$.

[3] For the definition of $\mathcal{F}(\mathcal{E},\mathcal{R})$, see Definition 5.2.1.

(ii) A *reduced Fell bundle structure* for A is a pair $(\mathcal{B}, \pi)$, where $\mathcal{B}$ is a Fell bundle over G and $\pi : C^*_{\mathrm{r}}(\mathcal{B}) \to A$ is a $\widehat{G}$-equivariant $*$-isomorphism.

Since $C^*_{\mathrm{r}}(\mathcal{B})$ is a reduced $\widehat{G}$-C^*-algebra, reduced Fell bundle structures can only exist for reduced $\widehat{G}$-C^*-algebras, that is, for $\widehat{G}$-C^*-algebras with an injective (continuous) coaction of G. A full Fell bundle structure for a $\widehat{G}$-C^*-algebra A can only exist if the coaction is maximal, at least if G is discrete. In fact, for discrete groups it was proved in [14] that the dual coaction on $C^*(\mathcal{B})$ is maximal and that any maximal coaction of a discrete group has this form. However, for non-discrete groups, it is not known whether the dual coaction on $C^*(\mathcal{B})$ is maximal.

Of course, if G is amenable, then all these problems disappear. In this case, all the coactions of G are reduced and maximal at the same time and all Fell bundles over G are amenable, that is, $\lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_{\mathrm{r}}(\mathcal{B})$ is a $\widehat{G}$-equivariant isomorphism. Therefore, in this case, there is no difference between full and reduced Fell bundle structures and we can forget the words full and reduced.

In [9] we consider only Abelian groups and view Fell bundle structures for a $\widehat{G}$-C^*-algebra A as "continuous spectral decompositions" of the underlying action of $\widehat{G}$ on A, following Exel's treatment in [19]. However, for non-Abelian groups the interpretation as a spectral decomposition is missing, and we prefer to use the terminology of Fell bundle structures.

Let $(\mathcal{E}, \mathcal{R})$ be a continuously square-integrable Hilbert $B, \widehat{G}$-module. In Section 6.8, we have constructed a Fell bundle $\mathcal{B} := \mathcal{B}(\mathcal{E}, \mathcal{R})$ over G whose fibers are given by

$$\mathcal{B}_t = \overline{\operatorname{span}}\{|\xi\rangle\!\rangle V_t \langle\!\langle\eta| : \xi, \eta \in \mathcal{R}\} = \overline{\{E_t(a) : a \in \mathcal{W}_{\mathcal{R}}\}},$$

where $\mathcal{W}_{\mathcal{R}} := \operatorname{span}|\mathcal{R}\rangle\langle\mathcal{R}| \subseteq \mathcal{K}(\mathcal{E})$. The topology on $\mathcal{B}$ is determined by the continuous sections $t \mapsto E_t(a)$ for $a \in \mathcal{W}_{\mathcal{R}}$.

In what follows, we show that $\mathcal{B}$ determines a full (resp. reduced) Fell bundle structure for the $\widehat{G}$-C^*-algebra $A := \mathcal{K}(\mathcal{E})$, provided the coaction on A is maximal (resp. reduced).

By definition, each fiber of $\mathcal{B}$ is contained in $\mathcal{L}(\mathcal{E})$. Thus we can consider the inclusion map

$$\kappa : \mathcal{B} \to \mathcal{L}(\mathcal{E}).$$

Proposition 6.9.2. *With the notations above, the map κ is a representation of $\mathcal{B}$ on $\mathcal{E}$. Its integrated form $\kappa : C^*(\mathcal{B}) \to \mathcal{L}(\mathcal{E})$ is given by the formula*

$$\kappa(\xi) = \int_G^{\mathrm{s}} \xi(t)\,\mathrm{d}t \quad \textit{for all } \xi \in L^1(\mathcal{B}),$$

where we view $\mathcal{B}_t$ as a subspace of $\mathcal{L}(\mathcal{E})$, so that the strict integral above gives rise to an element in $\mathcal{M}\big(\mathcal{K}(\mathcal{E})\big) = \mathcal{L}(\mathcal{E})$.

Proof. The only non-trivial axiom is 6.3.3(iv). To prove it, let $\eta \in \mathcal{E}$ and suppose that $\{b_i\}$ is a net in $\mathcal{B}$ converging to $b_0 \in \mathcal{B}$. Let $t_i \in G$ with $b_i \in \mathcal{B}_{t_i}$. Then we have $t_i \to t_0$.

Fix $\epsilon > 0$ and take a continuous section ξ of $\mathcal{B}$ of the form $\xi(t) = E_t(a)$ with $a \in \mathcal{W}_{\mathcal{R}}$ such that $\|\xi(t_0) - b_0\| < \epsilon$. Corollary 6.6.10 yields

$$\begin{aligned}\|b_i\eta - b_0\eta\| &\leq \|b_i\eta - \xi(t_i)\eta\| + \|\xi(t_i)\eta - \xi(t_0)\eta\| + \|\xi(t_0)\eta - b_0\eta\| \\ &\leq \|b_i - \xi(t_i)\|\|\eta\| + \|E_{t_i}(a)\eta - E_{t_0}(a)\eta\| + \|\xi(t_0) - b_0\|\|\eta\| \\ &\to \|b_0 - \xi(t_0)\|\|\eta\| + 0 + \|b_0 - \xi(t_0)\|\|\eta\| < 2\epsilon\|\eta\|.\end{aligned}$$

The formula for the integrated form follows from the general formula

$$\kappa(\xi) = \int_G^{\mathrm{s}} \kappa(\xi(t))\,\mathrm{d}t$$

which holds for any representation κ (see Equation (6.7)). Since we are identifying $\mathcal{B}_t \subseteq \mathcal{L}(\mathcal{E})$, we have $\kappa(\xi(t)) = \xi(t)$ by definition of κ. $\square$

Lemma 6.9.3. *Let $A_c(G)$ denote the space of compactly supported functions in $A(G)$ (the Fourier algebra). The space $\mathcal{J}(\mathcal{B})$ spanned by the sections*

$$\{t \mapsto \omega(t)E_t(a) : \omega \in A_c(G), a \in \mathcal{W}_{\mathcal{R}}\}$$

is dense in $\mathcal{C}_c(\mathcal{B})$ with respect to the inductive limit topology.

Proof. Since $t \mapsto E_t(a)$ is a continuous section of $\mathcal{B}$, and since the space of continuous sections is a $\mathcal{C}(G)$-module, it follows that $\mathcal{J}(\mathcal{B}) \subseteq \mathcal{C}_c(\mathcal{B})$. Since $A_c(G)$ contains $\mathcal{C}_c(G) * \mathcal{C}_c(G)$, which is dense in $\mathcal{C}_c(G)$ with respect to the inductive limit topology, it follows that the closure of $\mathcal{J}(\mathcal{B})$ in $\mathcal{C}_c(\mathcal{B})$ contains the space spanned by the sections $t \mapsto f(t)E_t(a)$, where $f \in \mathcal{C}_c(G)$ and $a \in \mathcal{W}$. This space is dense in $\mathcal{C}_c(\mathcal{B})$ by [23, II.14.6]. $\square$

Proposition 6.9.4. *Let $(\mathcal{E}, \mathcal{R})$ be a continuously square-integrable Hilbert $B, \widehat{G}$-module. Then, with the notations above, $\kappa : C^*(\mathcal{B}) \to \mathcal{L}(\mathcal{E})$ is a $\widehat{G}$-equivariant nondegenerate $*$-homomorphism, whose image is the $\widehat{G}$-C^*-algebra $A := \mathcal{K}(\mathcal{E})$. Moreover, we have*

$$\kappa\left(\mathcal{J}(\mathcal{B})\right) = \operatorname{span}\left(A_c(G) * \mathcal{W}_{\mathcal{R}}\right).$$

Proof. Consider the section $\xi \in \mathcal{J}(\mathcal{B})$ given by $\xi(t) := \omega(t)E_t(a) = E_t(\omega * a)$, where $\omega \in A_c(G)$ and $a \in \mathcal{W}_{\mathcal{R}}$. Theorem 6.7.2 and Proposition 6.9.2 yield

$$\kappa(\xi) = \int_G^{\mathrm{s}} \xi(t)\,\mathrm{d}t = \int_G^{\mathrm{s}} E_t(\omega * a)\,\mathrm{d}t = \omega * a.$$

Thus

$$\kappa\left(\mathcal{J}(\mathcal{B})\right) = \operatorname{span}\left(A_c(G) * \mathcal{W}_{\mathcal{R}}\right).$$

Lemma 6.9.3 yields $\kappa(C^*(\mathcal{B})) = \mathcal{K}(\mathcal{E})$. In particular, κ is nondegenerate. To prove the $\widehat{G}$-equivariance of κ, we use Theorem 6.7.2 again:

$$\begin{aligned}(\kappa \otimes \mathrm{id})\big(\gamma_{\mathcal{B}}(\xi)\big) &= (\kappa \otimes \mathrm{id}) \left(\int_G^{\mathrm{s}} \xi(t) \otimes \lambda_t \,\mathrm{d}t \right) \\ &= \int_G^{\mathrm{s}} \kappa\big(\xi(t)\big) \otimes \lambda_t \,\mathrm{d}t \\ &= \int_G^{\mathrm{s}} E_t(\omega * a) \otimes \lambda_t \,\mathrm{d}t \\ &= \gamma_{\mathcal{K}(\mathcal{E})}(\omega * a) = \gamma_{\mathcal{K}(\mathcal{E})}\big(\kappa(\xi)\big). \end{aligned}$$ □

We need the following result from [21, Corollary 2.15]:

Lemma 6.9.5. *Let $\mathcal{B}$ be a Fell bundle over G and let $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{E})$ be a representation of $\mathcal{B}$ on a Hilbert B-module $\mathcal{E}$. Then the representation $\rho \otimes \lambda : \mathcal{B} \to \mathcal{L}(\mathcal{E} \otimes C^*_{\mathrm{r}}(G))$ given by $(\rho \otimes \lambda)(b_t) = b_t \otimes \lambda_t$ factors through $C^*_{\mathrm{r}}(\mathcal{B})$, that is, there is a $*$-homomorphism $\varrho : C^*_{\mathrm{r}}(\mathcal{B}) \to \mathcal{L}(\mathcal{E} \otimes C^*_{\mathrm{r}}(G))$ such that $\varrho \circ \lambda_{\mathcal{B}} = \rho \otimes \lambda$. Moreover, if $\rho|_{\mathcal{B}_e}$ is faithful, then so is ϱ.*

Proposition 6.9.6. *In the situation of Proposition 6.9.4, suppose that A is a reduced $\widehat{G}$-C^*-algebra. Then the $*$-homomorphism $\kappa : C^*(\mathcal{B}) \to A$ factors through a $\widehat{G}$-equivariant $*$-isomorphism $\pi : C^*_{\mathrm{r}}(\mathcal{B}) \to A$. In other words, $(\mathcal{B}, \pi)$ is a reduced Fell bundle structure for A such that the following diagram commutes:*

$$\begin{tikzcd} C^*(\mathcal{B}) \arrow[r, "\kappa"] \arrow[d, "\lambda_{\mathcal{B}}"'] & A \\ C^*_{\mathrm{r}}(\mathcal{B}) \arrow[ur, "\pi"'] & \end{tikzcd}$$

Proof. Consider the representation

$$\kappa \otimes \lambda : \mathcal{B} \to \mathcal{M}\big(A \otimes C^*_{\mathrm{r}}(G)\big) = \mathcal{L}\big(\mathcal{E} \otimes C^*_{\mathrm{r}}(G)\big)$$

defined by $(\kappa \otimes \lambda)(b_t) := \kappa(b_t) \otimes \lambda_t$. Let $\xi\mathcal{C}_c(\mathcal{B})$ be the section defined by $\xi(t) := \omega(t)E_t(a)$, where $\omega \in A_c(G)$ and $a \in \mathcal{W}_{\mathcal{R}}$. Lemma 6.7.1 and Theorem 6.7.2 yield

$$\begin{aligned}(\kappa \otimes \lambda)\xi &= \int_G^{\mathrm{s}} \kappa(\xi(t)) \otimes \lambda_t \,\mathrm{d}t \\ &= \int_G^{\mathrm{s}} \omega(t) E_t(a) \otimes \lambda_t \,\mathrm{d}t \\ &= \int_G^{\mathrm{s}} E_t(\omega * a) \otimes \lambda_t \,\mathrm{d}t \\ &= \gamma_A(\omega * a) = \gamma_A(\kappa(\xi)). \end{aligned}$$

By Lemma 6.9.3, the space $\mathcal{J}(\mathcal{B})$ spanned by the section of the form $t \mapsto \omega(t)E_t(a)$ above is dense in $C^*(\mathcal{B})$. The calculation above implies $\kappa \otimes \lambda = \gamma_A \circ \kappa$. This yields the following commutative diagram:

$$\begin{tikzcd}
C^*(\mathcal{B}) \arrow[r, "\kappa\otimes\lambda"] \arrow[d, "\kappa"'] & \mathcal{M}\big(A \otimes C^*_{\mathrm{r}}(G)\big) \\
A \arrow[ur, "\gamma_A"] &
\end{tikzcd}$$

By definition, κ is the inclusion map $\mathcal{B} \to \mathcal{M}(A)$. In particular, $\kappa_{|\mathcal{B}_e} : \mathcal{B}_e \to \mathcal{M}(A)$ is faithful. Lemma 6.9.5 implies that the representation $\kappa \otimes \lambda$ factors faithfully through $C^*_{\mathrm{r}}(\mathcal{B})$, that is, there is a faithful $*$-homomorphism $\varrho : C^*_{\mathrm{r}}(\mathcal{B}) \to \mathcal{M}\big(A \otimes C^*_{\mathrm{r}}(G)\big)$ making the following diagram commute:

$$\begin{tikzcd}
C^*_{\mathrm{r}}(\mathcal{B}) \arrow[dr, dotted, "\varrho"] & \\
C^*(\mathcal{B}) \arrow[u, "\lambda_{\mathcal{B}}"] \arrow[r, "\kappa\otimes\lambda"] \arrow[d, "\kappa"'] & \mathcal{M}\big(A \otimes C^*_{\mathrm{r}}(G)\big) \\
A \arrow[ur, "\gamma_A"] &
\end{tikzcd}$$

Since $\lambda_{\mathcal{B}}$ and κ are surjective, the diagram above implies that $\varrho\big(C^*_{\mathrm{r}}(\mathcal{B})\big) = \gamma_A(A)$. Since γ_A is injective, the equation $\gamma_A\big(\pi(x)\big) = \varrho(x)$ well-defines a surjective $*$-homomorphism $\pi : C^*_{\mathrm{r}}(\mathcal{B}) \to A$. Since ϱ and γ_A are injective, π is injective as well. Moreover, it is easy to see from the diagram above that $\pi \circ \lambda_{\mathcal{B}} = \kappa$, that is, the diagram

$$\begin{tikzcd}
C^*(\mathcal{B}) \arrow[r, "\kappa"] \arrow[d, "\lambda_{\mathcal{B}}"'] & A \\
C^*_{\mathrm{r}}(\mathcal{B}) \arrow[ur, "\pi"] &
\end{tikzcd}$$

commutes. Since κ and $\lambda_{\mathcal{B}}$ are equivariant, and since $\lambda_{\mathcal{B}}$ is surjective, π is necessarily equivariant. $\square$

Let (A, γ_A) be a $\widehat{G}$-C^*-algebra. Recall that a reduction of A is a reduced $\widehat{G}$-C^*-algebra $(A_{\mathrm{r}}, \gamma_A^{\mathrm{r}})$ together with a $\widehat{G}$-equivariant surjection $\vartheta : A \to A_{\mathrm{r}}$ such that the induced map $\vartheta \rtimes G : A \rtimes G \to A_{\mathrm{r}} \rtimes G$ is an isomorphism.

Proposition 6.9.6 can be generalized in the following way:

Proposition 6.9.7. *In the situation of Proposition* 6.9.4*, let* $\vartheta : A \to A_{\mathrm{r}}$ *be a reduction of* A*. Then there is a* $\widehat{G}$*-equivariant isomorphism* $\pi : C^*_{\mathrm{r}}(\mathcal{B}) \to A_{\mathrm{r}}$ *making the following diagram commute:*

$$\begin{tikzcd} C^*(\mathcal{B}) \arrow[r, "\kappa"] \arrow[d, "\lambda_{\mathcal{B}}"'] & A \arrow[d, "\vartheta"] \\ C^*_{\mathrm{r}}(\mathcal{B}) \arrow[r, "\pi"] & A_{\mathrm{r}} \end{tikzcd}$$

Proof. Recall that κ denotes the inclusion map $\mathcal{B} \to \mathcal{M}(A)$. Let $\tilde{\kappa} := \vartheta \circ \kappa : \mathcal{B} \to \mathcal{M}(A_{\mathrm{r}})$. Then $\tilde{\kappa}$ is a representation of $\mathcal{B}$ on A_{r}. Its integrated form $\tilde{\kappa} : C^*(\mathcal{B}) \to \mathcal{M}(A_{\mathrm{r}})$ has image $\tilde{\kappa}\big(C^*(\mathcal{B})\big) = \vartheta(A) = A_{\mathrm{r}}$. In particular, $\tilde{\kappa}$ is nondegenerate. We claim that $\tilde{\kappa}|_{\mathcal{B}_e}$ is faithful. In fact, suppose that $b \in \mathcal{B}_e$ and $\tilde{\kappa}(b) = \vartheta(b) = 0$. Then $(\vartheta \rtimes G)\big(\gamma_{\mathcal{B}}(b)\big) = \gamma_{A_{\mathrm{r}}}\big(\vartheta(b)\big) = 0$. Since $\vartheta \rtimes G$ is an isomorphism, we get $\gamma_{\mathcal{B}}(b) = 0$. But $b \in \mathcal{B}_e \subseteq \mathcal{M}_e\big(C^*(\mathcal{B})\big) = \mathcal{M}_1\big(C^*(\mathcal{B})\big)$, the fixed point algebra, and hence $\gamma_{\mathcal{B}}(b) = b \otimes 1$. Thus $b = 0$, proving our claim. Now we can define the representation $\tilde{\kappa} \otimes \lambda : \mathcal{B} \to \mathcal{M}\big(A_{\mathrm{r}} \otimes C^*_{\mathrm{r}}(G)\big)$ of $\mathcal{B}$ and follow the same idea as in the proof of Proposition 6.9.6 to get the desired isomorphism $\pi : C^*_{\mathrm{r}}(\mathcal{B}) \to A_{\mathrm{r}}$. ☐

Proposition 6.9.8. *Let* $\mathcal{B}$ *be a Fell bundle over* G *and consider the* $\widehat{G}$*-*C^**-algebra* $A := C^*(\mathcal{B})$*. Then the reduction* A_{r} *of* A *is isomorphic to* $C^*_{\mathrm{r}}(\mathcal{B})$*. The quotient map* $\vartheta : A \to A_{\mathrm{r}}$ *is identified with* $\lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_{\mathrm{r}}(\mathcal{B})$*. In particular,* $\ker(\gamma_{\mathcal{B}}) = \ker(\lambda_{\mathcal{B}})$*, and*

$$\lambda_{\mathcal{B}} \rtimes G : C^*(\mathcal{B}) \rtimes G \to C^*_{\mathrm{r}}(\mathcal{B}) \rtimes G$$

is an isomorphism.

Proof. Consider the inclusion map $\kappa : \mathcal{B} \to \mathcal{M}(A)$. As we have seen in the proof of Proposition 6.9.7, the composition $\tilde{\kappa} := \vartheta \circ \kappa : \mathcal{B} \to \mathcal{M}(A_{\mathrm{r}})$ is faithful on $\mathcal{B}_e$. Now considering the representation $\tilde{\kappa} \otimes \lambda : \mathcal{B} \to \mathcal{M}\big(A \otimes C^*_{\mathrm{r}}(G)\big)$, the same idea as in the proof of Proposition 6.9.6 shows that there is a $\widehat{G}$-equivariant isomorphism $\pi : C^*_{\mathrm{r}}(\mathcal{B}) \to A_{\mathrm{r}}$ with $\pi \circ \lambda_{\mathcal{B}} = \vartheta$. ☐

Recall that a Fell bundle is called amenable if $\lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_{\mathrm{r}}(\mathcal{B})$ is injective.

Corollary 6.9.9. *The dual coaction* $(C^*(\mathcal{B}), \gamma_{\mathcal{B}})$ *is reduced if and only if* $\mathcal{B}$ *is an amenable Fell bundle.*

The following theorem summarizes the results of this section. If G is discrete, then this has been proved in [13, Lemma 2.1].

