Ramsey theory and the group of homeomorphisms of the Lelek fan

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Topological dynamics

G-flow

$G \times X \longrightarrow X$ - a continuous action

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$$\begin{tikzcd}[column sep=small] G & \times & X \arrow[r] & X & \text{- a continuous action} \\ \text{topological group} \arrow[u] & & & \text{compact Hausdorff space} \arrow[u] & \end{tikzcd}$$

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G is extremely amenable $\longleftrightarrow$ its universal minimal flow is a singleton ($\longleftrightarrow$ every G-flow has a fixed point).

Structural Ramsey property

Theorem (Ramsey)

For every $k \leq m$ and $r \geq 2$, there exists n such that for every colouring of k-element subsets of n with r-many colours there is a subset X of n of size m such that all k-element subsets of X have the same colour.

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for every colouring of copies of A in C by r colours, there is a copy B' of B in C, such that all copies of A in B' have the same colour.

Ramsey classes and extremely amenable groups

Ramsey classes

- finite linear orders (Ramsey)
- finite linearly ordered graphs (Nešetřil and Rödl)
- finite linearly ordered metric spaces (Nešetřil)
- finite Boolean algebras (Graham and Rothschild)

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Extremely amenable groups

- $\mathrm{Aut}(\mathbb{Q},<)$ (Pestov)
- $\mathrm{Aut}(\mathcal{OR})$ – $\mathcal{OR}$ the random ordered graph (Kechris, Pestov & Todorčević)
- $\mathrm{Iso}(\mathbb{U},d)$ (Pestov)
- $\mathrm{Homeo}(C,\mathcal{C})$ – $(C,\mathcal{C})$ the Cantor space with a generic maximal chain of closed subsets (KPT; Glasner & Weiss)

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Theorem (KPT; NvT)

Aut($\mathcal{A}$) *is extremely amenable* $\longleftrightarrow$ *finitely-generated substructures of* $\mathcal{A}$ *satisfy the Ramsey property and are rigid.*

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Structure $\mathcal{A}$	$M(\mathrm{Aut}(\mathcal{A}))$	authors
$\mathbb{N}$	linear orders on $\mathbb{N}$	Glasner and Weiss
random graph $\mathcal{R}$	linear orders on $\mathcal{R}$	KPT
Cantor space C	maximal chains of closed subsets of C	Glasner and Weiss

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Structure...	...homogeneous w.r.t.
$\mathbb{N}$, $\mathcal{R}$	embeddings
Lelek fan	epimorphisms
Gurarij space	linear isometric embeddings
Poulsen simplex	affine epimorphisms

Lelek fan L

= unique non-trivial subcontinuum of the Cantor fan with a dense set of endpoints (Bula-Oversteegen, Charatonik)

continuum = connected compact metric Hausdorff space

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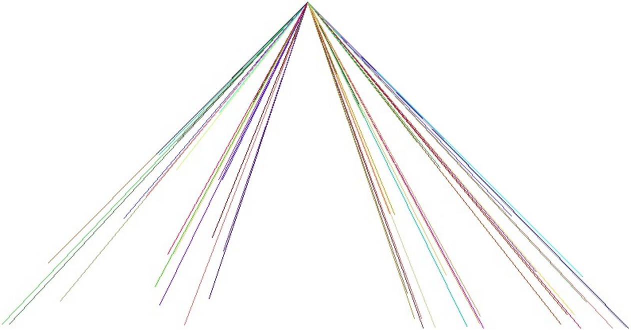

fan.jpg

Pre-Lelek fan

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$\mathcal{F} = \{\text{finite fans}\} + \text{surjective homomorphisms}$

(U) $T \in \mathcal{F} \rightsquigarrow \exists \phi : (\mathbb{L}, R^{\mathbb{L}}) \longrightarrow T$ - continuous surjective homomorphism

(R) X finite, $f : \mathbb{L} \longrightarrow X$ continuous $\rightsquigarrow \exists T \in \mathcal{F}$, $\phi : \mathbb{L} \longrightarrow T$ and $g : T \longrightarrow X$ such that $f = g \circ \phi$

(PU) $T \in \mathcal{F}$, $\phi_1, \phi_2 : \mathbb{L} \longrightarrow T \rightsquigarrow \exists g : \mathbb{L} \longrightarrow \mathbb{L}$ automorphism with $\phi_1 = \phi_2 \circ g$

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$$\begin{aligned} h &\mapsto h^* \\ \pi \circ h &= h^* \circ \pi. \end{aligned}$$

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$$c : \{C \longrightarrow A\} \longrightarrow \{1, 2, \ldots, r\}$$

there exists $f : C \longrightarrow B$ such that $\{B \longrightarrow A\} \circ f$ is monochromatic.

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Theorem (B-K)

Let $\mathbb{L}_<$ be the limit of $\mathcal{F}_<$. Then $\mathrm{Aut}(\mathbb{L}_<)$ is extremely amenable.

