Freeness and Graph Sums

Jamie Mingo (Queen's University)

based on joint work with Roland Speicher and Mihai Popa

Análise funcional e sistemas dinâmicos

Universidade Federal de Santa Catarina

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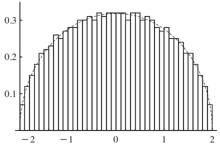
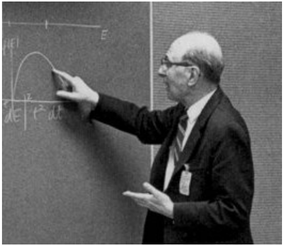
GUE random matrices

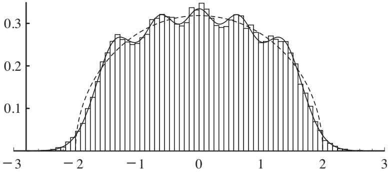
- (Ω, P) is a probability space
- $X_N : \Omega \to M_N(\mathbf{C})$ is a random matrix
- $X_N = X_N^* = \frac{1}{\sqrt{N}}(x_{ij})_{ij}$ a $N \times N$ self-adjoint random matrix with x_{ij} independent complex Gaussians with $\mathrm{E}(x_{ij}) = 0$ and $\mathrm{E}(|x_{ij}|^2) = 1$ (*modulo* self-adjointness)
- $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$ eigenvalues of X_N, $\mu_N = \frac{1}{N}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_N})$ is the spectral measure of X_N, $\int t^k \, d\mu_N(t) = \mathrm{tr}(X_N^k)$
- X_N is the $N \times N$ GUE with limiting eigenvalue distribution given by Wigner's semi-circle law



Wigner and Universality

- in the physics literature *universality* refers to the fact that the limiting eigenvalue distribution is semi-circular even if we don't assume the entries are Gaussian





random variables and their distributions

- $(\mathcal{A}, \varphi)$ unital algebra with state;
- $\mathbf{C}\langle x_1, \ldots, x_s \rangle$ is the unital algebra generated by the non-commuting variables $x_1, \ldots, x_s$
- the *distribution* of $a_1, \ldots, a_s \in (\mathcal{A}, \varphi)$ is the state $\mu : \mathbf{C}\langle x_1, \ldots, x_s \rangle \to \mathbf{C}$ given by $\mu(p) = \varphi(p(a_1, \ldots, a_s))$
- convergence in distribution of $\{a_1^{(N)}, \ldots, a_s^{(N)}\} \subset (\mathcal{A}_N, \varphi_N)$ to $\{a_1, \ldots, a_s\} \subset (\mathcal{A}, \varphi)$ means pointwise convergence of distributions: $\mu_N(p) \to \mu(p)$ for $p \in \mathbf{C}\langle x_1, \ldots, x_s \rangle$.
- let $f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ be the density of the Gauss law
- then $\log(\hat{f}(is)) = \dfrac{s^2}{2} = \displaystyle\sum_{n=1}^{\infty} k_n \frac{s^n}{n!}$ with $k_2 = 1$ and $k_n = 0$ for $n \neq 2$, so the Gauss law is characterized by having all cumulants except k_1 and k_2 equal to 0

Moments and Cumulants

- $a_1, \ldots, a_s \in (\mathcal{A}, \varphi)$ random variables
- a partition, $\pi = \{V_1, \ldots, V_k\}$, of $[n] = \{1, 2, 3, \ldots, n\}$ is a decomposition of $[n]$ into a disjoint union of subsets: $V_i \cap V_j = \emptyset$ for $i \neq j$ and $[n] = V_1 \cup \cdots \cup V_k$.
- $\mathcal{P}(n)$ is set of all partitions of $[n]$
- given a family of maps $\{k_1, k_2, k_3, \ldots, \}$ with $k_n : \mathcal{A}^{\otimes n} \to \mathbf{C}$ we define

$$k_\pi(a_1, \ldots, a_n) = \prod_{\substack{V \in \pi \\ V = \{i_1, \ldots, i_j\}}} k_j(a_{i_1}, \ldots, a_{i_j})$$

- in general moments are defined by the *moment-cumulant formula*

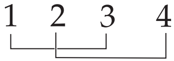
$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in \mathcal{P}(n)} k_\pi(a_1, \ldots, a_n)$$

- $k_1(a_1) = \varphi(a_1)$ and $\varphi(a_1 a_2) = k_2(a_1, a_2) + k_1(a_1) k_1(a_2)$

cumulants and independence

- $a \in \mathcal{A}$, n^{th} cumulant of a is $k_n^{(a)} = k_n(a, \dots, a)$
- if a_1 and a_2 are (classically) independent then $k_n^{(a_1+a_2)} = k_n^{(a_1)} + k_n^{(a_2)}$ for all n
- if $k_n(a_{i_1}, \dots, a_{i_n}) = 0$ unless $i_1 = \cdots i_n$ we say *mixed cumulants vanish*
- if mixed cumulants vanish then a_1 and a_2 are independent

free cumulants and free independence (*R. Speicher*)

- partition with a crossing: 

- non-crossing partition: 