Theorem 6.9.10. *Let* $(\mathcal{E}, \mathcal{R})$ *be a continuously square-integrable Hilbert* $B, \widehat{G}$*-module. Define* $\mathcal{B} := \mathcal{B}(\mathcal{E}, \mathcal{R})$ *and* $A := \mathcal{K}(\mathcal{E})$*, and let* $\kappa : \mathcal{B} \to \mathcal{M}(A)$ *be the inclusion map. Then* *the integrated form of κ is a $\widehat{G}$-equivariant surjection $\kappa : C^*(\mathcal{B}) \to A$, and there is a $\widehat{G}$-equivariant surjection $\nu : A \to C^*_{\mathrm{r}}(\mathcal{B})$ with $\lambda_{\mathcal{B}} = \nu \circ \kappa$. Moreover, the induced maps*

$$\kappa \rtimes G : C^*(\mathcal{B}) \rtimes G \to A \rtimes G \qquad \nu \rtimes G : A \rtimes G \to C^*_{\mathrm{r}}(\mathcal{B}) \rtimes G$$

$$\lambda_{\mathcal{B}} \rtimes G : C^*(\mathcal{B}) \rtimes G \to C^*_{\mathrm{r}}(\mathcal{B}) \rtimes G$$

are isomorphisms and we have $\lambda_{\mathcal{B}} \rtimes G = (\nu \rtimes G) \circ (\kappa \rtimes G)$. If (A, γ_A) is reduced, then ν is an isomorphism, that is, $(\mathcal{B}, \nu)$ is a reduced Fell bundle structure for A. And if (A, γ_A) is maximal, then κ is an isomorphism, that is, $(\mathcal{B}, \kappa)$ is a full Fell bundle structure for A.

Proof. Let ν be the composition $\pi^{-1} \circ \vartheta$ of Proposition 6.9.7. The equality $\lambda_{\mathcal{B}} = \nu \circ \kappa$ implies $\lambda_{\mathcal{B}} \rtimes G = (\nu \rtimes G) \circ (\kappa \rtimes G)$. Since $\lambda_{\mathcal{B}} \rtimes G$ is injective, so are $\nu \rtimes G$ and $\kappa \rtimes G$, and therefore they are isomorphisms. Since $C^*_{\mathrm{r}}(\mathcal{B})$ is reduced, $\nu : A \to C^*_{\mathrm{r}}(\mathcal{B})$ is a reduction of A. In particular, if A is reduced, then ν is an isomorphism. Now suppose that A is maximal. Take a maximalization $\mu : A_m \to C^*(\mathcal{B})$ of $(C^*(\mathcal{B}), \gamma_{\mathcal{B}})$. Since $\kappa \rtimes G : C^*(\mathcal{B}) \rtimes G \to A \rtimes G$ is an isomorphism, $\kappa \circ \mu : A_m \to A$ is a maximalization of (A, γ_A). Since (A, γ_A) is assumed to be maximal, uniqueness of maximalizations implies that $\kappa \circ \mu$ is an isomorphism (see also [34, Proposition 3.3]). Hence κ is an isomorphism as well. □

Note that in the situation above, $\nu : A \to C^*_{\mathrm{r}}(\mathcal{B})$ is always a reduction of (A, γ_A) and $\kappa : C^*(\mathcal{B}) \to A$ is maximalization of (A, γ_A) whenever $(C^*(\mathcal{B}), \gamma_{\mathcal{B}})$ is maximal. As already mentioned, it is not known whether $(C^*(\mathcal{B}), \gamma_{\mathcal{B}})$ is maximal for every Fell bundle over G. This is true for discrete groups (see [14]) and, of course, also for amenable groups. In the case of classical dual coactions (that is, if $\mathcal{B} = C \times_\gamma G$ is the semidirect product of some action (C, γ) of G) it is also true for all locally compact groups. We shall use the following terminology:

Definition 6.9.11. We say that a Fell bundle $\mathcal{B}$ over G has the *maximality property* if the dual coaction $(C^*(\mathcal{B}), \gamma_{\mathcal{B}})$ is maximal. We also say that G has the *maximality property* if every Fell bundle $\mathcal{B}$ over G has the maximality property.

By Proposition 6.9.8, the map $\lambda_{\mathcal{B}} \rtimes G : C^*(\mathcal{B}) \rtimes G \to C^*_{\mathrm{r}}(\mathcal{B}) \rtimes G$ is an isomorphism. Thus, if $\mathcal{B}$ has the maximality property, then $\lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_{\mathrm{r}}(\mathcal{B})$ is a maximalization of $(C^*_{\mathrm{r}}(\mathcal{B}), \gamma^{\mathrm{r}}_{\mathcal{B}})$.

The following result follows directly from Theorem 6.9.10.

Corollary 6.9.12. *Let $\mathcal{B}$ and A be as in Theorem* 6.9.10. *If A is reduced and maximal, then $\mathcal{B}$ is amenable. Conversely, if $\mathcal{B}$ has the maximality property and if $\mathcal{B}$ is amenable, then A is reduced and maximal.*

6.10 Fell bundles and continuously square-integrable C^*-algebras

Let $\mathcal{B}$ be a Fell bundle over G. By Theorem 6.5.3 and Corollary 6.5.4, we can associate to $\mathcal{B}$ two continuously square-integrable $\widehat{G}$-C^*-algebras

$$\big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big) := \big(C^*(\mathcal{B}), \overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}}\big) \quad \text{and} \quad \big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big) := \big(C^*_{\mathrm{r}}(\mathcal{B}), \overline{\lambda_{\mathcal{B}}(\mathcal{C}_c(\mathcal{B}))}^{\mathrm{si}}\big),$$

where we always furnish $C^*(\mathcal{B})$ and $C^*_{\mathrm{r}}(\mathcal{B})$ with the dual coactions of G. Thus we can consider the maps

$$\mathcal{B} \mapsto \big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big) \quad \text{and} \quad \mathcal{B} \mapsto \big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big).$$

These two maps are considered between Fell bundles over G and continuously square-integrable $\widehat{G}$-C^*-algebras. The aim of this section is to prove that, under certain hypotheses, they are equivalences between suitable categories. Firstly, we define the categories and prove that the maps above give rise to functors between these categories. The first one is the category of Fell bundles over G with the usual morphisms:

Definition 6.10.1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Fell bundles over G. A *morphism* from $\mathcal{B}_1$ to $\mathcal{B}_2$ is a continuous map $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ satisfying $\phi(\mathcal{B}_{1,t}) \subseteq \mathcal{B}_{2,t}$ for all $t \in G$, which is linear on the fibers, norm decreasing, and preserves multiplication and involution.

The second category that we need is the category of continuously square-integrable $\widehat{G}$-C^*-algebras with morphisms as follows:

Definition 6.10.2. Let $(A_1, \mathcal{R}_1)$ and $(A_2, \mathcal{R}_2)$ be continuously square-integrable $\widehat{G}$-C^*-algebras. A *morphism* from $(A_1, \mathcal{R}_1)$ to $(A_2, \mathcal{R}_2)$ is a $\widehat{G}$-equivariant $*$-homomorphism $\pi : A_1 \to A_2$ such that $\pi(\mathcal{R}_1) \subseteq \mathcal{R}_2$.

Notice that we do not require $\pi : A_1 \to A_2$ to be nondegenerate. Thus it need not extend to a strictly continuous $*$-homomorphism between the multiplier algebras. To circumvent this issue, we shall work with the bidual von Neumann algebras. Given a C^*-algebra A, we denote its bidual von Neumann algebra by A'' (also called the von Neumann enveloping algebra of A, see [58]). This can be concretely defined as the bicommutant of A in its universal representation. As a Banach space, A'' is isomorphic to the second dual A^{**} ([58, Proposition 3.7.8]). We always identify $A \subseteq \mathcal{M}(A) \subseteq A''$ in the usual way ([58, 3.12.4]). The assignment $A \mapsto A''$ is functorial. Given a $*$-homomorphism $\pi : A_1 \to A_2$, we denote its bi-transpose by $\pi'' : A_1'' \to A_2''$. It is the unique weakly continuous $*$-homomorphism extending π.

Lemma 6.10.3. *Let A_1 and A_2 be $\widehat{G}$-C^*-algebras, let $\mathcal{R}_1 \subseteq A_1$ and $\mathcal{R}_2 \subseteq A_2$ be relatively continuous subspaces, and assume that $\pi : A_1 \to A_2$ is a $\widehat{G}$-equivariant $*$-homomorphism with $\pi(\mathcal{R}_1) \subseteq \mathcal{R}_2$. Then*:

(i) $\pi''\big(E_t(a)\big) = E_t\big(\pi(a)\big)$ *for all $a \in \mathcal{R}_1\mathcal{R}_1^*$.*

(ii) π *is contractive for the* si*-norm on $\mathcal{R}_1$, that is, $\|\pi(\xi)\|_{\mathrm{si}} \leq \|\xi\|_{\mathrm{si}}$ for all $\xi \in \mathcal{R}_1$.*

Proof. (i) By definition, we have

$$E_t(a) = (\mathrm{id} \otimes \varphi)\big((1 \otimes \lambda_t^{-1})\gamma_{A_1}(a)\big) = \operatorname*{s-lim}_{\omega \in \mathcal{G}_\varphi} (\mathrm{id} \otimes \omega)\big((1 \otimes \lambda_t^{-1})\gamma_{A_1}(a)\big).^4$$

[4] Recall that the script "s" denotes strict limit. See Section 2.4 for the definition of $\mathcal{G}_\varphi$.

Note that $(\mathrm{id}\otimes\omega)\big((1\otimes\lambda_t^{-1})\gamma_{A_1}(a)\big)\in A_1$ for all $\omega\in\mathcal{G}_\varphi$. Since π'' is weakly continuous, and since the weak topology is weaker than the strict topology, we get

$$\pi''\big(E_t(a)\big)=\operatorname*{w-lim}_{\omega\in\mathcal{G}_\varphi}\pi\Big((\mathrm{id}\otimes\omega)\big(1\otimes\lambda_t^{-1})\gamma_{A_1}(a)\big)\Big),$$

where the script "w" stands for weak limit. Finally, since π is equivariant, we have

$$\begin{aligned}E_t\big(\pi(a)\big)&=\operatorname*{s-lim}_{\omega\in\mathcal{G}_\varphi}(\mathrm{id}\otimes\omega)\big(\gamma_{A_2}(\pi(a))\big)\\&=\operatorname*{s-lim}_{\omega\in\mathcal{G}_\varphi}(\mathrm{id}\otimes\omega)\big((\pi\otimes\mathrm{id})\gamma_{A_1}(a)\big)\\&=\operatorname*{s-lim}_{\omega\in\mathcal{G}_\varphi}\pi\Big((\mathrm{id}\otimes\omega)\big((1\otimes\lambda_t^{-1})\gamma_{A_1}(a)\big)\Big).\end{aligned}$$

Therefore, $E_t\big(\pi(a)\big)=\pi''\big(E_t(a)\big)$ as desired.

(ii) It follows from (i) that

$$\big\|E_e\big(\pi(\xi\xi^*)\big)\big\|=\big\|\pi''\big(E_e(\xi\xi^*)\big)\big\|\leq\|E_e(\xi\xi^*)\|\,.$$

Using Proposition 4.1.10(i), we conclude the proof:

$$\begin{aligned}\|\pi(\xi)\|_{\mathrm{si}}&=\|\pi(\xi)\|+\big\||\pi(\xi)\rangle\!\rangle\big\|=\|\pi(\xi)\|+\big\||\pi(\xi)\rangle\!\rangle\langle\!\langle\pi(\xi)|\big\|^{\frac{1}{2}}\\&=\|\pi(\xi)\|+\big\|E_e\big(\pi(\xi\xi^*)\big)\big\|^{\frac{1}{2}}\leq\|\xi\|+\|E_e(\xi\xi^*)\|^{\frac{1}{2}}=\|\xi\|_{\mathrm{si}}.\end{aligned}$$ □

Remark 6.10.4. If $\pi:A_1\to A_2$ in Lemma 6.10.3 is nondegenerate, then the hypothesis $\pi(\mathcal{R}_1)\subseteq\mathcal{R}_2$ is not necessary and the properties (i) and (ii) hold for any $a\in A_{1,\mathrm{i}}$ and $\xi\in A_{1,\mathrm{si}}$ (see Proposition 6.6.6). Moreover, in this case we know that $\pi(\mathcal{R}_1)\subseteq A_2$ is relatively continuous for any relatively continuous subspace $\mathcal{R}_1\subseteq A_1$ (see Corollary 3.3.3). However, in general, if π is degenerate, then it is not even clear whether $\pi(\mathcal{R}_1)\subseteq A_{2,\mathrm{si}}$.

Proposition 6.10.5. *The construction $(A,\mathcal{R})\mapsto\mathcal{B}(A,\mathcal{R})$ is a functor from the category of continuously square-integrable $\widehat{G}$-C^*-algebras to the category of Fell bundles over G. Given a morphism $\pi:(A_1,\mathcal{R}_1)\to(A_2,\mathcal{R}_2)$, the associated morphism $\phi:\mathcal{B}(A_1,\mathcal{R}_1)\to\mathcal{B}(A_2,\mathcal{R}_2)$ is given by $\phi\big(E_t(a)\big)=E_t\big(\pi(a)\big)$ for all $a\in\mathcal{R}_1\mathcal{R}_1^*$ and $t\in G$.*

Proof. Recall that the fibers are given by $\mathcal{B}_t(A_k,\mathcal{R}_k)=\overline{\operatorname{span}}\{E_t(a):a\in\mathcal{R}_k\mathcal{R}_k^*\}$ for all $t\in G$, where $k=1,2$ (see Corollary 6.8.7). By Lemma 6.10.3(i), the equation $\phi(b):=\pi''(b)$ defines a map $\phi:\mathcal{B}(A_1,\mathcal{R}_1)\to\mathcal{B}(A_2,\mathcal{R}_2)$. This maps respects the algebraic operations because they are inherited from the multiplier algebras. It remains to prove that ϕ is continuous. But this also follows from Lemma 6.10.3(i) because $\phi\big(E_t(a)\big)=E_t\big(\pi(a)\big)$ for all $a\in\mathcal{R}_1\mathcal{R}_1^*$. Since $\pi(\mathcal{R}_1)\subseteq\mathcal{R}_2$, this equation says that ϕ maps the generating space of continuous sections of the form $t\mapsto E_t(a)$ for $\mathcal{B}(A_1,\mathcal{R}_1)$ to the continuous sections of the form $t\mapsto E_t\big(\pi(a)\big)$ in $\mathcal{B}(A_2,\mathcal{R}_2)$. □

Proposition 6.10.6. *The maps $\mathcal{B}\mapsto\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ and $\mathcal{B}\mapsto\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$ are functors from the category of Fell bundles over G to the category of continuously square-integrable $\widehat{G}$-C^*-algebras. Given a morphism $\phi:\mathcal{B}_1\to\mathcal{B}_2$, the associated morphism*

$\pi : \big(A(\mathcal{B}_1), \mathcal{R}(\mathcal{B}_1)\big) \to \big(A(\mathcal{B}_2), \mathcal{R}(\mathcal{B}_2)\big)$ *is characterized by the formula* $\pi(\xi)(t) = \phi(\xi(t))$ *for all* $\xi \in \mathcal{C}_c(\mathcal{B}_1)$ *and* $t \in G$, *and the associated morphism* $\rho : \big(A_{\mathrm{r}}(\mathcal{B}_1), \mathcal{R}_{\mathrm{r}}(\mathcal{B}_1)\big) \to \big(A_{\mathrm{r}}(\mathcal{B}_2), \mathcal{R}_{\mathrm{r}}(\mathcal{B}_2)\big)$ *is uniquely determined by the following commutative diagram*:

$$\begin{tikzcd}
\big(A(\mathcal{B}_1), \mathcal{R}(\mathcal{B}_1)\big) \arrow[r, "\pi"] \arrow[d, "\lambda_{\mathcal{B}_1}"'] & \big(A(\mathcal{B}_2), \mathcal{R}(\mathcal{B}_2)\big) \arrow[d, "\lambda_{\mathcal{B}_2}"] \\
\big(A_{\mathrm{r}}(\mathcal{B}_2), \mathcal{R}_{\mathrm{r}}(\mathcal{B}_2)\big) \arrow[r, "\rho"] & \big(A_{\mathrm{r}}(\mathcal{B}_2), \mathcal{R}_{\mathrm{r}}(\mathcal{B}_2)\big).
\end{tikzcd}$$

Proof. The formula $\pi(\xi)(t) := \phi\big(\xi(t)\big)$ for $\xi \in \mathcal{C}_c(\mathcal{B}_1)$ and $t \in G$ defines a $*$-homomorphism $\pi : \mathcal{C}_c(\mathcal{B}_1) \to \mathcal{C}_c(\mathcal{B}_2)$. By the universal property of the cross-sectional C^*-algebras, this extends to a $*$-homomorphism $\pi : C^*(\mathcal{B}_1) \to C^*(\mathcal{B}_2)$. Note that Equation (6.10) implies $\pi(\omega * \xi) = \omega * \pi(\xi)$ for all $\xi \in \mathcal{C}_c(\mathcal{B}_1)$ and ω in the Fourier algebra $A(G)$. Hence π is $\widehat{G}$-equivariant. Since $\lambda_{\mathcal{B}_1} : C^*(\mathcal{B}_1) \to C^*_{\mathrm{r}}(\mathcal{B}_1)$ is a reduction of $(C^*(\mathcal{B}_1), \gamma_{\mathcal{B}_1})$ (see Proposition 6.9.8), Lemma 2.7.2 yields a unique $\widehat{G}$-equivariant $*$-homomorphism $\rho : C^*_{\mathrm{r}}(\mathcal{B}_1) \to C^*_{\mathrm{r}}(\mathcal{B}_2)$ satisfying $\rho \circ \lambda_{\mathcal{B}_1} = \lambda_{\mathcal{B}_2} \circ \pi$. It remains to prove that $\pi\big(\mathcal{R}(\mathcal{B}_1)\big) \subseteq \mathcal{R}(\mathcal{B}_2)$ and $\rho\big(\mathcal{R}_{\mathrm{r}}(\mathcal{B}_1)\big) \subseteq \mathcal{R}_{\mathrm{r}}(\mathcal{B}_2)$. Since $\pi\big(\mathcal{C}_c(\mathcal{B}_1)\big) \subseteq \mathcal{C}_c(\mathcal{B}_2)$ and $\rho\big(\lambda_{\mathcal{B}_1}(\mathcal{C}_c(\mathcal{B}))\big) \subseteq \lambda_{\mathcal{B}_2}\big(\mathcal{C}_c(\mathcal{B}_2)\big)$, this follows from Lemma 6.10.3(ii). $\square$

At this point we have three functors: $(A, \mathcal{R}) \mapsto \mathcal{B}(A, \mathcal{R})$ from the category of continuously square-integrable $\widehat{G}$-C^*-algebras to the category Fell bundles over G, as well as $\mathcal{B} \mapsto \big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big)$ and $\mathcal{B} \mapsto \big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$ in the opposite direction. We now analyze the compositions of these functors. First we show that starting from the category of Fell bundles over G and applying two of the functors consecutively, we get an equivalence on the category of Fell bundles over G. Under certain additional hypotheses, we also get an equivalence by starting from the category of continuously square-integrable $\widehat{G}$-C^*-algebras.

Lemma 6.10.7. *Let* $\mathcal{B}$ *be a Fell bundle over* G. *Then* $\{\xi(t) : \xi \in \mathcal{C}_c(\mathcal{B}) * \mathcal{C}_c(\mathcal{B})\}$ *is dense in* $\mathcal{B}_t$ *for all* $t \in G$.

Proof. It is clear that $\mathcal{B}_t = \overline{\{\xi(t) : \xi \in \mathcal{C}_c(\mathcal{B})\}}$ (see [23, Remark II.13.19]). The assertion now follows because $\mathcal{C}_c(\mathcal{B}) * \mathcal{C}_c(\mathcal{B})$ is dense in $\mathcal{C}_c(\mathcal{B})$ for the inductive limit topology (see [23, Remark VIII.5.12]). $\square$

Theorem 6.10.8. *Let* $\mathcal{B}$ *be a Fell bundle over* G. *Then the canonical inclusions*

$$\mathcal{B}_t \hookrightarrow \mathcal{M}\big(C^*(\mathcal{B})\big) \hookleftarrow \mathcal{B}_t\big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big) \quad \text{and} \quad \mathcal{B}_t \hookrightarrow \mathcal{M}\big(C^*_{\mathrm{r}}(\mathcal{B})\big) \hookleftarrow \mathcal{B}_t\big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$$

induce natural Fell bundle isomorphisms $\mathcal{B} \cong \mathcal{B}\big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big) \cong \mathcal{B}\big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$.