Universal minimal flow of $\mathrm{Homeo}(L)$

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Properties (1),(2) and (3) uniquely determine S up to an affine homeomorphism.

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FACT

$T : \{0,1\}^{\mathbb{Z}} \longrightarrow \{0,1\}^{\mathbb{Z}}$ the shift $\Rightarrow$ T-invariant probability measures forn P

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(U) $\forall n\ \exists \phi : P \longrightarrow S_n$ – continuous affine surjection

(APU) $\forall \varepsilon > 0\ \forall n\ \forall \phi_1, \phi_2 : P \longrightarrow S_n\ \exists f \in AH(P)$ with $d(\phi_1, \phi_2 \circ f) < \varepsilon$

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Theorem (B-LA-M)

(U) + (APU) characterize P among non-trivial metrizable simplexes up to affine homeomorphism.

Approximate Ramsey property for P

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there is $\pi \in \mathrm{Epi}_0(S_n, S_m)$ and $\alpha < r$ such that

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Theorem (B-LA-M)

$AH_s(P)$ is extremely amenable.

Universal minimal flow of $AH(P)$

Theorem (B-LA-M)

$$M(AH(P)) \cong AH(\widehat{P)/AH}_s(P) \cong P$$

$\mathrm{FIN}_k(n)$

$$= \{p : \{1, \ldots, n\} \longrightarrow \{0, 1, \ldots, k\} : \ \exists n \ (p(n) = k)\}$$

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$$\mathrm{supp}(p) \cap \mathrm{supp}(q) = \emptyset \ \longrightarrow \ (p + q)(n) = \max\{p(n), q(n)\}$$

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TETRIS-LIKE OPERATIONS

$T_i : \mathrm{FIN}_k \longrightarrow \mathrm{FIN}_{k-1}$

$$T_i(p)(n) = \begin{cases} p(n) & \text{if } p(n) < i \\ p(n) - 1 & \text{if } p(n) \geq i. \end{cases}$$

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$\bar{p} = (p_1, \ldots, p_m) \in \mathrm{FIN}_k^{(m)}(n)$

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$\mathrm{FIN}_k^{(d)}(n) = \{(p_1, \dots, p_d) : \mathrm{supp}(p_i) \cap \mathrm{supp}(p_j) = \emptyset$ for i $\neq$ j & $\mathrm{min(supp(p_i))) < min(supp(p_{i+1}))}$ & $\mathrm{min(supp_k(p_i)) < min(supp_k(p_{i+1}))}$ for i < d}

$\bar{p} = (p_1, \dots, p_m) \in \mathrm{FIN}_k^{(m)}(n)$

$$\langle \bar{p} \rangle = \left\{ \sum_{i=1}^{m} T_{\vec{i}_i}(p_i) : \vec{i}_i \in \prod_{j=1}^{k} \{0, 1, \dots, j\} \ \& \ \exists i \ \vec{i}_i = \vec{0} \right\}$$

Exact Ramsey Theorem

$\mathrm{supp}_k(p) = \{x : p(x) = k\}$

$\mathrm{FIN}_k^{(d)}(n) = \{(p_1, \dots, p_d) : \mathrm{supp}(p_i) \cap \mathrm{supp}(p_j) = \emptyset$ for i $\neq$ j & $\mathrm{min(supp(p_i)))} < \mathrm{min(supp(p_{i+1}))}$ & $\mathrm{min(supp_k(p_i))} < \mathrm{min(supp_k(p_{i+1}))}$ for i < d}

$\bar{p} = (p_1, \dots, p_m) \in \mathrm{FIN}_k^{(m)}(n)$

$$\langle \bar{p} \rangle = \left\{ \sum_{i=1}^{m} T_{\vec{i}_i}(p_i) : \vec{i}_i \in \prod_{j=1}^{k} \{0, 1, \dots, j\} \ \&\ \exists i \ \vec{i}_i = \vec{0} \right\}$$

For every $d \leq m$ and r there exists n such that for every colouring $c : \mathrm{FIN}_k^{(d)}(n) \longrightarrow \{0, 1, \dots, r-1\}$ there is $\bar{p} \in \mathrm{FIN}_k^{(m)}(n)$ such that $\langle \bar{p} \rangle^{(d)} \cap \mathrm{FIN}_k^{(d)}(n)$ is monochromatic.

Dual Ramsey Theorem

Theorem (Graham and Rothschild)

For every $k \leq m$ and $r \geq 2$, there exists n such that for every colouring of the k-element partitions of n by r-many colours there is an m-element partition X of n such that all k-element coarsenings of X have the same colour.

A mathematical question

Is there a non-trivial simplex with extremely amenable group of affine homeomorphisms?

A non-mathematical question

Does anyone want to share a cab to the ariport early tomorrow morning?

THANK YOU

OBRIGADA!