BROWN'S INVARIANT

Most of research about Connes' embedding conjecture has been focussing on impressive reformulations of it, that is, on finding apparently very far statements that turn out to be eventually equivalent to the original conjecture.

Over the last couple of years another point of view has been also taken, mostly due to Nate Brown's paper [1]. He assumes that a fixed separable II_1-factor M verifies Connes' embedding conjecture and tries to tell something interesting about M. In particular, he managed to associate an invariant to M, now called Brown's invariant, that carries information about rigidity properties of M. The purpose of this section is to introduce the reader to this invariant.

1. Convex combinations of representations into $R^{\mathcal{U}}$

Let M be a separable II_1-factor verifiying Connes' embedding conjecture and fix a free ultrafilter $\mathcal{U}$ on the natural numbers. The set $\mathbb{H}om(M, R^{\mathcal{U}})$ of unital morphisms $M \to R^{\mathcal{U}}$ modulo unitary equivalence is non-empty. We shall show that this set, that is infact Brown's invariant, has a surprisingly rich structure.

We can equip $\mathbb{H}om(M, R^{\mathcal{U}})$ with a metric in a reasonably simple way. Since M is separable, it is topologically generated by countably many elements $a_1, a_2 \ldots$, that we may assume to be contractions, that is $||a_i|| \leq 1$, for all i. So we can define a metric on $\mathbb{H}om(M, R^{\mathcal{U}})$ as follows

$$d([\pi],[\rho]) = \inf_{u \in U(R^{\mathcal{U}})} \left(\sum_{n=1}^{\infty} \frac{1}{2^{2n}} ||\pi(a_n) - u\rho(a_n)u^*||_2^2 \right)^{\frac{1}{2}},$$

since the series in the right hand side is convergent. A priori, d is just a pseudo-metric, but we can use Theorem 3.1 in [11] to say that approximately unitary equivalence is the same as unitary equivalence in separable subalgebras of $R^{\mathcal{U}}$. This means that d is actually a metric. Moreover, while this metric may depend on the generating set $\{a_1, a_2, \ldots\}$, the induced topology does not. It is indeed the point-wise convergence topology.

$\mathbb{H}om(M, R^{\mathcal{U}})$ does not carry any evident vector space structure, but Nate Brown's intuition was that one can however do convex combinations inside $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ in a formal way. There is indeed an obvious (and wrong) way to proceed: given *homomorphisms $\pi, \rho \colon M \to R^{\mathcal{U}}$ and $0 < t < 1$, take a projection $p_t \in (\pi(M) \cup \rho(M))' \cap R^{\mathcal{U}}$ such that $\tau(p_t) = t$ and define the "convex combination" $t\pi + (1-t)\rho$ to be

$$x \mapsto \pi(x)p_t + \rho(x)p_t^{\perp}.$$

Since the projection p_t is chosen in $(\pi(M) \cup \rho(M))'$, then $t\pi + (1-t)\rho$ is certainly a new unital morphism of M in $R^{\mathcal{U}}$. Unfortunately this procedure is not well defined on classes in $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ and the reason can be explained as follows: if $p \in R^{\mathcal{U}}$ is a projection, then the corner $pR^{\mathcal{U}}p$ is isomorphic to $R^{\mathcal{U}}$, by a well-known but not easy result. Thus the cut-down $p\pi$ can be seen as a new morphism $M \to R^{\mathcal{U}}$. The problem is that the isomorphism $pR^{\mathcal{U}}p \to R^{\mathcal{U}}$ is not canonical and this reflects on the fact that convex combinations as defined above are not well-defined on classes in $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$. The idea is to allow only particular isomorphisms $pR^{\mathcal{U}}p \to R^{\mathcal{U}}$ that are somehow fixed by conjugation by a unitary. This is done by using the so-called standard isomorphisms, that represent Nate Brown's main technical innovation.