Proof. Let $\Phi : \mathcal{B} \to \mathcal{M}\big(C^*(\mathcal{B})\big)$ and $\Psi : \mathcal{B} \to \mathcal{M}\big(C^*_{\mathrm{r}}(\mathcal{B})\big)$ be the canonical inclusions. Recall that $\Psi = \lambda_{\mathcal{B}} \circ \Phi$ (see Proposition 6.3.4). Theorem 6.6.4, Corollary 6.6.7 and Lemma 6.10.7 yield the equalities

$$\Phi(\mathcal{B}_t) = \overline{\{E_t(a) : a \in \mathcal{C}_c(\mathcal{B})^2\}} \quad \text{and} \quad \Psi(\mathcal{B}_t) = \overline{\{E_t(a) : a \in \lambda_{\mathcal{B}}(\mathcal{C}_c(\mathcal{B}))^2\}}.$$

Corollary 6.8.7 implies that

$$\mathcal{B}_t\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)=\Phi_t(\mathcal{B}_t)\quad\text{and}\quad\mathcal{B}_t\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)=\Psi_t(\mathcal{B}_t).$$

Therefore, we get fiber-preserving bijective maps $\Phi:\mathcal{B}\to\mathcal{B}\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ and $\Psi:\mathcal{B}\to\mathcal{B}\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$. These maps preserve all the algebraic operations because, by definition of $\mathcal{B}\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ and $\mathcal{B}\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$, all these operations are inherited from $\mathcal{M}\big(C^*(\mathcal{B})\big)$ and $\mathcal{M}\big(C^*_{\mathrm{r}}(\mathcal{B})\big)$, respectively. Moreover, again by Theorem 6.6.4, we have $\Phi(\xi(t))=E_t(\xi)$ for all $\xi\in\mathcal{C}_c(\mathcal{B})^2$. This equation says that Φ maps the pointwise-dense subspace of continuous sections $\mathcal{C}_c(\mathcal{B})^2$ for $\mathcal{B}$ (here we use Lemma 6.10.7) onto the pointwise-dense subspace of continuous sections $t\mapsto E_t(\xi)$, $\xi\in\mathcal{C}_c(\mathcal{B})^2$, for $\mathcal{B}\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$. This implies that Φ preserves the topologies, that is, Φ is a homeomorphism (see [23, II.13.16]). Analogously, Corollary 6.6.7 implies that Ψ is a homeomorphism. Therefore Φ and Ψ are isomorphisms of Fell bundles.

Finally, we prove that these isomorphisms are natural. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Fell bundles and suppose that $\phi:\mathcal{B}_1\to\mathcal{B}_2$ is a morphism. We have to prove that the diagram

$$\begin{tikzcd}
\mathcal{B}_1 \arrow[r, "\Phi_1"] \arrow[d, "\phi"'] & \mathcal{B}\big(A(\mathcal{B}_1),\mathcal{R}(\mathcal{B}_1)\big) \arrow[d, "\tilde\phi"] \\
\mathcal{B}_2 \arrow[r, "\Phi_2"'] & \mathcal{B}\big(A(\mathcal{B}_2),\mathcal{R}(\mathcal{B}_2)\big)
\end{tikzcd}$$

commutes, where Φ_k denotes the isomorphism $\mathcal{B}_k\cong\mathcal{B}_k\big(A(\mathcal{B}_k),\mathcal{R}(\mathcal{B}_k)\big)$ for $k=1,2$, and $\tilde\phi$ denotes the morphism induced by ϕ. The latter is given by $\tilde\phi\big(E_t(a)\big)=E_t\big(\pi(a)\big)$ for all $a\in\mathcal{R}(\mathcal{B}_1)\mathcal{R}(\mathcal{B}_1)^*$, where $\pi:\big(A(\mathcal{B}_1),\mathcal{R}(\mathcal{B}_1)\big)\to\big(A(\mathcal{B}_2),\mathcal{R}(\mathcal{B}_2)\big)$ is the morphism given by $\pi(\xi)|_t=\phi\big(\xi(t)\big)$ for every compactly supported continuous section ξ of $\mathcal{B}\big(A(\mathcal{B}_1),\mathcal{R}(\mathcal{B}_1)\big)$. Note that if $\xi\in\mathcal{C}_c(\mathcal{B}_1)^2$, then $\pi(\xi)=[t\mapsto\phi\big(\xi(t)\big)]\in\mathcal{C}_c(\mathcal{B}_2)^2$. Thus

$$\Phi_2\Big(\phi\big(\xi(t)\big)\Big)=E_t\big(\pi(\xi)\big)=\tilde\phi\big(E_t(\xi)\big)=\tilde\phi\Big(\Phi_1\big(\xi(t)\big)\Big).$$

This shows the commutativity of the diagram above, and therefore the naturality of the isomorphism $\mathcal{B}\cong\mathcal{B}\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$. The naturality of $\mathcal{B}\cong\mathcal{B}\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$ is shown in an analogous way. □

Before we proceed with the analysis of the functors, we describe the generalized fixed point algebra $\mathrm{Fix}(A,\mathcal{R})$, the Hilbert module $\mathcal{F}(A,\mathcal{R})$ and the ideal $\mathcal{I}(A,\mathcal{R})$ (see Definition 5.2.1) associated to the pairs $\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ and $\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$.

Corollary 6.10.9. *If $\mathcal{B}$ is a Fell bundle, then we have isomorphisms of C^*-algebras*

$$\mathcal{B}_e\cong\mathrm{Fix}\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)\cong\mathrm{Fix}\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big).$$

Proof. This follows from Theorem 6.10.8 and Corollary 6.8.8. □

Proposition 6.10.10. *Let $\mathcal{B}$ be a Fell bundle over G. Then*

$$\mathcal{F}\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)^* \cong \mathcal{F}\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)^* \cong L^2(\mathcal{B})$$

as Hilbert modules over $\mathrm{Fix}\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big) \cong \mathrm{Fix}\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big) \cong \mathcal{B}_e$.

Proof. Let us denote $(A,\mathcal{R}) := \big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ and $(A_{\mathrm{r}},\mathcal{R}_{\mathrm{r}}) := \big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$. Since $\mathcal{R}$ is the si-closure of $\mathcal{C}_c(\mathcal{B})$, we have $\mathcal{F}(A,\mathcal{R}) = \overline{|\mathcal{R}\rangle\!\rangle} = \overline{|\mathcal{C}_c(\mathcal{B})\rangle\!\rangle} \subseteq \mathcal{L}(A\otimes L^2(G),A)$. Hence

$$\mathcal{F}(A,\mathcal{R})^* = \overline{\langle\!\langle\mathcal{C}_c(\mathcal{B})|} \subseteq \mathcal{L}\big(A, A\otimes L^2(G)\big).$$

Proposition 6.4.2 yields $\langle\!\langle\xi^*| = T_\xi$ for all $\xi\in\mathcal{C}_c(\mathcal{B})$, where $T_\xi \in \mathcal{L}\big(A,A\otimes L^2(G)\big)$ is the operator defined by $T_\xi a|_t = \xi(t)a$ for all $a\in A$. Here we identify $\mathcal{B}_t\subseteq\mathcal{M}(A)$. Define $T:\mathcal{C}_c(\mathcal{B})\to\mathcal{F}(A,\mathcal{R})^*$ by $T(\xi) := T_\xi$. Note that

$$\langle T(\xi)|T(\eta)\rangle_{\mathrm{Fix}(A,\mathcal{R})} = T_\xi^*T_\eta = |\xi^*\rangle\!\rangle\langle\!\langle\eta^*| = E_e(\xi^**\eta) = (\xi^**\eta)(e) = \langle\xi|\eta\rangle_{\mathcal{B}_e}$$

for all $\xi,\eta\in\mathcal{C}_c(\mathcal{B})$, and

$$T(\xi\cdot b)\eta|_t = (\xi\cdot b)(t)\eta = \xi(t)b\eta = (T(\xi)\cdot b)\eta|_t$$

for all $\xi,\eta\in\mathcal{C}_c(\mathcal{B})$, $b\in\mathcal{B}_e$ and $t\in G$. Thus $T(\xi\cdot b) = T(\xi)\cdot b$. Therefore, T extends to an isomorphism $T: L^2(\mathcal{B})\to\mathcal{F}(A,\mathcal{R})^*$ of Hilbert modules over $\mathcal{B}_e\cong\mathrm{Fix}(A,\mathcal{R})$.

Similarly, since $\mathcal{R}_{\mathrm{r}}$ is the si-closure of $\lambda_{\mathcal{B}}(\mathcal{C}_c(\mathcal{B}))$, we have

$$\mathcal{F}(A_{\mathrm{r}},\mathcal{R}_{\mathrm{r}})^* = \overline{\langle\!\langle\lambda_{\mathcal{B}}(\mathcal{C}_c(\mathcal{B}))|} \subseteq \mathcal{L}\big(A_{\mathrm{r}}, A_{\mathrm{r}}\otimes L^2(G)\big).$$

Thus, we can also define a map $\tilde{T}:\mathcal{C}_c(\mathcal{B})\to\mathcal{F}(A_{\mathrm{r}},\mathcal{R}_{\mathrm{r}})^*$ by $\tilde{T}(\xi) := \langle\!\langle\lambda_{\mathcal{B}}(\xi^*)|$. Proposition 3.3.1(ii) gives

$$\tilde{T}(\xi) = \langle\!\langle\lambda_{\mathcal{B}}(\xi^*)| = (\lambda_{\mathcal{B}}\otimes\mathrm{id}_H)(\langle\!\langle\xi^*|) = (\lambda_{\mathcal{B}}\otimes\mathrm{id}_H)(T(\xi)),$$

where $H := L^2(G)$. Thus $\tilde{T} = (\lambda_{\mathcal{B}}\otimes\mathrm{id}_H)\circ T$. It follows that

$$\langle\tilde{T}(\xi)|\tilde{T}(\eta)\rangle_{\mathrm{Fix}(A_{\mathrm{r}},\mathcal{R}_{\mathrm{r}})} = \lambda_{\mathcal{B}}(\langle\xi|\eta\rangle_{\mathcal{B}_e})$$

for all $\xi,\eta\in\mathcal{C}_c(\mathcal{B})$, and

$$\tilde{T}(\xi\cdot b) = \tilde{T}(\xi)\cdot\lambda_{\mathcal{B}}(b)$$

for all $\xi\in\mathcal{C}_c(\mathcal{B})$, $b\in\mathcal{B}_e$. Since $\lambda_{\mathcal{B}}:\mathcal{B}_e\to\mathrm{Fix}(A_{\mathrm{r}},\mathcal{R}_{\mathrm{r}})$ is a $*$-isomorphism (here we identify $\mathcal{B}_e\subseteq\mathcal{M}(A)$), we conclude that $\tilde{T}$ extends to an isomorphism $\tilde{T}: L^2(\mathcal{B})\to\mathcal{F}(A_{\mathrm{r}},\mathcal{R}_{\mathrm{r}})$ of Hilbert modules over $\mathcal{B}_e\cong\mathrm{Fix}(A_{\mathrm{r}},\mathcal{R}_{\mathrm{r}})$. $\square$

Recall that $\lambda_{\mathcal{B}}\rtimes G: C^*(\mathcal{B})\rtimes G\to C^*_{\mathrm{r}}(\mathcal{B})\rtimes G$ is an isomorphism (Proposition 6.9.8). Given a continuously square-integrable $\widehat{G}$-C^*-algebra $(A,\mathcal{R})$, the ideal $\mathcal{I}(A,\mathcal{R})$ in $A\rtimes_{\mathrm{r}} G$ is, by definition, the ideal generated by the inner product of $\mathcal{F}(A,\mathcal{R})$, that is, it is the algebra of compact operators of the dual $\mathcal{F}(A,\mathcal{R})^*$. This yields the following result:

Corollary 6.10.11. *Let $\mathcal{B}$ be a Fell bundle over G. Then*

$$\mathcal{I}\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big) \cong \mathcal{I}\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big) \cong \mathcal{K}\big(L^2(\mathcal{B})\big).$$

In particular, $\mathcal{K}\big(L^2(\mathcal{B})\big)$ is (isomorphic to) an ideal of $C^*(\mathcal{B}) \rtimes G \cong C^*_{\mathrm{r}}(\mathcal{B}) \rtimes G$.

Now, we return to the analysis of our functors. Theorem 6.10.8 says that, up to natural isomorphism, the functor

$$(A,\mathcal{R}) \mapsto \mathcal{B}(A,\mathcal{R})$$

is a left inverse for the functors

$$\mathcal{B} \mapsto \big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big) \quad \text{and} \quad \mathcal{B} \mapsto \big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big).$$

We are going to prove that, under a certain hypothesis, it is a right inverse for first functor if we restrict to maximal coactions, and for the second functor if we restrict to reduced coactions. The missing hypothesis is the content of the next definition.

Definition 6.10.12. We say that a continuously square-integrable Hilbert $B,\widehat{G}$-module $(\mathcal{E},\mathcal{R})$ is *essential* if $\mathcal{R}$ is essential, that is, if $\overline{\operatorname{span}}^{\mathrm{si}}\, A(G) * \mathcal{R} = \mathcal{R}$.[5] In this case, we also call $(\mathcal{E},\mathcal{R})$ an *e-continuously square-integrable Hilbert $B,\widehat{G}$-module*. If $(A,\mathcal{R})$ is a continuously square-integrable $\widehat{G}$-C^*-algebra with $\mathcal{R}$ essential, we also say that $(A,\mathcal{R})$ is an *e-continuously square-integrable $\widehat{G}$-C^*-algebra*.

Proposition 6.10.13. *If G is discrete or amenable, then every continuously square-integrable Hilbert $B,\widehat{G}$-module $(\mathcal{E},\mathcal{R})$ is essential.*

Proof. If G is discrete, then $\mathcal{R} = \mathcal{E}$ and $\|\cdot\|_{\mathrm{si}}$ is equivalent to the norm on $\mathcal{E}$, and hence the result follows from Proposition 2.6.10. And if G is amenable, then $\mathcal{G} = C^*_{\mathrm{r}}(G)$ is a co-amenable quantum group and therefore the result follows from Proposition 5.3.10. □

Proposition 6.10.14. *Let $\mathcal{B}$ be a Fell bundle over G. Then, for all $\xi \in \mathcal{C}_c(\mathcal{B})$, we have*

$$\|\xi\|_{\mathrm{si}} = \|\xi\|_{C^*(\mathcal{B})} + \left\|\int_G \xi(t)\xi(t)^*\delta_G(t)\,\mathrm{d}t\right\|^{\frac{1}{2}} \leq \|\xi\|_{L^1(\mathcal{B})} + \left(\int_G \|\xi(t)\|^2\delta_G(t)\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Proof. Theorem 6.6.4 and Corollary 6.6.9 imply

$$\begin{aligned}\big\||\xi\rangle\!\rangle\big\|^2 &= \big\||\xi\rangle\!\rangle\langle\!\langle\xi|\big\| = \big\|E_e(\xi * \xi^*)\big\| = \big\|(\xi * \xi^*)(e)\big\| \\ &= \left\|\int_G \xi(t)\xi^*(t^{-1})\,\mathrm{d}t\right\| = \left\|\int_G \xi(t)\xi(t)^*\delta_G(t)\,\mathrm{d}t\right\|.\end{aligned}$$

The assertion now follows from the definition of $\|\cdot\|_{\mathrm{si}}$ (see Proposition 4.1.11). □

Proposition 6.10.15. *Let $\mathcal{B}$ be a Fell bundle over G. Then both pairs $\big(A(\mathcal{B}),\mathcal{R}(\mathcal{B})\big)$ and $\big(A_{\mathrm{r}}(\mathcal{B}),\mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$ are e-continuously square-integrable $\widehat{G}$-C^*-algebras.*

[5] Recall that $*$ denotes the Banach left action of the Fourier algebra $A(G)$ on $\mathcal{E}$ induced by the coaction of G on $\mathcal{E}$; see Equation (2.18).

Proof. We only have to show that $\mathcal{R}(\mathcal{B}) = \overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}}$ and $\mathcal{R}_{\mathrm{r}}(\mathcal{B}) = \overline{\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}}$ are essential. Equation (6.10) says that the left $A(G)$-action on $\mathcal{C}_c(\mathcal{B}) \subseteq C^*(\mathcal{B})$ is given by pointwise multiplication. Since $A(G)$ is dense in $\mathcal{C}_0(G)$, we can find a net (ω_i) in $A(G)$ such that $\omega_i(t) \to 1$ uniformly on compact subsets. As a consequence, if $\xi \in \mathcal{C}_c(\mathcal{B})$, then $(\omega_i * \xi)(t) = (\omega_i \cdot \xi)(t) \to \xi(t)$ uniformly on G. It follows from Proposition 6.10.14 that $\|\omega_i * \xi - \xi\|_{\mathrm{si}} \to 0$. Thus

$$\overline{A(G) * \overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}}}^{\mathrm{si}} = \overline{A(G) * \mathcal{C}_c(\mathcal{B})}^{\mathrm{si}} = \overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}}.$$

This shows that $\overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}}$ is essential. Now, since $\lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_{\mathrm{r}}(\mathcal{B})$ is equivariant, Proposition 3.3.1 yields $\|\lambda_{\mathcal{B}}(\xi)\|_{\mathrm{si}} \leq \|\xi\|_{\mathrm{si}}$ for all $\xi \in C^*(\mathcal{B})_{\mathrm{si}}$. Thus

$$\overline{\lambda_{\mathcal{B}}(\overline{\mathcal{R}_0}^{\mathrm{si}})}^{\mathrm{si}} = \overline{\lambda_{\mathcal{B}}(\mathcal{R}_0)}^{\mathrm{si}} \quad \text{for any subset } \mathcal{R}_0 \subseteq C^*(\mathcal{B})_{\mathrm{si}}. \tag{6.18}$$

Finally, note that

$$\begin{aligned} \overline{A(G) * \overline{\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}}}^{\mathrm{si}} &= \overline{A(G) * \lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}} = \overline{\lambda_{\mathcal{B}}\big(A(G) * \mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}} \\ &= \overline{\lambda_{\mathcal{B}}(\overline{A(G) * \mathcal{C}_c(\mathcal{B})}^{\mathrm{si}})}^{\mathrm{si}} = \overline{\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}}. \end{aligned}$$

Therefore, $\overline{\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}}$ is also essential. □

Let us fix a continuously square-integrable $\widehat{G}$-C^*-algebra $(A, \mathcal{R})$, and denote $\mathcal{B} := \mathcal{B}(A, \mathcal{R})$. As in Lemma 6.9.3, we define

$$\mathcal{J}(\mathcal{B}) := \operatorname{span}\{\eta : \eta(t) = \omega(t) E_t(a),\, a \in \mathcal{W}_{\mathcal{R}},\, \omega \in A_c(G)\} \subseteq \mathcal{C}_c(\mathcal{B}),$$

where $\mathcal{W}_{\mathcal{R}} := \operatorname{span} \mathcal{R}\mathcal{R}^*$.

Lemma 6.10.16. *We have* $\overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}} = \overline{\mathcal{J}(\mathcal{B})}^{\mathrm{si}}$ *and* $\overline{\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}} = \overline{\lambda_{\mathcal{B}}\big(\mathcal{J}(\mathcal{B})\big)}^{\mathrm{si}}$.

Proof. Since $\mathcal{J}(\mathcal{B}) \subseteq \mathcal{C}_c(\mathcal{B})$, we have $\overline{\mathcal{J}(\mathcal{B})}^{\mathrm{si}} \subseteq \overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}}$. On the other hand, by Lemma 6.9.3, $\mathcal{J}(\mathcal{B})$ is dense in $\mathcal{C}_c(\mathcal{B})$ with respect to the inductive limit topology. Thus, if $\xi \in \mathcal{C}_c(\mathcal{B})$, there is a net (ξ_n) in $\mathcal{J}(\mathcal{B})$ such that $\xi_n(t) \to \xi(t)$ uniformly on G and $\operatorname{supp}(\xi_n) \subseteq K$ for all n, where K is some fixed compact subset of G. It follows from Proposition 6.10.14 that $\|\xi_n - \xi\|_{\mathrm{si}} \to 0$. Therefore, $\overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}} = \overline{\mathcal{J}(\mathcal{B})}^{\mathrm{si}}$. As a consequence of Equation (6.18), we get

$$\overline{\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}} = \overline{\lambda_{\mathcal{B}}\big(\overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}}\big)}^{\mathrm{si}} = \overline{\lambda_{\mathcal{B}}\big(\overline{\mathcal{J}(\mathcal{B})}^{\mathrm{si}}\big)}^{\mathrm{si}} = \overline{\lambda_{\mathcal{B}}\big(\mathcal{J}(\mathcal{B})\big)}^{\mathrm{si}}. \qquad \square$$

Lemma 6.10.17. *Let $(A, \mathcal{R})$ be a continuously square-integrable $\widehat{G}$-C^*-algebra. Then the* si*-norm closure of $\mathcal{W}_{\mathcal{R}} = \operatorname{span} \mathcal{R}\mathcal{R}^*$ is equal to $\mathcal{R}$.*

Proof. Since $\mathcal{R}$ is complete, we have $\mathcal{W}_{\mathcal{R}} = \operatorname{span} \mathcal{R}\mathcal{R}^* \subseteq \operatorname{span} \mathcal{R}A \subseteq \mathcal{R}$, so that $\overline{\mathcal{W}_{\mathcal{R}}}^{\mathrm{si}} \subseteq \mathcal{R}$. Now, since $\mathcal{R}$ is dense in A, we can take a bounded approximate unit (e_i) for A contained in $\mathcal{R}^*$. Thus $e_i \to 1$ strictly in $\mathcal{M}(A)$. It follows that $\gamma_A(e_i) \to 1$ strictly in $\mathcal{M}(A \rtimes_{\mathrm{r}} G)$. Since $\mathcal{F}(A, \mathcal{R}) = \overline{|\mathcal{R}\rangle\!\rangle}$ is a Hilbert $A \rtimes_{\mathrm{r}} G$-module, Proposition 4.1.10(ii) yields $|\xi e_i\rangle\!\rangle = |\xi\rangle\!\rangle \gamma_A(e_i) \to |\xi\rangle\!\rangle$ for all $\xi \in \mathcal{R}$. Thus $\xi e_i \to \xi$ in the si-norm, and therefore $\mathcal{R} \subseteq \overline{\mathcal{W}_{\mathcal{R}}}^{\mathrm{si}}$. □

Theorem 6.10.18. *Let $(A,\mathcal{R})$ be a continuously square-integrable $\widehat{G}$-C^*-algebra.*

(i) *If (A,γ_A) is a maximal and $\mathcal{R}$ is essential, then there is a natural isomorphism*

$$(A,\mathcal{R}) \cong \Big(A\big(\mathcal{B}(A,\mathcal{R})\big), \mathcal{R}\big(\mathcal{B}(A,\mathcal{R})\big)\Big).$$

(ii) *If (A,γ_A) is reduced and $\mathcal{R}$ is essential, then there is a natural isomorphism*

$$(A,\mathcal{R}) \cong \Big(A_{\mathrm{r}}\big(\mathcal{B}(A,\mathcal{R})\big), \mathcal{R}_{\mathrm{r}}\big(\mathcal{B}(A,\mathcal{R})\big)\Big).$$

Proof. Define $\mathcal{B} := \mathcal{B}(A,\mathcal{R})$. From Theorem 6.9.10 we have a commutative diagram

$$\begin{tikzcd}
C^*(\mathcal{B}) \arrow[r, "\kappa"] \arrow[dr, "\lambda_{\mathcal{B}}"'] & A \arrow[d, "\nu"] \\
& C^*_{\mathrm{r}}(\mathcal{B})
\end{tikzcd}$$

where all the maps are equivariant surjections. If (A,γ_A) is maximal, then κ is an isomorphism and if (A,γ_A) is reduced, then ν is an isomorphism.