- $NC(n) = \{$ non-crossing partitions of $[n]\}$
- $\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \dots, a_n)$ defines the *free cumulants*: same rules apply as for classical independence.

freeness and asymptotic freeness

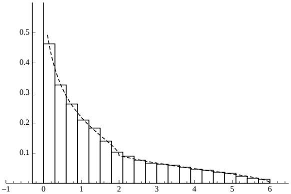
- if a and b are free with respect to φ then $\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2$
- in general if $a_1, \ldots, a_s$ are free then all *mixed moments* $\varphi(x_{i_1} \cdots x_{i_n})$ can be written as a polynomial in the moments of individual moments $\{\varphi(a_i^k)\}_{i,k}$.
- $\{a_1^{(N)}, \ldots, a_s^{(N)}\} \subset (\mathcal{A}_N, \varphi_N)$ are *asymptotically free* if $\mu_n \to \mu$ and $x_1, \ldots, x_s$ are free with respect to μ
- **in practice this means:** $a_1^{(N)}, \ldots, a_s^{(N)} \in (\mathcal{A}_n, \varphi_N)$ are asymptotically free if whenever we have $b_i^{(N)} \in \mathrm{alg}(1, a_{j_i}^{(N)})$ is such that $\varphi_N(b_i^{(N)}) = 0$ and $j_1 \neq j_2 \neq \cdots \neq j_m$ we have $\varphi_N(b_1^{(N)} \cdots b_m^{(N)}) \to 0$

simple distributions: Wigner and Marchenko-Pastur

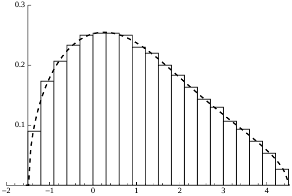
- let $f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ be the density of the Gauss law
- then $\log(\hat{f}(is)) = \frac{s^2}{2} = \sum_{n=1}^{\infty} k_n \frac{s^n}{n!}$ with $k_2 = 1$ and $k_n = 0$ for $n \neq 2$, so the Gauss law is characterized by having all cumulants except k_1 and k_2 equal to 0
- μ a probability measure on $\mathbb{R}$, $z \in \mathbf{C}^+$, $G(z) = \int (z-t)^{-1}\, d\mu(t)$ is the Cauchy transform of μ and $R(z) = G^{\langle -1 \rangle}(z) - \frac{1}{z} = \kappa_1 + \kappa_2 z + \kappa_3 z^2 + \cdots$ is the R-transform of μ
- if $d\mu(t) = \frac{1}{2\pi}\sqrt{4-t^2}\, dt$ is the *semi-circle* law we have $\kappa_n = 0$ except for $\kappa_2 = 1$
- if $1 < c$ and $a = (1-\sqrt{c})^2$ and $b = (1+\sqrt{c})^2$ we let $d\mu = \frac{\sqrt{(b-t)(t-a)}}{2\pi t}\, dt$, μ is the *Marchenko-Pastur* distribution: $\kappa_n = c$ for all n

random matrices and asymptotic freeness

- $X_N = X_N^* = \frac{1}{\sqrt{N}}(x_{ij})_{ij}$ a $N \times N$ self-adjoint random matrix with x_{ij} independent complex Gaussians with $\mathrm{E}(x_{ij}) = 0$ and $\mathrm{E}(|x_{ij}|^2) = 1$ (*modulo* self-adjointness)
- **Voiculescu's big theorem:** for large N mixed moments of X_N and Y_N are close to those of freely independent semi-circular operators (thus *asymptotically free*)



$X_{1000} + X_{1000}^2$



$X_{1000} + (X_{1000}^T)^2$

- (*with M. Popa*) transposing a matrix can free it from itself

Wishart Random Matrices

- Suppose $G_1, \ldots, G_{d_1}$ are $d_2 \times p$ random matrices where $G_i = (g_{jk}^{(i)})_{jk}$ and $g_{jk}^{(i)}$ are complex Gaussian random variables with mean 0 and (complex) variance 1, i.e. $\mathrm{E}(|g_{jk}^{(i)}|^2) = 1$. Moreover suppose that the random variables $\{g_{jk}^{(i)}\}_{i,j,k}$ are independent.
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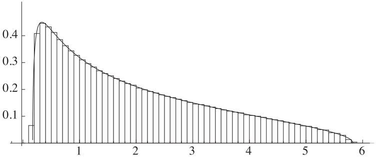
$$W = \frac{1}{d_1 d_2} \begin{pmatrix} G_1 \\ \hline \vdots \\ \hline G_{d_1} \end{pmatrix} \begin{pmatrix} G_1^* & | & \cdots & | & G_{d_1}^* \end{pmatrix} = \frac{1}{d_1 d_2} (G_i G_j^*)_{ij}$$

is a $d_1 d_2 \times d_1 d_2$ Wishart matrix. We write $W = d_1^{-1}(W(i,j))_{ij}$ as $d_1 \times d_1$ block matrix with each entry the $d_2 \times d_2$ matrix $d_2^{-1} G_i G_j^*$.



