MTM510037 — C*-Dynamical Systems and Crossed Products

Exercise Sheet 2 — Crossed Products

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(1) An extension of topological groups is an exact sequence of the form $N \hookrightarrow G \twoheadrightarrow K$ consisting of a continuous open surjective homomorphism $\pi\colon G \twoheadrightarrow K$ and an injective continuous homomorphism $\iota\colon N \hookrightarrow G$ which is open (and hence a topological isomorphism) onto its image $\mathrm{Im}(\iota) = \ker(\pi)$.

(a) Show that any sequence as above is "isomorphic" to a sequence of the form $N \hookrightarrow G \twoheadrightarrow G/N$, where N is a normal subgroup of G (considered as a topological group with the subspace topology), $N \hookrightarrow G$ is the embedding and $G \twoheadrightarrow G/N$ is the quotient map. It is part of the exercise to explain the meaning of "isomorphic" here.

(b) Show that an exact sequence of topological groups *splits* in the sense that there is a continuous homomorphism $\sigma\colon K \to G$ such that $\pi(\sigma(k)) = k$ if and only if there is a (continuous) action $\theta\colon K \to \mathrm{Aut}(N)$ of K on N by (continuous) group automorphisms such that G is isomorphic to the semidirect product $N \rtimes_\theta K$ and the original exact sequence is isomorphic to the canonical one $N \hookrightarrow N \rtimes_\theta K \twoheadrightarrow K$, where $N \hookrightarrow N \rtimes_\theta K$ sends $n \mapsto (n, e)$ and $N \rtimes_\theta K \twoheadrightarrow K$ sends $(n, k) \mapsto k$.

(2) With notations as in the previous exercise, show that the action $\theta\colon K \to \mathrm{Aut}(N)$ induces a C^*-action of $\alpha\colon K \to \mathrm{Aut}(C^*(N))$ and there exists a canonical isomorphism

$$C^*(N) \rtimes_\alpha K \cong C^*(N \rtimes_\theta K).$$

(3) Show that the construction of full crossed products $A \mapsto A \rtimes_\alpha G$ is functorial in the following sense: fix a locally compact G and suppose that (A, α) and (B, β) are C^*-algebras carrying G-actions α and β, respectively. Suppose that $\varphi\colon A \to B$ is a *-homomorphism which is G-equivariant, meaning that $\varphi(\alpha_t(a)) = \beta_t(\varphi(a))$ for all $a \in A$, $t \in G$. Show that φ induces a *-homomorphism $\varphi \rtimes G\colon A \rtimes_\alpha G \to B \rtimes_\beta G$. Moreover, if $\psi\colon B \to C$ is another G-equivariant *-homomorphism between C^*-algebras with G-actions, then $(\psi \circ \varphi) \rtimes G = (\psi \rtimes G) \circ (\varphi \rtimes G)$.

Formulate and prove an analogous result for reduced crossed products.

(4) Suppose that a locally compact group G acts on C^*-algebras A and B via actions α and β. Prove that there is a "tensor product action" $\alpha \otimes \beta$ of G on the minimal tensor product $A \otimes B$ given on elementary tensors by $(\alpha \otimes \beta)_t(a \otimes b) = \alpha_t(a) \otimes \beta_t(b)$.

If β is the trivial action, prove that there is a canonical isomorphism of C^*-algebras

$$(A \otimes B) \rtimes_{\alpha\otimes\beta,\mathrm{r}} G \cong (A \rtimes_{\alpha,\mathrm{r}} G) \otimes B.$$

Remark.: There is an analogous result for maximal tensor products and maximal crossed products that we will see in the lectures.

(5) Let (A, G, α) be a C^*-dynamical system. Let $(e_i)_{i\in I}$ be an approximate unit A and let $(\varphi_V)_{V\in\mathcal{V}}$ be the "standard" approximate unit for the group algebra $\mathbb{C}[G] = C_c(G)$ (with respect to the inductive limit topology) consisting of

functions $\varphi_V \in C_c^+(G)$ with $\operatorname{supp}(\varphi_V) \subseteq V$ and $\int_G \varphi_V(t)\,\mathrm{d}t = 1$. Here $\mathcal{V}$ denotes the directed set of all open neighborhoods of $e \in G$ with $V_1 \le V_2$ iff $V_2 \subseteq V_1$. Endow $\mathcal{V} \times I$ with the product (directed) order: $(V_1, i_1) \le (V_2, i_2)$ iff $V_1 \le V_2$ and $i_1 \le i_2$. Show that $(\varphi_V \otimes e_i)_{(V,i)\in\mathcal{V}\times I}$ is an approximate unit for $A \rtimes_{\alpha,\mathrm{alg}} G = C_c(G, A)$ with respect to the inductive limit topology, that is, prove that $(\varphi_V \otimes e_i) * f(t) \to f(t)$ uniformly with controlled supports. In particular $(\varphi_V \otimes e_i)_{(V,i)\in\mathcal{V}\times I}$ also serves as an approximate unit for $L^1(G, A)$, $A \rtimes_\alpha G$ or $A \rtimes_{\alpha,\mathrm{r}} G$.

(6) Let G be a locally compact group and consider its right regular representation $\rho\colon G \to \mathcal{U}(L^2(G))$, $\rho_t(\xi)(s) := \xi(st)\Delta(t)^{1/2}$. Show that ρ is unitarily equivalent to the left regular representation $\lambda\colon G \to \mathcal{U}(L^2(G))$, $\lambda_t(\xi)(s) := \xi(t^{-1}s)$, that is, there exists a unitary operator $U \in \mathcal{U}(L^2(G))$ satisfying

$$\lambda_t = U\rho_t U^* \quad \forall t \in G.$$

Conclude that if $C_r^*(G) := C_\lambda^*(G)$ and $C_\rho^*(G)$ are the C^*-algebras generated by the images of the integrated forms of λ and ρ, respectively. then

$$C_\lambda^*(G) \cong C_\rho^*(G).$$

Show that ρ and λ commute, meaning that $\lambda_t\rho_s = \rho_s\lambda_t$ for all $s, t \in G$. If $L(G) := C_\lambda^*(G)''$ and $R(G) := C_\rho^*(G)''$ are the von Neumann algebras generated by λ and ρ, respectively, conclude that one is the commutant of the other, that is,

$$L(G) = R(G)'.$$

(7) (Invariant ideals and quotients) Let (A, G, α) be a C^*-dynamical system and let $I \subseteq A$ be an *invariant* closed two-sided ideal, meaning that $\alpha_t(I) \subseteq I$ for all $t \in G$. In this case, show that α restricts to an action $\alpha|_I$ of G on I, and there is an induced action $\tilde{\alpha}$ of G on A/I.

(a) Show that there exists a canonical exact sequence of full crossed products:

$$0 \to I \rtimes_{\alpha|_I} G \to A \rtimes_\alpha G \to (A/I) \rtimes_{\tilde{\alpha}} G \to 0,$$

where the embedding $I \rtimes_{\alpha|_I} G \to A \rtimes_\alpha G$ is induced by the inclusion $C_c(G, I) \hookrightarrow C_c(G, A)$, and the quotient map $A \rtimes_\alpha G \to (A/I) \rtimes_{\tilde{\alpha}} G$ is induced by $C_c(G, A) \to C_c(G, A/I)$, $f \mapsto q \circ f$, where $q\colon A \to A/I$ denotes the quotient homomorphism.

(b) Discuss (without necessarily a full proof) if the same holds for reduced crossed products.