We know from Proposition 6.9.4 that $\kappa(\mathcal{J}(\mathcal{B})) = \mathrm{span}(A_c(G) * \mathcal{W}_{\mathcal{R}})$. Since $A_c(G)$ is dense in $A(G)$, Lemmas 6.10.16 and 6.10.17 yield

$$\overline{\kappa\big(C_c(\mathcal{B})\big)}^{\mathrm{si}} = \overline{\kappa\big(\mathcal{J}(\mathcal{B})\big)}^{\mathrm{si}} = \overline{\mathrm{span}}^{\mathrm{si}}\big(A_c(G) * \mathcal{R}\big) = \overline{\mathrm{span}}^{\mathrm{si}}\big(A(G) * \mathcal{R}\big). \tag{6.19}$$

Thus, if (A,γ_A) is maximal and $\mathcal{R}$ is essential, then (using that κ is an equivariant isomorphism)

$$\kappa(\overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}}) = \overline{\kappa(\mathcal{C}_c(\mathcal{B}))}^{\mathrm{si}} = \mathcal{R}.$$

Therefore, $\kappa : \big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big) \to (A,\mathcal{R})$ is an isomorphism of continuously square-integrable $\widehat{G}$-C^*-algebras. Similarly, if (A,γ_A) is reduced and $\mathcal{R}$ is essential, then (using that ν is an equivariant isomorphism)

$$\nu(\mathcal{R}) = \nu\big(\kappa(\overline{\mathcal{C}_c(\mathcal{B})}^{\mathrm{si}})\big) = \overline{\nu\big(\kappa(\mathcal{C}_c(\mathcal{B}))\big)}^{\mathrm{si}} = \overline{\lambda_{\mathcal{B}}\big(\mathcal{C}_c(\mathcal{B})\big)}^{\mathrm{si}}.$$

Therefore, ν is an isomorphism between $(A,\mathcal{R})$ and $\big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$.

Finally, we prove the naturality of the isomorphisms. Let $(A_1,\mathcal{R}_1)$ and $(A_2,\mathcal{R}_2)$ be continuously square-integrable $\widehat{G}$-C^*-algebras and suppose that $\pi : (A_1,\mathcal{R}_1) \to (A_2,\mathcal{R}_2)$ is a morphism. Define $\mathcal{B}_k := \mathcal{B}(A_k,\mathcal{R}_k)$ for $k = 1,2$, and let $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ be the morphism of Fell bundles induced by π. Recall that it is given by $\phi(E_t(a)) = E_t(\pi(a))$ for all $a \in \mathcal{R}_k\mathcal{R}_k^*$. Let $\tilde{\pi} : \big(A(\mathcal{B}_1), \mathcal{R}(\mathcal{B}_1)\big) \to \big(A(\mathcal{B}_2), \mathcal{R}(\mathcal{B}_2)\big)$ be the morphism induced by ϕ. It is given by

$\tilde{\pi}(\xi)|_t = \phi(\xi(t))$ for all $\xi \in \mathcal{C}_c(\mathcal{B}_1)$. In order to prove the naturality of the isomorphism $(A, \mathcal{R}) \cong \big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big)$, we have to show that the following diagram commutes:

$$\begin{tikzcd}
\big(A(\mathcal{B}_1), \mathcal{R}(\mathcal{B}_1)\big) \arrow[r, "\kappa_1"] \arrow[d, "\tilde{\pi}"'] & (A_1, \mathcal{R}_1) \arrow[d, "\pi"] \\
\big(A(\mathcal{B}_2), \mathcal{R}(\mathcal{B}_2)\big) \arrow[r, "\kappa_2"'] & (A_2, \mathcal{R}_2)
\end{tikzcd}$$

where $\kappa_k : \big(A(\mathcal{B}_k), \mathcal{R}(\mathcal{B}_k)\big) \to (A_k, \mathcal{R}_k)$ for $k = 1, 2$, are the canonical maps given by the integrated forms of the inclusions $\mathcal{B}_k \hookrightarrow \mathcal{M}(A_k)$. Thus they are given by $\kappa_k(\xi) = \int_G^{\mathrm{s}} \xi(t)\,\mathrm{d}t$ for all $\xi \in \mathcal{C}_c(\mathcal{B}_k)$. We have, for all $\xi \in \mathcal{C}_c(\mathcal{B}_1)$,

$$\begin{aligned}
\pi(\kappa_1(\xi)) &= \pi\left(\int_G^{\mathrm{s}} \xi(t)\,\mathrm{d}t\right) \\
&= \int_G^{\mathrm{s}} \pi''(\xi(t))\,\mathrm{d}t = \int_G \phi(\xi(t))\,\mathrm{d}t \\
&= \int_G^{\mathrm{s}} \tilde{\pi}(\xi)(t)\,\mathrm{d}t = \kappa_2(\tilde{\pi}(\xi)).
\end{aligned}$$

This shows that the diagram above commutes and, therefore, proves the naturality in the maximal case. The reduced case is similar. $\square$

With notation as above, note that given a continuously square-integrable $\widehat{G}$-C^*-algebra $(A, \mathcal{R})$, we have that $\nu : A \to C^*_{\mathrm{r}}(\mathcal{B})$ is a reduction of A, and if $\mathcal{B}$ has the maximality property (see Definition 6.9.11), then $\kappa : C^*(\mathcal{B}) \to A$ is a maximalization of A.

Therefore, the functor $(A, \mathcal{R}) \mapsto \big(A_{\mathrm{r}}(\mathcal{B}(A, \mathcal{R})), \mathcal{R}_{\mathrm{r}}(\mathcal{B}(A, \mathcal{R}))\big)$ is essentially the reduction functor, and, if G has the maximality property, then the functor $(A, \mathcal{R}) \mapsto \big(A(\mathcal{B}(A, \mathcal{R})), \mathcal{R}(\mathcal{B}(A, \mathcal{R}))\big)$ is essentially the maximalization functor, both acting on the category of continuously square-integrable $\widehat{G}$-C^*-algebras.

Combining Theorems 6.10.8 and 6.10.18, we immediately get the following result.

Theorem 6.10.19. *Let G be a locally compact group. Then the functor*

$$\mathcal{B} \mapsto \big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big)$$

is an equivalence between the category of Fell bundles over G and the category of e-continuously square-integrable reduced $\widehat{G}$-C^*-algebras. And if G has the maximality property, then the functor

$$\mathcal{B} \mapsto \big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big)$$

is an equivalence between the category of Fell bundles over G and the category of e-continuously square-integrable maximal $\widehat{G}$-C^*-algebras. The inverse of both functors is given by

$$(A, \mathcal{R}) \mapsto \mathcal{B}(A, \mathcal{R}).$$

If G is amenable, then every $\widehat{G}$-C^*-algebra is reduced and maximal at the same time, and every continuously square-integrable $\widehat{G}$-C^*-algebra is essential. Therefore, we get the following consequence.

Corollary 6.10.20. *Let G be an amenable locally compact group. Then the functor*

$$(A, \mathcal{R}) \mapsto \mathcal{B}(A, \mathcal{R})$$

is an equivalence from the category of continuously square-integrable $\widehat{G}$-C^*-algebras to the category of Fell bundles over G. The inverse is given by the equivalent functors

$$\mathcal{B} \mapsto \big(A(\mathcal{B}), \mathcal{R}(\mathcal{B})\big) \quad \textit{and} \quad \mathcal{B} \mapsto \big(A_{\mathrm{r}}(\mathcal{B}), \mathcal{R}_{\mathrm{r}}(\mathcal{B})\big).$$

The corollary above generalizes our main result for Abelian groups in [9, Theorem 38]. In particular, it also generalizes Exel's result in [19, Theorem 11.14].

If G is discrete, we can drop the hypothesis of essentialness in Theorem 6.10.19. Moreover, in this case we have a lot of simplifications, and we get the following well-known result (see [52, 62, 14]).

Corollary 6.10.21. *Let G be a discrete group. Then the functor $\mathcal{B} \mapsto C^*_{\mathrm{r}}(\mathcal{B})$ is an equivalence from the category of Fell bundles over G to the category of reduced $\widehat{G}$-C^*-algebras. The functor $\mathcal{B} \mapsto C^*(\mathcal{B})$ is an equivalence from the category of Fell bundles over G to the category of maximal $\widehat{G}$-C^*-algebras. Given a $\widehat{G}$-C^*-algebra (A, γ_A), the associated Fell bundle is given by $\mathcal{B}_t = \{a \in A : \gamma_A(a) = a \otimes \lambda_t\}$ for all $t \in G$.*

Proof. Since G is discrete, the quantum group $C^*_{\mathrm{r}}(G)$ is compact, and hence there is no difference between continuously square-integrable and arbitrary $\widehat{G}$-C^*-algebras. Given a $\widehat{G}$-C^*-algebra, the si-norm is equivalent to the norm of A, and hence $\mathcal{R} = A$ is the unique dense, complete (relatively continuous) subspace of A. Thus the first two assertions follow directly from Theorem 6.10.19 and the fact that discrete groups have the maximality property. If (A, γ_A) is a $\widehat{G}$-C^*-algebra A, the associated Fell bundle $\mathcal{B} = \mathcal{B}(A, A)$ over G is, by definition, given by $\mathcal{B}_t = \overline{\{E_t(a) : a \in A\}}$ for all $t \in G$. We have $E_t(a) = (\mathrm{id}_A \otimes \varphi)\big((1_A \otimes \lambda_t^{-1})\gamma_A(a)\big) \in A$ because φ is bounded and $\gamma_A(a) \in \tilde{\mathcal{M}}\big(A \otimes C^*_{\mathrm{r}}(G)\big)$. And Proposition 6.6.2 yields $\gamma_A\big(E_t(a)\big) = E_t(a) \otimes \lambda_t$. Now take any $b \in A$ with $\gamma_A(b) = b \otimes \lambda_t$. Then $E_t(b) = (\mathrm{id}_A \otimes \varphi)\big((1 \otimes \lambda_t^{-1})\gamma_A(b)\big) = b$. Therefore, $\mathcal{B}_t = \{a \in A : \gamma_A(a) = a \otimes \lambda_t\}$ for all $t \in G$. □

Remark 6.10.22. (1) Corollary 6.10.21 implies, in particular, that for a discrete group G, the categories of maximal and reduced coactions of G are equivalent. This is, in fact, true for any locally compact group ([34, Theorem 3.5]). The equivalences are given by applying the functors of reduction and maximalization. Note, however, that we are working with coactions of $C^*_{\mathrm{r}}(G)$, whereas in [34], coactions of $C^*(G)$ are used instead, that is, full coactions of G. In this setting, reduced coactions are replaced by normal coactions (and reduction by normalization).

(2) Let $\mathcal{B}$ be a Fell bundle over G with $\mathcal{B}_e \neq \{0\}$. It is well-known that $C^*(\mathcal{B})$ (resp. $C^*_{\mathrm{r}}(\mathcal{B})$) is a unital C^*-algebra if and only if G is discrete and the unit fiber $\mathcal{B}_e$ is unital (see

[23, Chapter XI, Exercise 39]). Using our results, we can give a simple proof of this fact. Indeed, let A be either $C^*(\mathcal{B})$ or $C^*_{\mathrm{r}}(\mathcal{B})$. We know that A is an integrable $\widehat{G}$-C^*-algebra. If it is, in addition, unital, then Proposition 3.2.5 implies that the quantum group $C^*_{\mathrm{r}}(G)$ is compact, that is, G is discrete. Since $\mathcal{B}_e \cong \{a \in A : \gamma_A(a) = a \otimes 1\}$, we also get that $\mathcal{B}_e$ is unital. Conversely, if G is discrete and $\mathcal{B}_e$ is unital, then it is clear that A is unital.

Our results can be used to classify the Fell bundle structures for a given $\widehat{G}$-C^*-algebra A. Recall that a full (resp. reduced) Fell bundle structure for A is a Fell bundle $\mathcal{B}$ over G together with a $\widehat{G}$-equivariant isomorphism $\pi : C^*(\mathcal{B}) \to A$ (resp. $\pi : C^*_{\mathrm{r}}(\mathcal{B}) \to A$).

Given a full (resp. reduced) Fell bundle structure $(\mathcal{B}, \pi)$ for A, we define $\mathcal{R}(\mathcal{B}, \pi)$ to be the dense, e-complete, relatively continuous subspace $\pi\big(\overline{C_c(\mathcal{B})}^{\mathrm{si}}\big)$ $\big($resp. $\pi\big(\overline{\lambda_{\mathcal{B}}(C_c(\mathcal{B}))}^{\mathrm{si}}\big)\big)$ of A.

We say that two full (resp. reduced) Fell bundle structures $(\mathcal{B}_1, \pi_1)$ and $(\mathcal{B}_2, \pi_2)$ for A are *isomorphic* if there is an isomorphism $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ such that $\pi_2 \circ \tilde{\phi} = \pi_1$, where $\tilde{\phi} : C^*(\mathcal{B}_1) \to C^*(\mathcal{B}_2)$ (resp. $\tilde{\phi} : C^*_{\mathrm{r}}(\mathcal{B}_1) \to C^*_{\mathrm{r}}(\mathcal{B}_2)$) is the isomorphism induced by ϕ. Note that in this case the corresponding relatively continuous subspaces $\mathcal{R}(\mathcal{B}_1, \pi_1)$ and $\mathcal{R}(\mathcal{B}_2, \pi_2)$ are equal.

Our results above now yield the following consequence:

Corollary 6.10.23. *Let A be a maximal* (*resp. reduced*) *$\widehat{G}$-C^*-algebra. Then isomorphism classes of full* (*resp. reduced*) *Fell bundle structures for (A, γ_A) correspond bijectively to dense, e-complete, relatively continuous subspaces of A via the map $(\mathcal{B}, \pi) \to \mathcal{R}(\mathcal{B}, \pi)$.*

Our results can also be used to classify Fell bundle structures $(\mathcal{B}, \pi)$ for a given $\widehat{G}$-C^*-algebra A if we disregard π as part of the data. It might happen that we have two dense, relatively continuous, complete subspaces $\mathcal{R}_1$ and $\mathcal{R}_2$ of A which are not equal, but there might be a $\widehat{G}$-equivariant automorphism π of A such that $\pi(\mathcal{R}_1) = \mathcal{R}_2$. In this case, we can regard π as an isomorphism between the continuously square-integrable $\widehat{G}$-C^*-algebras $(A, \mathcal{R}_1)$ and $(A, \mathcal{R}_2)$. Proposition 6.10.5 implies that the corresponding Fell bundles $\mathcal{B}(A, \mathcal{R}_1)$ and $\mathcal{B}(A, \mathcal{R}_2)$ are isomorphic.

Conversely, if we have two isomorphic Fell bundles $\mathcal{B}_1, \mathcal{B}_2$ over G which are part of full (resp. reduced) Fell bundle structures $(\mathcal{B}_1, \pi_1)$ and $(\mathcal{B}_2, \pi_2)$ for A and if we consider the associated relatively continuous subspaces $\mathcal{R}_1 := \mathcal{R}(\mathcal{B}_1, \pi_1)$ and $\mathcal{R}_2 := \mathcal{R}(\mathcal{B}_2, \pi_2)$ of A, then we have $\pi(\mathcal{R}_1) = \mathcal{R}_2$), where π is the $\widehat{G}$-equivariant automorphism $\pi := \pi_2 \circ \tilde{\phi} \circ \pi_1^{-1}$ of A. Here $\tilde{\phi}$ is the isomorphism $\tilde{\phi} : C^*(\mathcal{B}_1) \to C^*(\mathcal{B}_2)$ (resp. $\tilde{\phi} : C^*_{\mathrm{r}}(\mathcal{B}_1) \to C^*_{\mathrm{r}}(\mathcal{B}_2)$) induced by the given isomorphism $\phi : \mathcal{B}_1 \to \mathcal{B}_2$.

Let us formalize these observations.

Definition 6.10.24. Let A be a $\widehat{G}$-C^*-algebra and let $\mathrm{Aut}_{\widehat{G}}(A)$ denote the group of $\widehat{G}$-equivariant automorphisms of A. We say that $\mathcal{R}_1, \mathcal{R}_2 \subseteq A$ are $\mathrm{Aut}_{\widehat{G}}(A)$-conjugate if there is $\pi \in \mathrm{Aut}_{\widehat{G}}(A)$ with $\pi(\mathcal{R}_1) = \mathcal{R}_2$.

The discussion above together with our previous results yield the following:

Corollary 6.10.25. *Let A be a maximal (resp. reduced) $\widehat{G}$-C^*-algebra. Then isomorphism classes of Fell bundles $\mathcal{B}$ over G for which there is a $\widehat{G}$-equivariant isomorphism $C^*(\mathcal{B}) \cong A$ (resp. $C^*_{\mathrm{r}}(\mathcal{B}) \cong A$) correspond bijectively to $\mathrm{Aut}_{\widehat{G}}(A)$-conjugacy classes of dense, e-complete, relatively continuous subspaces of A.*

The corollaries above have been formulated in terms of dense, e-complete, relatively continuous subspaces. However, as we are going to see, one can also formulate them in terms of dense, s-complete, relatively continuous subspaces.

Definition 6.10.26. Let A be a $\widehat{G}$-C^*-algebra. Given a dense, complete, relatively continuous subspace $\mathcal{R} \subseteq A$, we define $\mathcal{R}_{\mathrm{ec}} := \overline{\mathrm{span}}^{\mathrm{si}}\big(A(G) * \mathcal{R}\big)$. We call $\mathcal{R}_{\mathrm{ec}}$ the *essential part* of $\mathcal{R}$.

The following result gives a partial solution to Question 5.4.13.

Proposition 6.10.27. *Let A be a $\widehat{G}$-C^*-algebra. The essential part of a dense, complete, relatively continuous subspace $\mathcal{R} \subseteq A$ is a dense, e-complete, relatively continuous subspace of A, and we have*

$$\mathcal{F}(A, \mathcal{R}_{\mathrm{ec}}) = \mathcal{F}(A, \mathcal{R}) \quad \textit{and} \quad \mathcal{B}(A, \mathcal{R}_{\mathrm{ec}}) = \mathcal{B}(A, \mathcal{R}).$$

Moreover, the map

$$\mathcal{C}_{\mathrm{sc}}(A) \ni \mathcal{R} \mapsto \mathcal{R}_{\mathrm{ec}} \in \mathcal{C}_{\mathrm{ec}}(A)$$

is a bijection, where $\mathcal{C}_{\mathrm{sc}}(A)$ and $\mathcal{C}_{\mathrm{ec}}(A)$ are the sets of all dense, s-complete and e-complete, relatively continuous subspaces of A, respectively. The inverse map is given by the map

$$\mathcal{C}_{\mathrm{ec}}(A) \ni \mathcal{R} \mapsto \mathcal{R}_{\mathrm{sc}} \in \mathcal{C}_{\mathrm{sc}}(A).^6$$

Proof. It follows from Equation (6.19) that $\mathcal{R}_{\mathrm{ec}}$ is complete (and, of course, also dense and relatively continuous). Since $A(G) \cdot A(G)$ is dense in the Fourier algebra $A(G)$ (see comments before Proposition 2.5.5), $\mathcal{R}_{\mathrm{ec}}$ is essential.