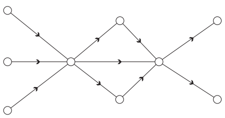
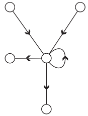
Partial Transposes on $M_{d_1}(\mathbf{C}) \otimes M_{d_2}(\mathbf{C})$

- G_i a $d_2 \times p$ matrix
- $W(i,j) = \frac{1}{d_2} G_i G_j^*$, a $d_2 \times d_2$ matrix,
- $W = \frac{1}{d_1}(W(i,j))_{ij}$ is a $d_1 \times d_1$ *block* matrix with entries $W(i,j)$
- $W^{\mathrm{T}} = \frac{1}{d_1}(W(j,i)^{\mathrm{T}})_{ij}$ is the "full" transpose
- $W^{\urcorner} = \frac{1}{d_1}(W(j,i))_{ij}$ is the "left" partial transpose
- $W^{\Gamma} = \frac{1}{d_1}(W(i,j)^{\mathrm{T}})_{ij}$ is the "right" partial transpose
- we **assume** that $\dfrac{p}{d_1 d_2} \to c$, $0 < c < \infty$
- eigenvalue distributions of W and W^{T} converge to Marchenko-Pastur with parameter c

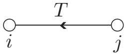
- eigenvalues of $W^{\urcorner}$ and W^{Γ} converge to a shifted semi-circular with mean c and variance c (Aubrun, 2012)
- W and W^{T} are asymptotically free (M. and Popa, 2014)
- (**main theorem**) the matrices $\{W, W^{\urcorner}, W^{\Gamma}, W^{\mathrm{T}}\}$ form an asymptotically free family

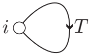
graphs and graphs sums (*with Roland Speicher*)

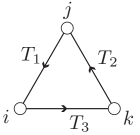
- a *graph* means a finite oriented graph with possibly loops and multiple edges


- a *graph sum* means attach a matrix to each edge and sum over vertices







$\sum_{i,j} t_{ij}$ $\qquad$ $\sum_{i} t_{ii}$ $\qquad$ $\sum_{i,j,k} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$

graph sums and their growth

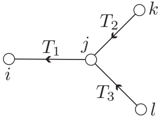
- given $G = (V, E)$ a graph and an assignment $e \mapsto T_e \in M_N(\mathbf{C})$ we have a graph sum

$$S_G(T) = \sum_{i:V\to[N]} \prod_{e\in E} t^{(e)}_{i_{t(e)} i_{s(e)}}$$

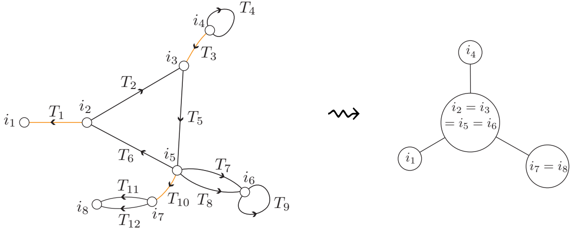
- problem find "best" $r(G) \in \mathbb{R}^+$ such that for all T we have

$$|S_G(T)| \leqslant N^{r(G)} \prod_{e\in E} \|T_e\|$$

- for example: $|S_G(T_1, T_2, T_3)| \leqslant N^{3/2} \|T_1\| \, \|T_2\| \, \|T_3\|$ when

$G =$ 

finding the growth (*J.F.A. 2012*)

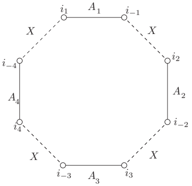


$\therefore \quad r = \dfrac{3}{2}$

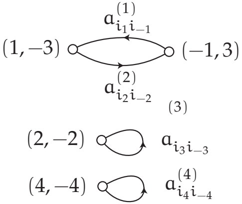
- a edge is *cutting* is its removal disconnects the graph
- a graph is *two-edge connected* if it has no cutting edge
- a *two-edge connected component* is a two-edge connected subgraph which is maximal
- we make a quotient graph whose vertices are the two-edge connected components on the old graph and the edges are the cutting edges of the old graph
- $r(G)$ is $\frac{1}{2}$ the number of leaves on the quotient graph (*always a union of trees*)

Conclusion: traces and graph sums

- $X = W^{\urcorner}$ is the partially transposed Wishart matrix, but now we **no longer** assume entries are Gaussian
- we let $A_1, A_2, \dots, A_n$ be $d_1 d_2 \times d_1 d_2$ constant matrices
- compute $\mathrm{E}(\mathrm{Tr}(XA_1XA_2 \cdots XA_n))$; when $A_i = I$ we get the n^{th} moment of the eigenvalue distribution
- integrating out the X's leaves a sum of graph sums, one for each partition $\pi \in \mathcal{P}(n)$



$$\pi = (1,-3)(-1,3)(2,-2)(4,-4)$$



THM: the only π's for which $r(G_\pi)$ is large enough ($n/2+1$ in this case) are non-crossing partitions with blocks of size 1 or 2 (corresponding to the free cumulants κ_1 and κ_2)

THM: method extends to showing that $\{W, W^{\urcorner}, W^{\Gamma}, W^{\mathrm{T}}\}$ ass. free