Definition 1.1. (N.P.Brown[1]**)** Let $p \in R^{\mathcal{U}}$ be a projection. A *standard isomorphism* $\theta_p \colon pR^{\mathcal{U}}p \to R^{\mathcal{U}}$ is any map gotten in the following way: lift p to a projection $(p_n) \in \ell^\infty(R)$ such that $\tau_R(p_n) = \tau_{R^{\mathcal{U}}}(p)$, for all $n \in \mathbb{N}$, fix isomorphisms $\theta_n \colon p_nRp_n \to R$ and define θ_p to be the isomorphism on the right hand side of the commutative diagram

$$\begin{array}{ccc} \ell^\infty(p_nRp_n) & \longrightarrow & pR^{\mathcal{U}}p \\ \big\downarrow \oplus\theta_n & & \big\downarrow \cong \\ \ell^\infty(R) & \longrightarrow & R^{\mathcal{U}} \end{array}$$

Definition 1.2. Given $[\pi_1], \ldots, [\pi_n] \in \mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ and $t_1, \ldots, t_n \in [0,1]$ such that $\sum t_i = 1$, we define

$$\sum_{i=1}^n t_i[\pi_i] := \left[\sum_{i=1}^n \left(\theta_i^{-1} \circ \pi_i \right) \right],$$

where $\theta_i \colon p_iR^{\mathcal{U}}p_i \to R^{\mathcal{U}}$ are standard isomorphisms and $p_1, \ldots, p_n \in R^{\mathcal{U}}$ are orthogonal projections such that $\tau(p_i) = t_i$ for $i \in \{1, \ldots, n\}$.

We can explain in a few words why this procedure of using standard isomorphisms works. It has been originally proven by Murray and von Neumann that there is a unique unital embedding of $M_n(\mathbb{C})$ into R up to unitary equivalence. Since R contains an increasing chain of matrix algebras whose union is weakly dense, it follows that all unital endomorphisms of R are approximately inner. Now, if we take an automorphism Θ of $R^{\mathcal{U}}$ that can be lifted (i.e., it is of the form $(\theta_n)_{n\in\mathbb{N}}$ where θ_n is an automorphism of $\ell^\infty(R)$), it follows that Θ is just the conjugation by some unitary, when restricted to a separable subalgebras or $R^{\mathcal{U}}$. Now, Nate Brown's standard isomorphisms are exactly those isomorphisms from a corner $pR^{\mathcal{U}}p \to R^{\mathcal{U}}$ that are liftable and therefore it is intuitively clear that after quotienting out the unitary equivalence, the choice of the standard isomorphism should not affect the result. The formalization of this rough idea leads to the following theorem.

Theorem 1.3. (N.P. Brown[1]**)** $\sum_{i=1}^n t_i[\pi_i]$ *is well defined, i.e., independent of the projections p_i, the standard isomorphisms θ_i and the representatives π_i.*

To prove this result we need some preliminary observations.

Lemma 1.4. *Let $p, q \in R$ be projections of the same trace and $\theta : pRp \to qRq$ be a unital *homomorphism, that is $\theta(p) = q$. Then there is a sequence of partial isometries $v_n \in R$ such that:*

(1) $v_n^* v_n = p$,
(2) $v_n v_n^* = q$,
(3) $\theta(x) = \lim_{n\to\infty} v_n x v_n^*$,

where the limit is taken in the 2-norm.

Proof. Since p, q have the same trace, we can find a partial isometry w such that $w^*w = q$ and $ww^* = p$. Consider the unital endomorphism $\theta_w : pRp \to pRp$ defined by $\theta_w(x) = w\theta(x)w^*$. Since R is hyperfinite, every endomorphism is approximatively inner in the 2-norm, that is, we can find unitaries $u_n \in pRp$ such that $w\theta(x)w^* = \lim_{n\to\infty} u_n x u_n^*$. Defining $v_n = w^* u_n$ completes the proof. □

Proposition 1.5. *Let $p, q \in R^{\mathcal{U}}$ be projections of the same trace, $M \subseteq pR^{\mathcal{U}}p$ be a separable von Neumann subalgebra and $\Theta : pR^{\mathcal{U}}p \to qR^{\mathcal{U}}q$ be a unital *homomorphism. Assume there exist projections $(p_i), (q_i) \in \ell^\infty(R)$ which are lifts of p and q, respectively, such that $\tau_R(p_i) = \tau_R(q_i) = \tau_{R^{\mathcal{U}}}(p)$, for all $i \in \mathbb{N}$, and there exist unital *homomorphisms $\theta_i : p_iRp_i \to q_iRq_i$ such that $(\theta_i(x_i))$ is a lift of $\Theta(x)$, whenever $(x_i) \in \Pi p_iRp_i$ is a lift of $x \in M$.*

Then there exists a partial isometry $v \in R^{\mathcal{U}}$ such that:

(1) $v^*v = p$,
(2) $vv^* = q$,
(3) $\Theta(x) = vxv^*$, *for all* $x \in M$.