By Proposition 5.3.2, $\mathcal{F}(A, \mathcal{R}_{\mathrm{ec}}) = \mathcal{F}(A, \mathcal{R})$. This also implies $\mathcal{B}(A, \mathcal{R}_{\mathrm{ec}}) = \mathcal{B}(A, \mathcal{R})$. By Corollary 5.4.6, the s-completions of $\mathcal{R}_{\mathrm{ec}}$ and $\mathcal{R}$ coincide. By Corollary 5.4.11, the essential part of $\mathcal{R}_{\mathrm{sc}}$ coincides with $\mathcal{R}_{\mathrm{ec}}$. The last assertion now follows. □

Proposition 6.10.27 allows us to reformulate Corollaries 6.10.23 and 6.10.24 in terms of dense, s-complete, relatively continuous subspaces.

Remark 6.10.28. Let $\pi : (A, \mathcal{R}) \to (A', \mathcal{R}')$ be a morphism of continuously square-integrable $\widehat{G}$-C^*-algebras. This means that $\pi : A \to A'$ is a $\widehat{G}$-equivariant $*$-homomorphism satisfying $\pi(\mathcal{R}) \subseteq \mathcal{R}'$. Since π is $\widehat{G}$-equivariant, we have $\pi(\omega * \xi) = \omega * \pi(\xi)$ for all $\omega \in A(G)$ and $\xi \in A$. Hence $\pi(\mathcal{R}_{\mathrm{ec}}) \subseteq \mathcal{R}'_{\mathrm{ec}}$. Therefore, the assignment $\mathbb{F} : (A, \mathcal{R}) \mapsto (A, \mathcal{R}_{\mathrm{ec}})$ is a functor from the category of all continuously square-integrable $\widehat{G}$-C^*-algebras to the full subcategory of e-continuously square-integrable $\widehat{G}$-C^*-algebras. Moreover, when restricted

[6] Recall that $\mathcal{R}_{\mathrm{sc}}$ denotes the s-completion of $\mathcal{R}$. See Definition 5.3.6.

to the subcategory of s-continuously square-integrable $\widehat{G}$-C^*-algebras, $\mathbb{F}$ is injective. This follows from Proposition 6.10.27.

It is not clear to me whether the assignment $\mathbb{G} : (A, \mathcal{R}) \mapsto (A, \mathcal{R}_{\mathrm{sc}})$ is a functor, that is, it is not clear whether, given a morphism $\pi : (A, \mathcal{R}) \to (A', \mathcal{R}')$, we get $\pi(\mathcal{R}_{\mathrm{sc}}) \subseteq \mathcal{R}'_{\mathrm{sc}}$. However, this is the case if we restrict to nondegenerate homomorphisms. If π is nondegenerate, then $\pi(A_{\mathrm{si}}) \subseteq A'_{\mathrm{si}}$, and if $\xi \overset{rc}{\sim} \xi$, then $\pi(\xi) \overset{rc}{\sim} \pi(\xi)$ (see Corollaries 3.3.2 and 3.3.3). Thus, the result follows from the description of the s-completion in Corollary 5.4.9. But in general, if π is degenerate, then it is not even clear whether $\pi(A_{\mathrm{si}}) \subseteq A'_{\mathrm{si}}$. If this were true, then $\mathbb{F}$ would be an invertible functor having $\mathbb{G}$ as its inverse and, therefore, the categories of e-continuously and s-continuously $\widehat{G}$-C^*-algebras would be isomorphic.

It is not clear either whether we can reformulate our theorems above in terms of s-continuously square-integrable $\widehat{G}$-C^*-algebras. However, since isomorphisms are nondegenerate, we can reformulate them in terms of isomorphism classes of s-continuously square-integrable $\widehat{G}$-C^*-algebras. For example, we can say that isomorphism classes of s-continuously square-integrable reduced $\widehat{G}$-C^*-algebras correspond bijectively to isomorphism classes of Fell bundles over G.

Recall that a $\widehat{G}$-C^*-algebra A is $\mathcal{R}$-proper if there is a unique dense, s-complete, relatively continuous subspace of A. By Proposition 6.10.27, this is equivalent to say that there is a unique dense, e-complete, relatively continuous subspace of A. This fact together with Corollary 6.10.23 yields:

Corollary 6.10.29. *Let A be a maximal (or reduced) $\widehat{G}$-C^*-algebra. Then A is $\mathcal{R}$-proper if and only if there is, up to isomorphism, a unique full (or reduced) Fell bundle structure for A. In particular, if A is $\mathcal{R}$-proper, then there is, up to isomorphism, a unique Fell bundle $\mathcal{B}$ over G such that $C^*(\mathcal{B}) \cong A$ (or $C^*_{\mathrm{r}}(\mathcal{B}) \cong A$) as $\widehat{G}$-C^*-algebras.*

6.11 Some examples and counterexamples

Let G be a locally compact group, and consider the quantum group $\mathcal{G} = C^*_{\mathrm{r}}(G)$.

Example 6.11.1. We begin with one of the most basic (and important) examples in the theory of continuously square-integrable coactions of groups, namely, we consider the $\widehat{G}$-Hilbert space $L^2(G)$ with the usual coaction given by $\gamma_{L^2(G)}(\xi) := \hat{W}^*(\xi \otimes 1)$ for all $\xi \in L^2(G)$, where $\hat{W} \in \mathcal{L}\big(L^2(G \times G)\big)$ is the unitary defined by $\hat{W}\zeta(s,t) := \zeta(s, s^{-1}t)$ for all $\zeta \in L^2(G \times G)$ and $s, t \in G$. Recall that $\hat{W}$ is the left regular corepresentation of the dual of $\mathcal{G}$, that is, of $\widehat{\mathcal{G}} = M\big(\mathcal{C}_0(G)\big) \cong \mathcal{C}_0(G)$. Therefore, $\gamma_{L^2(G)}$ is the same coaction we have considered in the general case of locally compact quantum groups. Here $M : \mathcal{C}_0(G) \to \mathcal{L}\big(L^2(G)\big)$ denotes the multiplication representation. As we already know from the general theory (see Proposition 5.2.8), there is at least one dense, relatively continuous subspace of $L^2(G)$, namely, $\mathcal{R}_0 = \hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})$. Recall that $\mathcal{T}_{\hat{\varphi}}$ denotes the Tomita $*$-algebra of the left Haar weight of $\widehat{\mathcal{G}} = \mathcal{C}_0(G)$. The modular group of $\mathcal{C}_0(G)$ is trivial, and therefore the Tomita $*$-algebra is $\mathcal{N}_{\hat{\varphi}} \cap \mathcal{N}^*_{\hat{\varphi}} = \mathcal{C}_0(G) \cap L^2(G)$. Since $\hat{\Lambda}$ is simply the inclusion of $C_0(G) \cap L^2(G)$ into $L^2(G)$, the general theory shows that $\mathcal{R}_0 = \mathcal{C}_0(G) \cap L^2(G)$ is a (dense) relatively continuous subspace of $L^2(G)$ and that $\mathcal{F}\big(L^2(G), \mathcal{R}_0\big) = \widehat{\mathcal{G}}^{\mathrm{c}} = \widehat{\mathcal{G}} = M\big(\mathcal{C}_0(G)\big) \cong \mathcal{C}_0(G)$.

We are going to prove these facts directly from the definitions. For all $\xi, \eta, f \in L^2(G)$ and $t \in G$, we have

$$\gamma_{L^2(G)}(\xi)^*(\eta \otimes 1)f|_t = (\xi^* \otimes 1)\hat{W}(\eta \otimes f)|_t = \int_G \overline{\xi(s)}\hat{W}(\eta \otimes f)(s,t)\,\mathrm{d}s$$
$$= \int_G \overline{\xi(s)}\eta(s)f(s^{-1}t)\,\mathrm{d}s = \lambda(\overline{\xi} \cdot \eta)f|_t,$$

where $\cdot$ denotes the pointwise product of functions. Hence

$$\gamma_{L^2(G)}(\xi)^*(\eta \otimes 1) = \lambda(\overline{\xi} \cdot \eta) \quad \text{for all } \xi, \eta \in L^2(G).$$

The pointwise product $\overline{\xi} \cdot \eta$ belongs to $L^1(G)$ and hence is always a left bounded function. Thus, Equation (6.1) yields

$$\xi \in L^2(G)_{\mathrm{si}} \Leftrightarrow \lambda(\overline{\xi} \cdot \eta) \in \mathcal{N}_\varphi \text{ for all } \eta \in L^2(G) \Leftrightarrow \overline{\xi} \cdot \eta \in L^2(G) \text{ for all } \eta \in L^2(G).$$

The last condition above is true if and only if $\xi \in L^\infty(G)$ (see [28, Problem 51]). This means that $L^2(G)_{\mathrm{si}} = L^2(G) \cap L^\infty(G)$ and (recall that $\Lambda(\lambda(\zeta)) = \zeta$ for every left bounded function $\zeta \in L^2(G)$ with $\lambda(\zeta) \in C^*_{\mathrm{r}}(G)$; see Section 6.1)

$$\langle\!\langle \xi | \eta = \Lambda\big(\lambda(\overline{\xi} \cdot \eta)\big) = \overline{\xi} \cdot \eta = M_{\overline{\xi}}\eta, \quad \text{for all } \xi \in L^2(G)_{\mathrm{si}},\ \eta \in L^2(G).$$

In other words, $\langle\!\langle \xi | = M_{\overline{\xi}}$ or, equivalently, $|\xi\rangle\!\rangle = M_\xi$. In particular, we get:

$$\|\xi\|_{\mathrm{si}} = \|\xi\| + \||\xi\rangle\!\rangle\| = \|\xi\|_2 + \|\xi\|_\infty \quad \text{for all } \xi \in L^2(G)_{\mathrm{si}} = L^2(G) \cap L^\infty(G).$$

With the above description of the bra-ket operators, we can now describe relative continuity. By definition, given $\xi, \eta \in L^2(G)_{\mathrm{si}}$, we have $\xi \overset{rc}{\sim} \eta$ if and only if $\langle\!\langle \xi \mid \eta \rangle\!\rangle = M_{\overline{\xi}\cdot\eta}$ belongs to the reduced crossed product $\mathbb{C} \rtimes_{\mathrm{r}} \widehat{\mathcal{G}}^{\mathrm{c}} = M\big(\mathcal{C}_0(G)\big) \cong \mathcal{C}_0(G)$. Thus

$$\xi \overset{rc}{\sim} \eta \Longleftrightarrow \overline{\xi} \cdot \eta \in \mathcal{C}_0(G).$$

With this criterion, it follows immediately that $\mathcal{R}_0 = \mathcal{C}_0(G) \cap L^2(G)$ is relatively continuous. By the formula $|\xi\rangle\!\rangle = M_\xi$, we also have $\mathcal{F}\big(L^2(G), \mathcal{R}_0\big) = M\big(\mathcal{C}_0(G)\big) \cong \mathcal{C}_0(G)$.

What is more interesting is that the criterion above allows us to find other (dense) relatively continuous subspaces of $L^2(G)$. One possible choice is $\mathcal{R}_\mu := M_\mu(\mathcal{R}_0)$ for any function $\mu \in L^\infty(G)$ with $|\mu|^2 = \overline{\mu} \cdot \mu = 1$. By the same criterion above, $\mathcal{R}_\mu$ is also a (dense) relatively continuous subspace of $L^2(G)$ and

$$\mathcal{F}(L^2(G), \mathcal{R}_\mu) = \{M_{\mu\cdot\xi} : \xi \in \mathcal{C}_0(G)\} \cong \mathcal{C}_0(G).$$

In particular, we have $\mathrm{Fix}(L^2(G), \mathcal{R}_\mu) \cong \mathcal{I}(L^2(G, \mathcal{R}_\mu)) \cong \mathcal{C}_0(G)$.

Note that the union $\mathcal{R}_0 \cup \mathcal{R}_\mu$ is relatively continuous if and only if $\mu \in \mathcal{C}_b(G)$. Thus the union of relatively continuous subspaces need not be relatively continuous (of course, this can only happen if G is not discrete, that is, if $C^*_{\mathrm{r}}(G)$ is not compact).

Another source of relatively continuous subspaces of $L^2(G)$ is the following. Let $S \subseteq G$ be an open subset of *full measure*, that is, the complement $G \backslash S$ has measure zero (for instance, if G is not discrete, then one can take S to be the complement of any finite subset) and define $\mathcal{R}_S := L^2(G) \cap \mathcal{C}_0(S)$, that is, the space of all functions in $L^2(G) \cap \mathcal{C}_0(G)$ vanishing outside of S. Note that $\mathcal{R}_S \subseteq \mathcal{R}_0$ is a dense, relatively continuous subspace of $L^2(G)$. Moreover, we have

$$\mathcal{F}(L^2(G), \mathcal{R}_S) = M(\mathcal{C}_0(S)) \cong \mathcal{C}_0(S),$$

where we consider the ideal $\mathcal{C}_0(S) \subseteq \mathcal{C}_0(G)$ as a Hilbert $\mathcal{C}_0(G)$-module in the obvious way. In particular, $\mathrm{Fix}(L^2(G), \mathcal{R}_S) \cong \mathcal{I}(L^2(G), \mathcal{R}_S) \cong \mathcal{C}_0(S)$. Thus, in general, the generalized fixed point algebra need not be isomorphic to $\mathcal{C}_0(G)$ (not even Morita equivalent).

More generally, we can consider $\mathcal{R}_{S,\mu} := M_\mu(\mathcal{R}_S)$, with μ as above. The subspace $\mathcal{R}_{S,\mu} \subseteq L^2(G)$ is also relatively continuous with $\mathcal{F}(L^2(G), \mathcal{R}_{S,\mu}) \cong \mathcal{C}_0(S)$ for all μ. Note that all the examples of dense, relatively continuous subspaces considered above are of the form $\mathcal{R}_{S,\mu}$ (just consider the special cases $S = G$ and $\mu = 1$).

For all the relatively continuous subspaces considered above, it is easy to see that $\mathcal{R} = \mathcal{R}_{\mathcal{F}(L^2(G),\mathcal{R})}$, that is, all the subspaces are s-complete. Let us now describe explicitly what completeness means. For this, one has to describe the left $A(G)$-action. Given $f, g, \xi, \eta \in L^2(G)$, we have

$$\langle \xi | (\mathrm{id} \otimes \omega_{f,g})(\hat{W}^*)\eta \rangle = \langle \xi \otimes f | \hat{W}^*(\eta \otimes g) \rangle = \int_G \int_G \overline{\xi(s) f(t)} \eta(s) g(st) \, \mathrm{d}s \, \mathrm{d}t$$

$$= \int_G \int_G \overline{\xi(s) f(s^{-1}t)} g(t) \eta(s) \, \mathrm{d}s \, \mathrm{d}t = \int_G \overline{\xi(s)} (g * \tilde{f})(s) \eta(s) \, \mathrm{d}s = \langle \xi | (g * \tilde{f}) \cdot \eta \rangle,$$

where $\tilde{f}$ is the function $\tilde{f}(r) := \overline{f(r^{-1})}$. Thus $(\mathrm{id} \otimes \omega_{f,g})(\hat{W}^*) = M_{g * \tilde{f}}$. The functional $\omega_{f,g}$ is identified with the function $t \mapsto \omega_{f,g}(\lambda_t)$ in $A(G)$. Note that $\omega_{f,g}(\lambda_t) = (g * \tilde{f})\check{\ }(t)$, where we write $\check{h}(t) := h(t^{-1})$ for a function h on G. It follows that

$$\omega * \xi = (\mathrm{id} \otimes \omega)(\gamma_{L^2(G)}(\xi)) = (\mathrm{id} \otimes \omega)(\hat{W}^*(\xi \otimes 1)) = (\mathrm{id} \otimes \omega)(\hat{W}^*)\xi = M_{\check{\omega}} \xi = \check{\omega} \cdot \xi$$

for all $\omega \in A(G)$ and $\xi \in L^2(G)$. Thus the action of $A(G)$ on $L^2(G)$ induced by $\gamma_{L^2(G)}$ is, up to the operation $\check{\ }$, given by pointwise multiplication. The Fourier algebra $A(G)$ is invariant under the operation $\check{\ }$, that is, $A(G)\check{\ } = A(G)$ (moreover, this operation is isometric; see [22, Remark 2.15]). Thus, by definition, a subspace $\mathcal{R} \subseteq L^2(G)_{\mathrm{si}} = L^2(G) \cap L^\infty(G)$ is complete if and only if it is closed with respect to the norm $\|\cdot\|_{\mathrm{si}} = \|\cdot\|_2 + \|\cdot\|_\infty$ and invariant under pointwise multiplication by functions in $A(G)$. Note that $\|\omega \cdot \xi\|_{\mathrm{si}} \leq \|\omega\|_\infty \|\xi\|_{\mathrm{si}}$ for all $\omega \in A(G)$ and $\xi \in L^2(G)$. Since $A(G)$ is dense in $\mathcal{C}_0(G)$, it follows that $\mathcal{R}$ is complete if and only if it is si-closed and invariant under pointwise multiplication by functions in $\mathcal{C}_0(G)$.

By this criterion, all the relatively continuous subspaces $\mathcal{R}_{S,\mu}$ considered above are complete. Indeed, as already mentioned, they are s-complete. Moreover, we claim that any complete subspace $\mathcal{R} \subseteq L^2(G)_{\mathrm{si}}$ is automatically s-complete. In fact, let $\xi \in L^2(G)_{\mathrm{si}}$ with $\xi \overset{rc}{\sim} \xi$ and $\omega * \xi = \check{\omega} \cdot \xi \in \mathcal{R}$ for all $\omega \in A(G)$. Since $A(G)$ is dense in $\mathcal{C}_0(G)$, there is a bounded approximate unit (ω_i) for $\mathcal{C}_0(G)$ with $\omega_i \in A(G)$ for all i. This means that (ω_i) is uniformly bounded and converges uniformly to 1 on compact subsets of G. Thus $\omega_i \cdot \xi \to \xi$ in $L^2(G)$ for all $\xi \in \mathcal{C}_c(G)$ and hence also for all $\xi \in L^2(G)$. Since $\xi \overset{rc}{\sim} \xi$, that is, since $|\xi|^2 \in \mathcal{C}_0(G)$, we also get $\omega_i \cdot \xi \to \xi$ in $L^\infty(G)$. It follows that $\omega_i \cdot \xi \to \xi$ in the si-norm and, therefore, $\xi \in \mathcal{R}$ because $\mathcal{R}$ is si-closed. This argument also shows that any complete, relatively continuous subspace of $L^2(G)$ is e-complete. We conclude that all the extra conditions (s-completess and e-completeness) are automatically satisfied by any complete, relatively continuous subspace of $L^2(G)$.

The condition $\xi \overset{rc}{\sim} \xi$ above was important in order to prove the s-completeness of any complete subspace. If we just suppose that $\omega \cdot \xi \in \mathcal{R}$ for all $\omega \in A(G)$ (or even in $\mathcal{C}_0(G)$), then this does not imply, in general, that $\xi \in \mathcal{R}$, even if $\mathcal{R}$ is s-complete and relatively continuous. Indeed, if we take $\mathcal{R} = L^2(G) \cap \mathcal{C}_0(G)$, then any function $\xi \in \tilde{\mathcal{R}} := L^2(G) \cap \mathcal{C}_b(G) \subseteq L^2(G)_{\mathrm{si}}$ satisfies the condition $\omega \cdot \xi \in \mathcal{R}$ for all $\omega \in \mathcal{C}_0(G)$, but, in general, $\tilde{\mathcal{R}}$ is not contained in $\mathcal{R}$ (consider, for example, $G = \mathbb{R}$). This also provides an example of a dense, complete subspace which is not e-complete and not relatively continuous. In fact, note that $\tilde{\mathcal{R}}$ is complete, and the si-closed linear span of $A(G) * \tilde{\mathcal{R}}$ is equal to $\mathcal{R}$. Thus $\tilde{\mathcal{R}}$ is not e-complete in general. It is also not relatively continuous in general either.

By Proposition 2.6.14, an operator $T \in \mathcal{L}\big(L^2(G)\big)$ is $\widehat{G}$-equivariant if and only if it commutes with all the multiplication operators M_ω for ω in $A(G)$ and hence also in $\mathcal{C}_0(G)$, that is, $T \in M\big(L^\infty(G)\big)$. In other words, $\mathcal{L}^{\widehat{G}}\big(L^2(G)\big) = M\big(L^\infty(G)\big)$. Thus, the $\widehat{G}$-equivariant unitaries on $L^2(G)$ are exactly the operators M_μ, where μ is some function in $L^\infty(G)$ with $|\mu|^2 = \overline{\mu} \cdot \mu = 1$.

In particular, $(L^2(G), \mathcal{R}_{S,\mu})$ and $(L^2(G), \mathcal{R}_S)$ are isomorphic as continuously square-integrable $\widehat{G}$-Hilbert spaces for all μ and S as above. And as we have already seen, the associated Hilbert $\mathcal{C}_0(G)$-module is $\mathcal{C}_0(S)$.

We claim that $\mathcal{R}_\mu$ is a maximal, relatively continuous subspace of $L^2(G)$ for all μ. In fact, since M_μ is an equivariant unitary and $\mathcal{R}_\mu = M_\mu(\mathcal{R}_0)$, it is enough to prove that $\mathcal{R}_0$ is maximal. Suppose that $\mathcal{R}$ is a relatively continuous subspace of $L^2(G)$ containing $\mathcal{R}_0$. Thus $\langle\!\langle \xi \,|\, \eta \rangle\!\rangle = M_{\overline{\xi}\eta} \in M\big(\mathcal{C}_0(G)\big)$ for all $\xi \in \mathcal{R}$, $\eta \in \mathcal{R}_0$. This means that $\overline{\xi}\eta \in \mathcal{C}_0(G)$ for all $\xi \in \mathcal{R}$ and $\eta \in \mathcal{R}_0$. This implies that ξ is a continuous function. Since $\langle\!\langle \xi \,|\, \xi \rangle\!\rangle = M_{|\xi|^2} \in M\big(\mathcal{C}_0(G)\big)$, we get $\xi \in \mathcal{C}_0(G) \cap L^2(G) = \mathcal{R}_0$. Therefore, $\mathcal{R}_0$ is maximal, proving our claim.

This simple example shows that, in general, there may be several maximal (and hence s-complete) relatively continuous subspaces of a Hilbert module.

Note, however, that in the example above the subspace $\mathcal{R}_0 = \mathcal{C}_0(G) \cap L^2(G)$ is, up to an equivariant unitary, the only maximal relatively continuous subspace of $L^2(G)$ we have so far. If we drop the maximality requirement, the only examples of dense, complete, relatively continuous subspaces of $L^2(G)$ we have so far are, up to equivariant unitaries, of the form $\mathcal{R}_S$, where S is some open subset of G of full measure.

Question 6.11.2. Is every dense, complete, relatively continuous subspaces of $L^2(G)$ of the form $\mathcal{R}_{S,\mu}$, for some S and μ as above?

The answer to this question depends on the topological structure of G. Even if G is not discrete, there are cases where the answer is affirmative and others cases where the answer is negative. However, this is not easy to see from the definition as in the examples above. In order to give a satisfactory answer to this question and also to produce new examples, we are going to use our general results. All this will be done in the next section.

6.11.1 Square-integrable $\widehat{G}$-Hilbert spaces

In this section, we not only provide an answer to Question 6.11.2, but also describe the class of separable (continuously) square-integrable $\widehat{G}$-Hilbert spaces in terms of (continuous) measurable fields of Hilbert spaces. The results we obtain here generalize those appearing in [48, Section 8] to the setting of non-Abelian groups.

By Theorem 5.5.6, the category of s-continuously square-integrable $\widehat{G}$-Hilbert spaces (that is, Hilbert $\mathbb{C},\widehat{G}$-Hilbert modules) is equivalent to the category of Hilbert $\mathcal{C}_0(G)$-modules. Given a Hilbert $\mathcal{C}_0(G)$-module $\mathcal{F}$, the associated s-continuously square-integrable $\widehat{G}$-Hilbert space is $(\mathcal{E}_{\mathcal{F}},\mathcal{R}_{\mathcal{F}})$, where $\mathcal{E}_{\mathcal{F}}=\mathcal{F}\otimes_{\mathcal{C}_0(G)}L^2(G)$ and $\mathcal{R}_{\mathcal{F}}$ is the s-completion of $\mathcal{F}\odot_{\mathcal{C}_0(G)}\mathcal{R}_0$. Here $\mathcal{R}_0=\hat{\Lambda}(\mathcal{T}_{\hat{\varphi}})=\mathcal{C}_0(G)\cap L^2(G)$ as in Example 6.11.1.

Isomorphism classes of Hilbert $\mathcal{C}_0(G)$-modules correspond bijectively to isomorphism classes of *continuous fields of Hilbert spaces* over G (see [78]). [7] Given a continuous field of Hilbert spaces $\mathcal{H}=\{\mathcal{H}_t\}_{t\in G}$, the associated Hilbert $\mathcal{C}_0(G)$-module is the space $\mathcal{C}_0(\mathcal{H})$ of continuous sections of $\mathcal{H}$ vanishing at infinity (with the canonical structure of a Hilbert $\mathcal{C}_0(G)$-module).

Let $L^2(\mathcal{H})$ be the Hilbert space of square-integrable sections of $\mathcal{H}$ (as usual, we identify any two sections that coincide almost everywhere). Note that

$$\mathcal{C}_0(\mathcal{H})\otimes_{\mathcal{C}_0(G)}L^2(G)\cong L^2(\mathcal{H})$$

(as Hilbert spaces) via the map $f\otimes_{\mathcal{C}_0(G)}\xi\mapsto f\cdot\xi$, where $\cdot$ denotes pointwise multiplication. Via this isomorphism, we can therefore endow $L^2(\mathcal{H})$ with a coaction of G turning it into a $\widehat{G}$-Hilbert space. It is easy to see that this coaction is given by the formula $\gamma_{L^2(\mathcal{H})}(\xi)=U(\xi\otimes 1)$ for all $\xi\in L^2(\mathcal{H})$, where $U\in\mathcal{L}(L^2(\mathcal{H})\otimes L^2(G))$ is the unitary defined by $U(\xi\otimes\eta)(s,t):=\xi(s)\eta(st)$ for all $\xi\in L^2(\mathcal{H})$ and $\eta\in L^2(G)$. Here we identify $L^2(\mathcal{H})\otimes L^2(G)\cong L^2(\mathcal{H}\times G)$ in the canonical way, where $\mathcal{H}\times G$ denotes the pull-back of $\mathcal{H}$ along the projection $G\times G\to G$, $(s,t)\mapsto s$). Note that U is the corepresentation of $\mathcal{G}=C^*_{\mathrm{r}}(G)$ associated to $\gamma_{L^2(\mathcal{H})}$. If $\mathcal{H}$ is the trivial bundle $\mathcal{H}=G\times\mathbb{C}$, then $L^2(\mathcal{H})=L^2(G)$ and U is the corepresentation $\hat{W}^*$ already considered in Example 6.11.1.

If G is Abelian, then the coaction $\gamma_{L^2(\mathcal{H})}$ corresponds to the action γ of $\widehat{G}$ on $L^2(\mathcal{H})$ given by the formula

$$\gamma_x(\xi)|_t=\overline{\langle x\,|\,t\rangle}\xi(t)\quad\text{for all }\xi\in L^2(\mathcal{H}),\ x\in\widehat{G},\ t\in G.\tag{6.20}$$

Basically, the same the same considerations for $L^2(G)$ in the previous section can also be done in the general case of $L^2(\mathcal{H})$. In fact, we have the following result.

[7] We refer to [10, 11, 23] for more details on fields of Hilbert spaces. In [23, 78], continuous fields of Hilbert spaces are also called (*continuous*) *Hilbert bundles.*

Proposition 6.11.3. *Let $\mathcal{H}=\{\mathcal{H}_t\}_{t\in G}$ be a continuous field of Hilbert spaces and consider the $\widehat{G}$-Hilbert space $L^2(\mathcal{H})$ as above.*

(i) *An element $\xi\in L^2(\mathcal{H})$ is square-integrable if and only if $\xi\in L^\infty(\mathcal{H})$ (the space of essentially bounded measurable sections of $\mathcal{H}$), that is, $L^2(\mathcal{H})_{\mathrm{si}}=L^2(\mathcal{H})\cap L^\infty(\mathcal{H})$.*

(ii) *For $\xi\in L^2(\mathcal{H})_{\mathrm{si}}$, the operators $\langle\!\langle\xi|\in\mathcal{L}(L^2(\mathcal{H}),L^2(G))$ and $|\xi\rangle\!\rangle\in\mathcal{L}\big(L^2(G),L^2(\mathcal{H})\big)$ are given by*

$$\langle\!\langle\xi|\eta|_t=\langle\xi(t)\,|\,\eta(t)\rangle\quad\textit{and}\quad|\xi\rangle\!\rangle f|_t=\xi(t)f(t)$$

for all $\eta\in L^2(\mathcal{H})$, $f\in L^2(G)$ and $t\in G$.

(iii) *For $\xi,\eta\in L^2(\mathcal{H})_{\mathrm{si}}$, the operators $\langle\!\langle\xi\,|\,\eta\rangle\!\rangle\in\mathcal{L}\big(L^2(G)\big)$ and $|\xi\rangle\!\rangle\langle\!\langle\eta|\in\mathcal{L}\big(L^2(\mathcal{H})\big)$ are given by*

$$\langle\!\langle\xi\,|\,\eta\rangle\!\rangle f|_t=\langle\xi(t)\,|\,\eta(t)\rangle f(t)=M_{\langle\xi|\eta\rangle_0}f|_t\quad\textit{and}\quad|\xi\rangle\!\rangle\langle\!\langle\eta|\zeta|_t=\xi(t)\langle\eta(t)\,|\,\zeta(t)\rangle$$

for all $f\in L^2(G)$, $\zeta\in L^2(\mathcal{H})$ and $t\in G$, where $\langle\xi\mid\eta\rangle_0(t):=\langle\xi(t)\mid\eta(t)\rangle$ and $M:L^\infty(G)\to\mathcal{L}\big(L^2(G)\big)$ denotes the multiplication representation. In particular,

$$\xi\overset{rc}{\sim}\eta\Longleftrightarrow\langle\xi\,|\,\eta\rangle_0\in\mathcal{C}_0(G).$$

Proof. Take $\xi\in L^2(\mathcal{H})$. Then, for all $\eta\in L^2(\mathcal{H})$, $f\in L^2(G)$ and $t\in G$, we have

$$\begin{aligned}\gamma_K(\xi)^*(\eta\otimes 1)f|_t&=(\xi^*\otimes 1)U^*(\eta\otimes f)|_t\\&=\int_G\langle\xi(s)\,|\,U^*(\eta\otimes f)(s,t)\rangle\,\mathrm{d}s\\&=\int_G\langle\xi(s)\,|\,\eta(s)\rangle f(s^{-1}t)\,\mathrm{d}s\\&=\lambda(\langle\xi\,|\,\eta\rangle_0)f|_t.\end{aligned}$$

Thus $\gamma_{L^2(\mathcal{H})}(\xi)^*(\eta\otimes 1)=\lambda(\langle\xi\,|\,\eta\rangle_0)$. Note that $\langle\xi\,|\,\eta\rangle_0\in L^1(G)$. Equation (6.1) yields

$$\xi\in L^2(\mathcal{H})_{\mathrm{si}}\Longleftrightarrow\lambda(\langle\xi\,|\,\eta\rangle_0)\in\bar{\mathcal{N}}_\varphi\quad\forall\,\eta\in L^2(\mathcal{H})\Longleftrightarrow\langle\xi\,|\,\eta\rangle_0\in L^2(G)\quad\forall\,\eta\in L^2(\mathcal{H}),$$

and, in this case, $\langle\!\langle\xi|\eta=\langle\xi\,|\,\eta\rangle_0$ for all $\eta\in L^2(\mathcal{H})$. Since $\langle\xi\,|\,\eta\rangle_0\in L^2(G)$ for all $\xi\in L^\infty(\mathcal{H})$, we get $L^2(\mathcal{H})\cap L^\infty(\mathcal{H})\subseteq L^2(\mathcal{H})_{\mathrm{si}}$. Conversely, if ξ is any element of $L^2(\mathcal{H})$, we can define the linear map

$$S_\xi:\mathcal{C}_c(G)\to L^2(\mathcal{H}),\quad S_\xi(f)|_t:=\xi(t)f(t).\tag{6.21}$$

Suppose that ξ is square-integrable. It is easy to show that $\langle\langle\!\langle\xi|\eta\mid f\rangle=\langle\eta\mid S_\xi f\rangle$ for all $\eta\in L^2(\mathcal{H})$ and $f\in\mathcal{C}_c(G)$. It follows that $|\xi\rangle\!\rangle f=S_\xi f$ for all $f\in\mathcal{C}_c(G)$. Thus $|\xi\rangle\!\rangle$ extends S_ξ to a bounded operator $L^2(G)\to L^2(\mathcal{H})$. This can only happen if $\xi\in L^\infty(\mathcal{H})$ and, in this case, the same formula (6.21) for the operator S_ξ also holds for any function $f\in L^2(G)$. Therefore, $L^2(\mathcal{H})_{\mathrm{si}}=L^2(\mathcal{H})\cap L^\infty(\mathcal{H})$, and $|\xi\rangle\!\rangle=S_\xi$ for all $\xi\in L^2(\mathcal{H})_{\mathrm{si}}$. All the other assertions now follow. $\square$

Now we describe completeness for subspaces of $L^2(\mathcal{H})_{\mathrm{si}}$.

Proposition 6.11.4. *The* si-*norm on* $L^2(\mathcal{H})_{\mathrm{si}} = L^2(\mathcal{H}) \cap L^\infty(\mathcal{H})$ *is given by*

$$\|\xi\|_{\mathrm{si}} = \|\xi\|_2 + \|\xi\|_\infty \quad \text{for all } \xi \in L^2(\mathcal{H})_{\mathrm{si}},$$

where $\|\xi\|_2$ *(resp.* $\|\xi\|_\infty$*) denotes the norm of* ξ *in* $L^2(\mathcal{H})$ *(resp.* $L^\infty(\mathcal{H})$*). The left* $A(G)$*-action on* $L^2(\mathcal{H})$ *induced by the coaction of* G *is given by*

$$\omega * \xi = \check{\omega} \cdot \xi \quad \text{for all } \omega \in A(G),\ \xi \in L^2(\mathcal{H}), \tag{6.22}$$

where $\cdot$ *denotes pointwise multiplication and* $\check{\omega}(t) := \omega(t^{-1})$ *for all* $t \in G$.

A subspace $\mathcal{R} \subseteq L^2(\mathcal{H})_{\mathrm{si}}$ *is complete if and only if it is* si-*closed and* $\omega \cdot \xi \in \mathcal{R}$ *for all* $\omega \in \mathcal{C}_0(G)$.

Any complete subspace of $L^2(\mathcal{H})_{\mathrm{si}}$ *is automatically s-complete, and any complete, relatively continuous subspace of* $L^2(\mathcal{H})$ *is automatically e-complete.*

Proof. Let $\xi \in L^2(\mathcal{H})_{\mathrm{si}}$. Since $\langle\langle \xi \,|\, \xi \rangle\rangle = M_{\langle \xi | \xi \rangle_0}$, we have

$$\|\langle\langle \xi \,|\, \xi \rangle\rangle\| = \|\langle \xi \,|\, \xi \rangle_0\|_\infty = \|\xi\|_\infty^2.$$

The formula for the si-norm now follows. The formula for the $A(G)$-action is proved as in the case of $L^2(G)$ in Example 6.11.1. Thus $\mathcal{R} \subseteq L^2(\mathcal{H})_{\mathrm{si}}$ is complete if and only if it is si-closed and $\omega \cdot \xi \in \mathcal{R}$ for all $\omega \in A(G)$ and $\xi \in \mathcal{R}$. Since $\|\omega \cdot \xi\|_{\mathrm{si}} \le \|\omega\|_\infty \|\xi\|_{\mathrm{si}}$, and since $A(G)$ is dense in $\mathcal{C}_0(G)$, this last condition is equivalent to the requirement that $\omega \cdot \xi \in \mathcal{R}$ for all $\omega \in \mathcal{C}_0(G)$ and $\xi \in \mathcal{R}$. The last assertion also follows in the same way as in the case of $L^2(G)$ in Example 6.11.1. □

As already seen in Example 6.11.1, the relative continuity in the last assertion of the proposition above is really necessary.

Theorem 6.11.5. *Let* G *be a locally compact group. Given a continuous field of Hilbert spaces* $\mathcal{H}$ *over* G*, we define* $\mathcal{R}(\mathcal{H})$ *to be the subspace* $L^2(\mathcal{H}) \cap \mathcal{C}_0(\mathcal{H}) \subseteq L^2(\mathcal{H})$*. Then the assignment* $\mathcal{H} \mapsto (L^2(\mathcal{H}), \mathcal{R}(\mathcal{H}))$ *is a bijection between isomorphism classes of continuous fields of Hilbert spaces over* G *and isomorphism classes of continuously square-integrable* $\widehat{G}$*-Hilbert spaces.*

Proof. By Theorem 5.5.6, we know that isomorphism classes of Hilbert $\mathcal{C}_0(G)$-modules correspond bijectively to isomorphism classes of s-continuously square-integrable $\widehat{G}$-Hilbert spaces via the map $\mathcal{F} \mapsto (\mathcal{E}_\mathcal{F}, \mathcal{R}_\mathcal{F})$. As already mentioned, isomorphism classes of Hilbert $\mathcal{C}_0(G)$-modules correspond bijectively to isomorphism classes of continuous fields of Hilbert spaces over G. Given such a field $\mathcal{H}$, the corresponding Hilbert $\mathcal{C}_0(G)$-module is $\mathcal{F} = \mathcal{C}_0(\mathcal{H})$. As already observed, $\mathcal{E}_\mathcal{F} = \mathcal{C}_0(\mathcal{H}) \otimes_{\mathcal{C}_0(G)} L^2(G)$ is isomorphic to $L^2(\mathcal{H})$ via the map $f \otimes_{\mathcal{C}_0(G)} \xi \mapsto f \cdot \xi$. By definition of the coaction on $L^2(\mathcal{H})$, this is an isomorphism of $\widehat{G}$-Hilbert spaces. In this picture, the relatively continuous subspace $\mathcal{R}_\mathcal{F}$ corresponds to the s-completion of $\mathcal{C}_0(\mathcal{H}) \cdot \mathcal{R}_0$ in $L^2(\mathcal{H})$, where $\mathcal{R}_0 = \mathcal{C}_0(G) \cap L^2(G)$. By Proposition 6.11.4, any complete subspace is automatically s-complete. Thus $\mathcal{R}_\mathcal{F}$ is just the completion of $\mathcal{C}_0(\mathcal{H}) \cdot \mathcal{R}_0$ which is equal to $\mathcal{R}(\mathcal{H})$. In fact, $\mathcal{R}(\mathcal{H})$ is equal to the completion of $\mathcal{C}_c(\mathcal{H})$. □

In the situation above, we have

$$\mathcal{F}\big(L^2(\mathcal{H}), \mathcal{R}(\mathcal{H})\big) \cong \mathcal{C}_0(\mathcal{H}). \tag{6.23}$$

This is a special case of the general result $\mathcal{F}(\mathcal{E}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}}) \cong \mathcal{F}$ (see Corollary 5.5.5). The C^*-algebra of compact operators on the Hilbert $\mathcal{C}_0(G)$-module $\mathcal{C}_0(\mathcal{H})$ is isomorphic to $\mathcal{C}_0(\mathcal{K}(\mathcal{H}))$ (see [27]), where $\mathcal{K}(\mathcal{H}) = \{\mathcal{K}(\mathcal{H}_t)\}_{t\in G}$ denotes the C^*-bundle of compact operators with the canonical structure. In particular, we get

$$\operatorname{Fix}\big(L^2(\mathcal{H}), \mathcal{R}(\mathcal{H})\big) \cong \mathcal{C}_0\big(\mathcal{K}(\mathcal{H})\big). \tag{6.24}$$

This C^*-algebra is Morita equivalent to the ideal in $\mathcal{C}_0(G)$ generated by the inner product of $\mathcal{C}_0(\mathcal{H})$, which is easily seen to be equal to $\mathcal{C}_0(S)$, where S is the open subset of G consisting of all $t \in G$ with $\mathcal{H}_t \neq \{0\}$. In other words, we have

$$\mathcal{I}\big(L^2(\mathcal{H}), \mathcal{R}(\mathcal{H})\big) \cong \mathcal{C}_0(S). \tag{6.25}$$

In particular, $\mathcal{R}(\mathcal{H})$ is saturated if and only if all the fibers $\mathcal{H}_t$ are non-zero.

Now we consider an even more general class of $\widehat{G}$-Hilbert spaces. Note that in order to consider the space $L^2(\mathcal{H})$, we do not need a continuous field, but just a *measurable field* of Hilbert spaces $\mathcal{H} = \{\mathcal{H}_t\}_{t\in G}$ over G (see [10, 11]). However, to avoid measure theoretic difficulties, we always assume that G is second countable and that the fibers $\mathcal{H}_t$ are separable when working with measurable fields. In this case, we also say that $\mathcal{H}$ is a measurable field of separable Hilbert spaces over G.