Proof. We shall prove the proposition only in the case $M = W^*(X)$ is singly generated.

Let $(x_i) \in \Pi p_iRp_i$ be a lift of X. By Lemma 1.4, we can find partial isometries $v_i \in R$ such that $v_i^*v_i = p_i$, $v_iv_i^* = q_i$ and $||\theta_i(x_i) - v_ix_iv_i^*||_2 < 1/i$. Observe that $(v_i) \in \ell^\infty(R)$ drops to a partial isometry $v \in R^{\mathcal{U}}$ with support p and range q. To show that $\Theta(X) = vXv^*$, fix $\varepsilon > 0$ and consider the set

$$S_\varepsilon = \{i \in \mathbb{N} : ||\theta_i(x_i) - v_ix_iv_i^*||_2 < \varepsilon\}.$$

This set contains the cofinite set $\{i \in \mathbb{N} : i \geq \frac{1}{\varepsilon}\}$ and therefore $S_\varepsilon \in \mathcal{U}$. □

Exercise 1.6. Prove Proposition 1.5 in the general case. (Hint: pick the v_i's arranging inequalities of the form $||\theta_i(Y_i) - v_iY_iv_i^*||_2 < 1/i$ on a finite set of Y_i's corresponding to lifts of a finite subset of a generating set of M).

Proof of Theorem 1.3. Assume $\sigma_i : q_iR^{\mathcal{U}}q_i \to R^{\mathcal{U}}$ are standard isomorphisms, where the q_i's are orthogonal projections of trace t_i and $[\rho_i] = [\pi_i]$. By Proposition 1.5, applied to the standard isomorphism $\sigma_i^{-1} \circ \theta_i : p_iR^{\mathcal{U}}p_i \to q_iR^{\mathcal{U}}q_i$, we can find partial isometries $v_i \in R^{\mathcal{U}}$ such that $v_i^*v_i = p_i$, $v_iv_i^* = q_i$ and

$$v_i\left(\theta_i^{-1} \circ \pi_i\right)(x)v_i^* = \left(\sigma_i^{-1} \circ \pi_i\right)(x), \qquad \text{for all } x \in M.$$

Now since $[\pi_i] = [\rho_i]$, we can find unitaries u_i such that $\rho_i = u_i\pi_iu_i^*$. The proof is then completed by the following exercise.

Exercise 1.7. Show that $u := \sum \sigma_i^{-1}(u_i)v_i$ is a unitary conjugating $\sum \theta_i^{-1} \circ \pi_i$ over to $\sum \sigma_i^{-1} \circ \rho_i$.

□

2. Convex-like structures

Having a notion of convex combinations, it is natural to ask whether it verifies the obvious properties that it would satisfy if $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ were a convex subset of a Banach space.

An axiomatization of convex subsets of a Banach space has been proposed by Nate Brown himself through the notion of convex-like structures. The proof that every convex-like structure is in fact a convex subset of a Banach space has been given by Capraro and Fritz in [3].

Let (X, d) be a complete and bounded metric space. Denote by $X^{(n)} = X \times \cdots \times X$ the n-fold Cartesian product and let $Prob_n$ be the set of probability measures on the n-point set $\{1, 2, \ldots, n\}$, endowed with the ℓ_1-metric $\|\mu - \tilde{\mu}\| = \sum_{i=1}^n |\mu(i) - \tilde{\mu}(i)|$.