Measurable fields of separable Hilbert spaces are classified by the *dimension function*:

$$d(t) := \dim(\mathcal{H}_t) \quad \text{for all } t \in G.$$

This is always a measurable function $d : G \to \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, and any such function appears as the dimension function of some measurable field of separable Hilbert spaces. Two measurable fields $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic if and only if the respective dimension functions d and d' are equal almost everywhere.

The dimension function of a continuous field of separable Hilbert spaces is always lower semi-continuous, that is, $\{t \in G : d(t) > n\}$ is open in G for all $n \in \mathbb{N}$. This is one of the basic differences between measurable and continuous fields. Another difference is that continuous fields are not determined by the dimension function. If two continuous fields of separable Hilbert spaces are isomorphic (as continuous fields), then the dimension functions are equal, but the converse does *not* hold in general. This will become clear by the end of this section.

Given a measurable field of separable Hilbert spaces $\mathcal{H}$, one can always decompose the space $L^2(\mathcal{H})$ in a canonical way. Given $n \in \bar{\mathbb{N}}$, we define $S_n := \{t \in G : d(t) = n\}$ and let $\mathcal{H}_n$ be the measurable field whose dimension function is $n \cdot 1_{S_n}$ (where 1_S denotes the characteristic function of a subset $S \subseteq G$ and we use the convention $\infty \cdot 0 = 0$). This gives a partition of G into measurable subsets such that

$$L^2(\mathcal{H}) \cong \bigoplus_{n\in\bar{\mathbb{N}}} L^2(\mathcal{H}_n).$$

Using this decomposition, we can now define a coaction of G on $L^2(\mathcal{H})$. Indeed, each $L^2(\mathcal{H}_n)$ is canonically isomorphic to $L^2(S_n)^n \cong L^2(S_n) \otimes \mathbb{C}^n$ (where $\mathbb{C}^\infty$ is, by definition, $l^2\mathbb{N}$). Now, for each measurable subset $S \subseteq G$, the Hilbert space $L^2(S)$ is a $\widehat{G}$-invariant direct summand of $L^2(G)$ (remember that all the multiplication operators on $L^2(G)$ are $\widehat{G}$-equivariant). In particular, we have a coaction of G on $L^2(S)$. The corresponding corepresentation of $\mathcal{G} = C^*_\mathrm{r}(G)$ on $L^2(S)$ is the unitary

$$U_S \in \mathcal{L}\big(L^2(S) \otimes \mathcal{G}\big) \subseteq \mathcal{L}\big(L^2(S \times G)\big) \quad \text{given by} \quad U_S\zeta(s,t) := \zeta(s, st).$$

Taking direct sums, we get a corepresentation

$$U \in \mathcal{L}\big(L^2(\mathcal{H}) \otimes \mathcal{G}\big) \subseteq \mathcal{L}\big(L^2(\mathcal{H} \times G)\big)$$

of $\mathcal{G}$ on $L^2(\mathcal{H})$, which is given by the formula:

$$U\zeta(s,t) = \zeta(s, st) \quad \text{for all } \zeta \in L^2(\mathcal{H} \times G),\ s,t \in G.$$

Thus we have a coaction $\gamma_{L^2(\mathcal{H})}$ of G on $L^2(\mathcal{H})$ given by

$$\gamma_{L^2(\mathcal{H})}(\xi) = U(\xi \otimes 1) \quad \text{for all } \xi \in L^2(\mathcal{H}).$$

Note that this coaction is given by the same formula as in the case of continuous fields. Thus Propositions 6.11.3 and 6.11.4 also hold for measurable fields (and the proof is exactly the same). In particular, $L^2(\mathcal{H})_{\mathrm{si}} = L^2(\mathcal{H}) \cap L^\infty(\mathcal{H})$, and hence $L^2(\mathcal{H})$ is square-integrable. Using Kasparov's Stabilization Theorem, we can now prove that any separable square-integrable $\widehat{G}$-Hilbert space is of this form:

Theorem 6.11.6. *Let G be a second countable locally compact group and let K be a separable square-integrable $\widehat{G}$-Hilbert space. Then there is, up to isomorphism, a unique measurable field $\mathcal{H} = \{\mathcal{H}_t\}_{t\in G}$ of separable Hilbert spaces over G such that K and $L^2(\mathcal{H})$ are isomorphic as $\widehat{G}$-Hilbert spaces. Hence, isomorphism classes of separable square-integrable $\widehat{G}$-Hilbert spaces correspond bijectively to isomorphism classes of measurable fields of separable Hilbert spaces.*

Proof. By Kasparov's Stabilization Theorem (see Theorem 4.5.6), K is a $\widehat{G}$-invariant direct summand of $L^2(G)^\infty$. Note that $L^2(G)^\infty$ is isomorphic to the $\widehat{G}$-Hilbert space $L^2(G, l^2\mathbb{N})$ of L^2-sections of the constant field with fiber $l^2\mathbb{N}$. By Equation (6.22) the left $A(G)$-action on $L^2(G)^\infty$ (and hence also on K) is, up to the operation $\omega \mapsto \check{\omega}$, given by pointwise multiplication. Note that this action is a representation of $A(G)$ on $L^2(G)^\infty$, and it extends to a normal representation of $L^\infty(G)$ on $L^2(G)^\infty$ (also by pointwise multiplication). The fact that K is a $\widehat{G}$-invariant direct summand of $L^2(G)^\infty$ just means that K is a (normal) subrepresentation of $L^2(G)^\infty$. Any separable normal representation of $L^\infty(G)$ has the form $L^2(\mathcal{H})$ for some measurable field of separable Hilbert spaces $\mathcal{H}$ over G, where $L^\infty(G)$ acts by multiplication. This field is uniquely determined up to isomorphism by the representation of $L^\infty(G)$. Thus K, considered as a representation of $L^\infty(G)$, is isomorphic to some $L^2(\mathcal{H})$ as in the statement of the theorem. This implies, in particular, that the representation of $A(G)$ and therefore (by Lemma 2.6.14) the coactions of G are isomorphic. Conversely, if the coactions are isomorphic, then so are the representations of $A(G)$ and hence also of $L^\infty(G)$. Therefore $\mathcal{H}$ is uniquely determined up to isomorphism by the coaction of G. □

As already mentioned, measurable fields of separable Hilbert spaces are classified by the dimension function. As a consequence, we get that isomorphism classes of separable square-integrable $\widehat{G}$-Hilbert spaces correspond bijectively to (almost everywhere) equivalence classes of measurable functions $G \to \bar{\mathbb{N}}$.

By a *continuous structure* for a measurable field $\mathcal{H}$ over G we mean a continuous field $\mathcal{H}'$ over G together with an isomorphism $\mathcal{H} \cong \mathcal{H}'$ of measurable fields or, equivalently, an isomorphism $L^2(\mathcal{H}) \cong L^2(\mathcal{H}')$ of $\widehat{G}$-Hilbert spaces.

Our results can be used to classify the continuous structures for a given measurable field in the following way:

Theorem 6.11.7. *Let G be a second countable locally compact group, and let $K := L^2(\mathcal{H})$ be a separable square-integrable $\widehat{G}$-Hilbert space, where $\mathcal{H}$ is some measurable field of separable Hilbert spaces. Then there is a bijective correspondence between dense, complete, relatively continuous subspaces $\mathcal{R} \subseteq K$ and isomorphism classes of continuous structures for $\mathcal{H}$.*

Proof. By Proposition 6.11.4, any complete subspace of K_{si} is s-complete. Theorem 5.4.4 implies that dense, complete, relatively continuous subspaces $\mathcal{R} \subseteq K$ correspond to essential, concrete Hilbert $\mathcal{C}_0(G)$-modules in $\mathcal{L}^{\widehat{G}}(L^2(G), K)$. Alternatively, by Theorem 5.1.2, we can describe essential, concrete Hilbert $\mathcal{C}_0(G)$-modules in $\mathcal{L}^{\widehat{G}}(L^2(G), K)$ by isomorphism classes of pairs $(\mathcal{F}, u)$, where $\mathcal{F}$ is an abstract Hilbert $\mathcal{C}_0(G)$-module and u is a $\widehat{G}$-equivariant unitary operator $u : \mathcal{F} \otimes_{\mathcal{C}_0(G)} L^2(G) \to K$. In this picture, the corresponding relatively continuous subspace is the completion of $u(\mathcal{F} \odot_{\mathcal{C}_0(G)} \mathcal{C}_c(G))$. This only depends on the isomorphism class of the pair $(\mathcal{F}, u)$.

As already mentioned, isomorphism classes of continuous fields of Hilbert spaces over G correspond to isomorphism classes of Hilbert $\mathcal{C}_0(G)$-modules via the assignment $\mathcal{H}' \mapsto \mathcal{C}_0(\mathcal{H}')$. Since $\mathcal{C}_0(\mathcal{H}') \otimes_{\mathcal{C}_0(G)} L^2(G) \cong L^2(\mathcal{H}')$, we conclude that dense, complete, relatively continuous subspaces $\mathcal{R} \subseteq L^2(\mathcal{H})$ correspond to isomorphism classes of pairs $(\mathcal{H}', u)$, where $\mathcal{H}'$ is a continuous field of Hilbert spaces over G and u is a $\widehat{G}$-equivariant unitary $u : L^2(\mathcal{H}') \to L^2(\mathcal{H})$. This yields the result. □

Example 6.11.8. Suppose that $C \subseteq G$ is a closed subset with non-zero measure and empty interior (for instance, a Cantor subset of $G = \mathbb{R}$ with positive measure). Then the subspace $L^2(C)$ of functions in $L^2(G)$ vanishing outside C is a $\widehat{G}$-invariant Hilbert subspace. In particular, it is square-integrable. The dimension function of the underlying measurable field coincides with the characteristic function 1_C of C. It is easy to see that there is no lower semi-continuous function that is almost everywhere equal to 1_C (see [48, Section 8]). Theorem 6.11.7 shows that there is no continuous structure for the measurable field underlying $L^2(S)$ or, equivalently, there is no dense relatively continuous subspace of

$L^2(C)$. This can also be seen directly from Proposition 6.11.3. In fact, this proposition says that for $\xi, \eta \in L^2(C)_{\mathrm{si}}$, we have $\xi \overset{rc}{\sim} \eta$ if and only if the function $t \mapsto \overline{\xi(t)}\eta(t)$ belongs to $\mathcal{C}_0(G)$. Since C has empty interior, this function must be zero. Hence $\{0\}$ is the only relatively continuous subset of the square-integrable $\widehat{G}$-Hilbert space $L^2(C)$.

We can also classify the relatively continuous subspaces up to $\widehat{G}$-equivariant unitaries. We shall say that two dense, complete, relatively continuous subspaces $\mathcal{R}_1, \mathcal{R}_2$ in a $\widehat{G}$-Hilbert space K are *equivalent* if there is a $\widehat{G}$-equivariant unitary u on K such that $u(\mathcal{R}_1) = \mathcal{R}_2$. Note that, in this case, u implements an isomorphism $(K, \mathcal{R}_1) \cong (K, \mathcal{R}_2)$ of continuously square-integrable $\widehat{G}$-Hilbert spaces. Thus the corresponding Hilbert $\mathcal{C}_0(G)$-modules are isomorphic. Conversely, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two isomorphic Hilbert $\mathcal{C}_0(G)$-modules for which there is a $\widehat{G}$-equivariant unitary $u_k : \mathcal{F}_k \otimes_{\mathcal{C}_0(G)} L^2(G) \to K$ of $\widehat{G}$-Hilbert spaces ($k = 1, 2$), then the corresponding induced relatively continuous subspaces $\mathcal{R}_k \subseteq K$, given as completion of $u_k\big(\mathcal{F}_k \odot_{\mathcal{C}_0(G)} \mathcal{C}_c(G)\big)$, are equivalent via the $\widehat{G}$-equivariant unitary $u := u_2 \circ (v \otimes_{\mathcal{C}_0(G)} \mathrm{id}) \circ u_1^{-1}$ on K, where v denotes the given isomorphism $\mathcal{F}_1 \to \mathcal{F}_2$. This yields the following result:

Corollary 6.11.9. *Let $K = L^2(\mathcal{H})$ be a separable square-integrable $\widehat{G}$-Hilbert space. Then equivalence classes of dense, complete, relatively continuous subspaces $\mathcal{R} \subseteq K$ correspond bijectively to isomorphism classes of continuous fields of separable Hilbert spaces $\mathcal{H}'$ for which there is an isomorphism $L^2(\mathcal{H}') \cong K$ of $\widehat{G}$-Hilbert spaces (or, equivalently, an isomorphism $\mathcal{H}' \cong \mathcal{H}$ of measurable fields).*

As an application, we can now give an answer to our Question 6.11.2, describing all dense, complete, relatively continuous subspaces of $L^2(G)$ or, equivalently, all continuous structures for the trivial 1-dimensional bundle $G \times \mathbb{C}$.

Corollary 6.11.10. *There is a bijective correspondence between dense, complete, relatively continuous subspaces $\mathcal{R} \subseteq L^2(G)$ and isomorphism classes of triples (S, V, ψ), where S is an open subset of $\widehat{G}$ of full measure, V is a Hermitian complex line bundle over S, and ψ is a measurable section of V with $|\psi(t)| = 1$ for almost every $t \in S$.*

Two triples (S_1, V_1, ψ_1) and (S_2, V_2, ψ_2) are isomorphic if and only if $S_1 = S_2$ and there is an isomorphism $\phi\colon V_1 \to V_2$ of Hermitian complex line bundles with $\phi_*(\psi_1) = \psi_2$, that is, $\phi \circ \psi_1(t) = \psi_2(t)$ for all $t \in G$.

Equivalence classes of dense, complete, relatively continuous subspaces $\mathcal{R} \subseteq L^2(G)$ correspond bijectively to isomorphism classes of pairs (S, V) with S and V as above, where two pairs (S_1, V_1) and (S_2, V_2) are isomorphic if and only if $S_1 = S_2$ and $V_1 \cong V_2$ as Hermitian complex line bundles.

Proof. Theorem 6.11.7 and Corollary 6.11.9 reduce the problem to that of classifying the continuous structures for the trivial field $G \times \mathbb{C}$, that is, for the measurable field of Hilbert spaces over G underlying $L^2(G)$. In other words, we must consider pairs $(\mathcal{H}, u)$, where $\mathcal{H} = \{\mathcal{H}_t\}_{t \in G}$ is a continuous field of separable Hilbert spaces and $u : L^2(\mathcal{H}) \to L^2(G)$ is a $\widehat{G}$-equivariant unitary. Since the dimension function of $G \times \mathbb{C}$ is constant equal to 1, we have $\dim \mathcal{H}_t = 1$ for almost every $t \in G$. Since the map $t \mapsto \dim \mathcal{H}_t$ is lower semi-continuous, this implies that the set S of all $t \in G$ with $\dim \mathcal{H}_t = 1$ is an open subset of

G of full measure. In fact, S is equal to $\{t \in G : \dim \mathcal{H}_t > 0\} = \{t \in G : \dim \mathcal{H}_t \geq 1\}$ because the set where $\dim \mathcal{H}_t > 1$ is also open and therefore empty since it has measure zero. Note that S is an invariant of the continuous field.

If $t \in S$, then there is a non-zero continuous section in a neighborhood of t. This provides a local trivialization of $\mathcal{H}$ near t because $\dim \mathcal{H} = 1$ on S. Thus $\mathcal{H}$ is locally trivial on S (see also [23, Remark II.13.9]). Equivalently, it is a Hermitian complex line bundle V over S, that is, a complex line bundle with a continuously varying family of inner products on the fibers. We conclude that $\mathcal{H}$ is uniquely determined by the pair (S, V) up to isomorphism. In this picture, $L^2(\mathcal{H})$ corresponds to the space $L^2(V)$ of square-integrable sections of V.

Any measurable section ψ of V with $|\psi| = 1$ almost everywhere determines an equivariant unitary $f \mapsto f \cdot \psi$ from $L^2(G)$ to $L^2(V)$. Conversely, any unitary $L^2(G) \to L^2(V)$ is of this form for some ψ as above. Hence, continuous structures $(\mathcal{H}, u)$ for $G \times \mathbb{C}$ correspond to triples (S, V, ψ) as in the statement. By definition, two continuous structures $(\mathcal{H}_1, u_1)$ and $(\mathcal{H}_2, u_2)$ are isomorphic if and only if there is an isomorphism $\mathcal{H}_1 \cong \mathcal{H}_2$ which is compatible with the isomorphisms $L^2(\mathcal{H}_1) \cong L^2(G) \cong L^2(\mathcal{H}_2)$. This is translated into the fact that the corresponding triples (S_1, V_1, ψ_1) and (S_2, V_2, ψ_2) are isomorphic. $\square$

Recall that the first Chern class classifies isomorphism classes of complex line bundles over S by elements of the cohomology group $H^2(S; \mathbb{Z})$. The Hermitian inner product on a complex vector bundle is unique up to isomorphism. Thus the result above can also be rephrased in terms of pairs (S, x) with S as above and x an element of $H^2(S; \mathbb{Z})$.

Given a triple (S, V, ψ) as above, the corresponding relatively continuous subspace of $L^2(G)$ is, by construction,

$$\mathcal{R}_{S,V,\psi} = \{f \in L^2(G) : f \cdot \psi \text{ is a } \mathcal{C}_0\text{-section of } V \text{ on } S\}.$$

Note that by Equation (6.23), the corresponding Hilbert $\mathcal{C}_0(G)$-module is (isomorphic to) the space $\mathcal{C}_0(V)$ of $\mathcal{C}_0$-sections of V. The generalized fixed point algebra is the algebra bundle of endomorphisms of V which is always trivial for a line bundle. The identity section provides a nowhere vanishing global section. Therefore,

$$\operatorname{Fix}\bigl(L^2(G), \mathcal{R}_{S,V,\psi}\bigr) \cong \mathcal{C}_0(S). \tag{6.26}$$

In particular, the generalized fixed point algebra is independent of V and ψ. By Equation (6.25), the same is true for the corresponding ideal in $\mathcal{C}_0(G)$:

$$\mathcal{I}\bigl(L^2(G), \mathcal{R}_{S,V,\psi}\bigr) \cong \mathcal{C}_0(S). \tag{6.27}$$

If V is the trivial line bundle $V = S \times \mathbb{C}$, then ψ is a unitary function in $L^\infty(S)$, and we get the relatively continuous subspace

$$\mathcal{R}_{S,\psi} = \{f \in L^2(G) : f \cdot \psi \in \mathcal{C}_0(S)\}.$$

Since S has full measure, unitary functions in $L^\infty(S)$ correspond to unitary functions in $L^\infty(G)$. Thus these spaces can be equivalently described by relatively continuous subspaces of the form $\mathcal{R}_{S,\mu} = M_\mu(\mathcal{R}_S)$ as in Example 6.11.1, where μ is a unitary function in $L^\infty(G)$ and $\mathcal{R}_S = L^2(G) \cap \mathcal{C}_0(S)$. This describes all the relatively continuous subspaces corresponding to trivial line bundles.

Therefore, the answer to Question 6.11.2 is affirmative if and only if all the open subsets of full measure in G carry no non-trivial complex line bundles, that is, whenever $H^2(S, \mathbb{Z})$ is trivial for all open subsets $S \subseteq G$ of full measure.

Example 6.11.11. Consider the case where G is the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Since $\mathbb{T}$ is 1-dimensional, subsets of $\mathbb{T}$ carry no non-trivial complex line bundles. Hence Question 6.11.2 is affirmative in this case, and therefore all dense, complete, relatively continuous subspaces of $L^2(\mathbb{T})$ are of the form $\mathcal{R}_{S,\mu}$ with S and μ as above. Up to equivalence, all dense, complete, relatively continuous subspaces are therefore of the form $\mathcal{R}_S$ for some S as above.

Example 6.11.12. Now consider the 2-dimensional torus $G = \mathbb{T}^2$. If $S \subseteq \mathbb{T}^2$ is a proper open subset, then $H^2(S; \mathbb{Z}) = 0$ because S is a *non-compact* oriented 2-dimensional manifold. Hence S supports no non-trivial complex line bundles in this case. The corresponding dense, complete, relatively continuous subspaces of $L^2(\mathbb{T}^2)$ are therefore of the form $\mathcal{R}_{S,\mu}$ as above.

However, $\mathbb{T}^2$ carries non-trivial line bundles because $H^2(\mathbb{T}^2; \mathbb{Z}) \cong \mathbb{Z}$. Hence we have countably many non-equivalent dense, complete, relatively continuous subspaces of $L^2(\mathbb{T}^2)$ of the form $\mathcal{R}_{\mathbb{T}^2,V_n,\psi_n}$, where V_n is a complex line bundle corresponding to $n \in \mathbb{Z}$ and ψ_n is a measurable unitary section of V_n. In [9] we give an explicit description of these line bundles (and also of the relatively continuous subspaces).

Therefore the answer to Question 6.11.2 is negative in this case. Note that by Equation (6.26), all the relatively continuous subspaces $\mathcal{R}_{\mathbb{T}^2,V_n,\psi_n}$ have the same generalized fixed point algebra $\mathcal{C}(\mathbb{T}^2)$.

As already mentioned, Corollary 6.11.10 also classifies the continuous structures for the trivial 1-dimensional bundle $G \times \mathbb{C}$. We can also consider the n-dimensional trivial bundle $G \times \mathbb{C}^n$, where n is some natural number. Theorem 6.11.7 provides a classification of all its continuous structures, namely, they correspond to dense, complete, relatively continuous subspaces of $L^2(G)^n \cong L^2(G) \otimes \mathbb{C}^n$. However, we have more variables to consider here, and an explicit description of all these structures becomes much more complicated. Let us just indicate some points here.

Let $(\mathcal{H}, u)$ be a continuous structure for $G \times \mathbb{C}^n$. Since the dimension function of $G \times \mathbb{C}^n$ is constant equal to n, we have $\dim \mathcal{H}_t = n$ for almost all $t \in G$. It follows that $S := \{t \in G : \dim \mathcal{H}_t = n\} = \{t \in G : \dim \mathcal{H} \geq n\}$ is an open subset of full measure in G. When restricted to S, $\mathcal{H}$ is locally trivial and therefore is an n-dimensional (Hermitian) complex vector bundle. However, on the complement $G \backslash S$ (which is a closed subset of G of measure zero) many things can happen because the dimension function of $\mathcal{H}$ can take values between 0 and $n - 1$.

Already the case $S = G$ is interesting. In this case, we have an n-dimensional vector bundle V over G which corresponds to a relatively continuous subspace of $L^2(G)^n$, and two non-isomorphic vector bundles yield non-equivalent relatively continuous subspaces.

For Abelian groups, this situation is analyzed in [48, Section 8]. As noted there, relatively continuous subspaces of $L^2(G)^n$ associated to vector bundles over G as above are always maximal (the proof in the non-Abelian case is exactly the same).

Note that by Equation (6.24), the generalized fixed point algebra associated to a vector bundle V as above is the algebra $\mathcal{C}_0\big(\mathrm{End}(V)\big)$ of continuous sections vanishing at infinity of the bundle $\mathrm{End}(V)$ of endomorphisms of V. In particular, for the n-dimensional trivial vector bundle $V = G \times \mathbb{C}^n$, we get the algebra $\mathcal{C}_0(G, \mathbb{M}_n)$ (where $\mathbb{M}_n$ denotes the algebra of $n \times n$ matrices). Note also that if V' is another n-dimensional complex vector bundle such that $V \cong V' \otimes L$ for some complex line bundle L, then the generalized fixed point algebras associated to V and V' are isomorphic because $\mathrm{End}(L)$ is always trivial. As noted in [48], the converse also holds, that is, if the generalized fixed point algebras associated to V and V' are isomorphic, then there is a complex line bundle L such that $V \cong V' \otimes L$. In particular, the generalized fixed point algebra associated to V is isomorphic to $\mathcal{C}_0(G, \mathbb{M}_n)$ if and only if $V \cong \mathbb{C}^n \otimes L \cong L \oplus L \oplus \ldots \oplus L$ is the direct sum of n copies of the same line bundle.

Using the criterion above, it is not difficult to find complex vectors bundles for which the generalized fixed point algebra is not isomorphic to $\mathcal{C}_0(G, \mathbb{M}_n)$. In fact, this is possible for $n = 2$ and $G = \mathbb{T}^2$ (see [48, Section 8]). Thus there are *maximal* relatively continuous subspaces of $L^2(G) \oplus L^2(G)$ whose generalized fixed point algebras are not isomorphic.

6.11.2 Some Fell bundle structures

Let G be a locally compact group, and consider the $\widehat{G}$-C^*-algebra of compact operators $A := \mathcal{K}\big(L^2(G)\big)$ endowed with the coaction of G induced by the coaction $\gamma_{L^2(G)}$ of G on $L^2(G)$ as in the previous sections.

Note that A is isomorphic to a classical dual coaction. In fact, if we let G act on $\mathcal{C}_0(G)$ by translation, then $\mathcal{K}\big(L^2(G)\big)$ is isomorphic to the (full and reduced) crossed product algebra:

$$A \cong C^*\big(G, \mathcal{C}_0(G)\big) \cong C^*_{\mathrm{r}}\big(G, \mathcal{C}_0(G)\big).$$

This already provides a full and a reduced Fell bundle structure for A. The underlying Fell bundle over G is, in both cases, the semidirect product $\mathcal{C}_0(G) \times_\tau G$ (see Example 6.3.2(1)), where $\tau_t(f)|_s := f(st)$ denotes action of G on $L^\infty(G)$ by translation. Moreover, since classical dual coactions on full (resp. reduced) crossed products are maximal (resp. reduced) coactions, we get that A is at the same time a maximal and a reduced $\widehat{G}$-C^*-algebra. Hence, by Corollary 6.9.12, the underlying Fell bundle of any full or reduced Fell bundle structure for A is amenable. Thus there is no difference between full and reduced Fell bundle structures in this case, and we can therefore forget the words full and reduced and just speak of Fell bundle structures.

The semidirect product $\mathcal{C}_0(G) \times_\tau G$ provides a canonical Fell bundle structure for A, but it is no longer unique in general. Indeed, as we have seen in the previous sections, many non-equivalent dense, complete, relatively continuous subspaces $\mathcal{R} \subseteq L^2(G)$ can be found in general. By Theorem 6.9.10, each $\mathcal{R}$ gives rise to a Fell bundle structure for A. As we are going to see, the underlying Fell bundles are not isomorphic in general.

Firstly, we give a general description of the Fell bundle $\mathcal{B} := \mathcal{B}(L^2(G), \mathcal{R})$ over G associated to a dense, complete, relatively continuous subspace $\mathcal{R} \subseteq L^2(G)$. By Corollary 6.8.7, the fibers of $\mathcal{B}$ are given by

$$\mathcal{B}_t = \overline{\operatorname{span}}\{|\xi\rangle\!\rangle V_t \langle\!\langle \eta| : \xi, \eta \in \mathcal{R}\} = \overline{\{E_t(a) : a \in \mathcal{W}\}} \subseteq \mathcal{L}\big(L^2(G)\big) = \mathcal{M}(A)$$

for all $t \in G$, where $\mathcal{W} := \operatorname{span}|\mathcal{R}\rangle\langle\mathcal{R}|$. Recall that V_t is the operator on $L^2(G)$ given by $V_t(\xi)|_s = \xi(st)$ for all $s, t \in G$ and $\xi \in L^2(G)$. By the same corollary, the topology of $\mathcal{B}$ is determined by the continuous sections $t \mapsto E_t(a)$ for $a \in \mathcal{W}$.

By Example 6.11.1 (or Proposition 6.11.3), we have $|\xi\rangle\!\rangle = M_\xi$ for all $\xi \in L^2(G)_{\mathrm{si}} = L^2(G) \cap L^\infty(G)$. A short computation shows that $V_t M_f = M_{\tau_t(f)} V_t$ for every function $f \in L^\infty(G)$ and $t \in G$. Thus

$$\mathcal{B}_t = \overline{\operatorname{span}}\{M_\xi V_t M_{\overline{\eta}} : \xi, \eta \in \mathcal{R}\} = \overline{\operatorname{span}}\{M_{\xi\tau_t(\overline{\eta})} V_t : \xi, \eta \in \mathcal{R}\}.$$

Hence $\mathcal{B}_t$ is, as a Banach space, isomorphic to the closed linear space of products $\xi\tau_t(\overline{\eta})$ in $L^\infty(G)$ with $\xi, \eta \in \mathcal{R}$. The product on $\mathcal{B}$ can be deduced from the relation $(M_f V_t)(M_g V_s) = M_{f\tau_t(g)} V_{ts}$ for all $f, g \in L^\infty(G)$ and $t, s \in G$. Analogously, the involution on $\mathcal{B}$ is deduced from $(M_f V_t)^* = \delta_G(t)^{-1} M_{\tau_{t^{-1}}(f)} V_{t^{-1}}$.

By Theorem 6.9.10, the Fell bundle $\mathcal{B}$ provides a Fell bundle structure for A, that is, there are $\widehat{G}$-equivariant isomorphisms $C^*(\mathcal{B}) \cong A \cong C^*_{\mathrm{r}}(\mathcal{B})$.

Example 6.11.13. Let us consider a special case of the situation above, namely, the case where $\mathcal{R} = \mathcal{R}_S = L^2(G) \cap \mathcal{C}_0(S)$ for an open subset $S \subseteq G$ of full measure.

In this case, we have

$$\mathcal{B}_t = \overline{\operatorname{span}}\big\{M_{\xi\tau_t(\overline{\eta})} V_t : \xi, \eta \in \mathcal{R}_S\big\} = M\big(\overline{\operatorname{span}}\{\xi\tau_t(\overline{\eta}) : \xi, \eta \in \mathcal{C}_0(S)\}\big) V_t = M(\mathcal{I}_t) V_t,$$

where $\mathcal{I}_t := \overline{\operatorname{span}}\big(\mathcal{C}_0(S) \cdot \tau_t(\mathcal{C}_0(S))\big) = \mathcal{C}_0(S) \cap \tau_t(\mathcal{C}_0(S)) = \mathcal{C}_0(S \cap S \cdot t)$ (recall that the product of two ideals in a C^*-algebra equals its intersection). Note that $\{\mathcal{I}_t\}_{t \in G}$ is a collection of ideals of $\mathcal{C}_0(S)$ and, as a Banach space, $\mathcal{B}_t$ is isomorphic to $\mathcal{I}_t$ for all $t \in G$.

If we define the map $\theta_t : \mathcal{I}_{t^{-1}} \to \mathcal{I}_t$ by $\theta_t(f) = \tau_t(f)$ for all $f \in \mathcal{I}_{t^{-1}}$, then the pair $\theta = \{\mathcal{I}_t, \theta_t\}_{t \in G}$ is a continuous partial action of G on $\mathcal{I}_e = \mathcal{C}_0(S)$ (see [20, 3.8]). In fact, it is the restriction of the global continuous action τ of G on $\mathcal{C}_0(G)$ to the ideal $\mathcal{C}_0(S)$ as in Example 6.3.2(2). Let $\mathcal{A} := \mathcal{C}_0(S) \times_\theta G$ be the semidirect product Fell bundle associated to θ as in Example 6.3.2(2). We claim that $\mathcal{B} \cong \mathcal{A}$ (as Fell bundles). In fact, recall that

$$\mathcal{A} = \{(f, t) \in \mathcal{C}_0(S) \times G : f \in \mathcal{I}_t\}$$

(considered as topological subspace of $\mathcal{C}_0(S) \times G$), and the operations are given by

$$(f, t) \cdot (g, s) = (f\tau_t(g), ts), \quad (f, t)^* = (\tau_{t^{-1}}(\overline{f}), t^{-1}), \quad \|(f, t)\| = \|f\|.$$

Now if we define the map $\Phi : \mathcal{A} \to \mathcal{B}$ by $\Phi(f, t) := \delta_G(t)^{\frac{1}{2}} M_f V_t$ for all $f \in \mathcal{I}_t$, then straightforward calculations show that Φ preserves all the operations of the bundles and is a Banach isomorphism when restricted to the fibers. Moreover, note that Φ sends sections of the form $t \mapsto (\xi\tau_t(\overline{\eta}), t)$, where $\xi, \eta \in \mathcal{R}$, to sections of the form $t \mapsto \delta_G(t)^{\frac{1}{2}} M_{\xi\tau_t(\overline{\eta})} V_t = \delta_G(t)^{\frac{1}{2}} E_t(|\xi\rangle\langle\eta|)$. These sections generate pointwise-dense spaces of continuous sections of $\mathcal{A}$ and $\mathcal{B}$, respectively. This implies that Φ is a homeomorphism (see [23, II.13.16, II.13.17]), and hence $\mathcal{A} \cong \mathcal{B}$ as claimed. As a consequence, we get

$$C^*_{(\mathrm{r})}\big(G, \mathcal{C}_0(S), \theta\big) \cong C^*_{(\mathrm{r})}(\mathcal{A}) \cong C^*_{(\mathrm{r})}(\mathcal{B}) \cong \mathcal{K}\big(L^2(G)\big),$$

where all the isomorphisms are $\widehat{G}$-equivariant with respect to the dual coactions.

In the situation above, the unit fiber of $\mathcal{B}$ is isomorphic to $\mathcal{C}_0(S)$ (which is also the generalized fixed point algebra). Thus this situation already provides examples of Fell bundle structures for $\mathcal{K}\big(L^2(G)\big)$ whose underlying Fell bundles are not isomorphic to the semidirect product $\mathcal{C}_0(G) \times_\tau G$ (which has $\mathcal{C}_0(G)$ as unit fiber).

Question 6.11.14. What happens if we have a Fell bundle structure for $\mathcal{K}\big(L^2(G)\big)$ whose unit fiber of the underlying Fell bundle is isomorphic to $\mathcal{C}_0(G)$? Is the Fell bundle in this case isomorphic to the semidirect product $\mathcal{C}_0(G) \times_\tau G$?

In general, the answer to this question is no. Counterexamples can be found already in the case of Abelian groups. Indeed, we have done this in [9]. In order to explain this, we assume from now on that G is Abelian. Recall that, in this case, the coaction of G on $L^2(G)$ corresponds to the action γ of $\widehat{G}$ given by the formula (6.20). Therefore, the coaction of G on $\mathcal{K}\big(L^2(G)\big)$ corresponds to the action $\alpha_x(T) = \gamma_x \circ T \circ \gamma_x^{-1}$ for all $x \in \widehat{G}$ and $T \in \mathcal{K}\big(L^2(G)\big)$.

Recall from Corollary 6.10.23 that isomorphism classes of Fell bundle structures for $\mathcal{K}\big(L^2(G)\big)$ correspond bijectively to dense, e-complete, relatively continuous subspaces of $\mathcal{K}\big(L^2(G)\big)$. Since G is Abelian (and in particular amenable), every complete, relatively continuous subspace is automatically e-complete (Proposition 5.3.10). Moreover, by [48, Theorem 7.2], there is a bijective correspondence between dense, complete, relatively continuous subspaces of $L^2(G)$ and $\mathcal{K}\big(L^2(G)\big)$. This combined with Corollary 6.11.10 yields the first part of the following result (Theorem 48 in [9]):

Theorem 6.11.15. *Let G be a locally compact Abelian group. Isomorphism classes of Fell bundle structures for $\mathcal{K}\big(L^2(G)\big)$ correspond bijectively to isomorphism classes of triples (S, V, ψ) as in Corollary* 6.11.10.

Moreover, isomorphism classes of Fell bundles $\mathcal{B}$ over G for which there is an isomorphism $C^*(\mathcal{B}) \cong \mathcal{K}\big(L^2(G)\big)$ correspond bijectively to conjugacy classes of pairs (S, V) with S and V as above, where (S_1, V_1) and (S_2, V_2) are conjugate *if and only if there is some $t \in G$ with $t \cdot S_1 = S_2$ and $t^*(V_2) \cong V_1$. Here $t^*(V_2)$ means that we pull back the line bundle V_2 on S_2 along the map $s \mapsto ts$ to a line bundle on S_1.*

The proof of the second part of the theorem uses Corollary 6.10.25 which says that the Fell bundles associated to dense, complete, relatively continuous subspaces $\mathcal{R}_1, \mathcal{R}_2 \subseteq A$ are isomorphic if and only if $\mathcal{R}_1$ and $\mathcal{R}_2$ are $\mathrm{Aut}_{\widehat{G}}(A)$-conjugate, that is, there is a $\widehat{G}$-equivariant automorphism π of A such that $\pi(\mathcal{R}_1) = \mathcal{R}_2$. In the case of $A = \mathcal{K}\big(L^2(G)\big)$, the group $\mathrm{Aut}_{\widehat{G}}(A)$ can be described explicitly, and here is where the translations $s \mapsto ts$ come into play. See [9] for further details.

Using the theorem above, we can give concrete counterexamples to Question 6.11.14. In fact, consider the case where G is the 2-dimensional torus $\mathbb{T}^2$, and take the Fell bundle $\mathcal{B}(V)$ associated to some complex line bundle V on $\mathbb{T}^2$. We know from the previous section that the corresponding generalized fixed point algebra, that is, the unit fiber of $\mathcal{B}(V)$, is isomorphic to $\mathcal{C}(\mathbb{T}^2)$. Since $\mathbb{T}^2$ is path-connected, we have $t^*(V) \cong V$ for every $t \in G$. By the above theorem, non-isomorphic line bundles yield non-isomorphic Fell bundles. As already mentioned in Example 6.11.12, we have $H^2(\mathbb{T}^2; \mathbb{Z}) \cong \mathbb{Z}$, and therefore there exist countably many non-isomorphic Fell bundle structures for $\mathcal{K}\big(L^2(\mathbb{T}^2)\big)$ whose underlying Fell bundles have the same unit fiber $\mathcal{C}(\mathbb{T}^2)$. In [9], we give an explicit description of these Fell bundles. Finally, if V is the trivial line bundle, the corresponding Fell bundle $\mathcal{B}(V)$ is isomorphic to the semidirect product $\mathcal{C}(\mathbb{T}^2) \times_\tau \mathbb{T}^2$ (this follows from Example 6.11.13).

As already mentioned at the end of the previous section, generalized fixed point algebras become more complicated when we consider higher dimensions and study the $\widehat{G}$-Hilbert space $L^2(G)^n$ for $n \in \mathbb{N}$. As we have seen, we can find maximal relatively continuous subspaces of $L^2(G)^n$ whose generalized fixed point algebras are not isomorphic (this happens for $n = 2$ and $G = \mathbb{T}^2$). In particular, the corresponding Fell bundle structures for the $\widehat{G}$-C^*-algebra $\mathcal{K}\big(L^2(G)^n\big)$ of compact operators on $L^2(G)^n$ are not isomorphic (because the generalized fixed point algebras always appear as the unit fibers of the corresponding Fell bundles).

More generally, one can consider Fell bundle structures for the $\widehat{G}$-C^*-algebra of compact operators $\mathcal{K}\big(L^2(\mathcal{H})\big)$, where $\mathcal{H} = \{\mathcal{H}_t\}_{t\in G}$ is some measurable field of separable Hilbert spaces over G and we endow $\mathcal{K}\big(L^2(\mathcal{H})\big)$ with the coaction of G induced by the coaction on $L^2(\mathcal{H})$ as in the previous section. Since G is Abelian, this coaction corresponds to the action of $\widehat{G}$ given by conjugation by the corresponding action on $L^2(\mathcal{H})$ as in Equation (6.20). Of course, here the situation is even more complicated, but at least we can say that isomorphism classes of Fell bundle structures for $\mathcal{K}\big(L^2(\mathcal{H})\big)$ correspond bijectively to isomorphism classes of continuous structures for the measurable field $\mathcal{H}$. This is Theorem 46 in [9] and it is a consequence of Theorem 6.11.7 and Theorem 7.2 in [48].

In particular, Example 6.11.8 shows that there are square-integrable $\widehat{G}$-C^*-algebras with no Fell bundle structure. This gives a negative answer to Question 11.16 in [19].

We conclude that, in general, there are square-integrable $\widehat{G}$-C^*-algebras without any or with several Fell bundle structures. Of course, all these problems disappear if G is discrete, that is, if $\widehat{G}$ is compact. In this case, we know that any $\widehat{G}$-C^*-algebra is square-integrable, and there is (up to canonical isomorphism) a unique Fell bundle structure for a given $\widehat{G}$-C^*-algebra.

More generally, we know that a $\widehat{G}$-C^*-algebra A has a unique Fell bundle structure if and only if A is $\mathcal{R}$-proper (see Corollary 6.10.29). In particular, this is the case if A is spectrally proper, that is, if the induced action of $\widehat{G}$ on the primitive ideal space $\mathrm{Prim}(A)$ is proper (see Definition 9.2 and Theorem 9.1 in [48]). Moreover, we prove in [9, Theorem 55] that the functor $\mathcal{B} \mapsto C^*(\mathcal{B})$ provides an equivalence from the category of *spectrally proper* Fell bundles over G (in the sense that the spectrum of $\mathcal{B}$ is a proper $\widehat{G}$-space with respect to the canonical action of $\widehat{G}$) to the category of spectrally proper $\widehat{G}$-C^*-algebras. In particular, two spectrally proper Fell bundles are isomorphic if and only if their cross-sectional C^*-algebras are equivariantly isomorphic.

If we specialize even further and assume that A is proper in the sense of Kasparov [35], then it is possible to give an explicit description of the associated Fell bundle; see [9, Theorem 56]. In particular, this gives a description of the Fell bundle if A is a commutative $\widehat{G}$-C^*-algebra $\mathcal{C}_0(X)$ for a proper $\widehat{G}$-space X ([9, Proposition 58]). In this last case, the underlying Fell bundle is necessarily commutative. This provides a bijective correspondence between commutative Fell bundles over G and proper $\widehat{G}$-spaces ([9, Theorem 57]).

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