Definition 2.1. (N.P.Brown[1]) We say (X, d) has a *convex-like structure* if for every $n \in \mathbb{N}$ and $\mu \in Prob_n$ there is a continuous map $\gamma_\mu \colon X^{(n)} \to X$ such that

(1) for each permutation $\sigma \in S_n$ and $x_1, \ldots, x_n \in X$,
$$\gamma_\mu(x_1, \ldots, x_n) = \gamma_{\mu\circ\sigma}(x_{\sigma(1)}, \ldots, x_{\sigma(n)});$$
(2) if $x_1 = x_2$, then $\gamma_\mu(x_1, x_2, \ldots, x_n) = \gamma_{\tilde{\mu}}(x_1, x_3, \ldots, x_n)$, where $\tilde{\mu} \in Prob_{n-1}$ is given by $\tilde{\mu}(1) = \mu(1) + \mu(2)$ and $\tilde{\mu}(j) = \mu(j+1)$ for $2 \leq j \leq n-1$;
(3) if $\mu(i) = 1$, then $\gamma_\mu(x_1, \ldots, x_n) = x_i$;
(4) There is a constant C such that for all $x_1, \ldots, x_n \in X$ and for all $\nu, \tilde{\nu} \in Prob_n$, one has
$$d(\gamma_\mu(x_1, \ldots, x_n), \gamma_{\tilde{\mu}}(x_1, \ldots, x_n)) \leq C\|\mu - \tilde{\mu}\|.$$
(5) For all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ and for all $\nu \in Prob_n$, one has
$$d(\gamma_\mu(x_1, \ldots, x_n), \gamma_\mu(y_1, \ldots, y_n)) \leq \sum_{i=1}^n \mu(i) d(x_i, y_i);$$
(6) for all $\nu \in Prob_2$, $\mu \in Prob_n$, $\tilde{\mu} \in Prob_m$ and $x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_m \in X$,
$$\gamma_\nu(\gamma_\mu(x_1, \ldots, x_n), \gamma_{\tilde{\mu}}(\tilde{x}_1, \ldots, \tilde{x}_m)) = \gamma_\eta(x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_m),$$
where $\eta \in Prob_{n+m}$ is given by $\eta(i) = \nu(1)\mu(i)$, if $1 \leq i \leq n$, and $\eta(j+n) = \nu(2)\tilde{\mu}(j)$, if $1 \leq j \leq m$.

Remark 2.2. To understand this definition we make a short parallelism with the usual notion of convex combinations in a Banach space equipped with a norm $||\cdot||$. An element $\mu \in Prob_n$ is uniquely represented by a n-tuple $(t_1, \ldots, t_n)$ of positive real numbers such that $\sum t_i = 1$. The element $\gamma_\mu(x_1, \ldots, x_n)$ in Definition 2.1 is exactly $t_1x_1 + \ldots + t_nx_n$.

It is now clear that the first axiom of a convex-like structure is the commutative property. The second axiom just formalizes the following associative property:

$$t_1x_1 + t_2x_1 + t_3x_3 + \ldots + t_nx_n = (t_1 + t_2)x_1 + t_3x_3 + \ldots + t_nx_n.$$

The third axiom formalizes the following property

$$0 \cdot x_1 + \ldots 0 \cdot x_{i-1} + 1 \cdot x_1 + 0 \cdot x_{i+1} + \ldots + 0 \cdot x_n = x_i.$$

The fourth axiom formalizes the following inequality:

$$||(t_1x_1 + \ldots t_nx_n) - (s_1x_1 + \ldots + s_nx_n)|| \leq \max_i ||x_i|| \sum_{i=1}^{n} |t_i - s_i|,$$

where $\max ||x_i||$ in our case is uniformly bounded, since the metric space is supposed to be bounded. The fifth axiom corresponds to the following inequality:

$$||(t_1x_1 + \ldots + t_nx_n) - (t_1y_1 + \ldots + t_ny_n)|| \leq \sum |t_i| d(x_i, y_i).$$

The last axiom requires the following algebraic property

$$\begin{aligned} &\alpha(t_1x_1 + \ldots + t_nx_n) + (1-\alpha)(s_1y_1 + \ldots + s_my_m) = \\ &= (\alpha t_1)x_1 + \ldots + (\alpha t_n)x_n + ((1-\alpha)s_1)y_1 + \ldots + ((1-\alpha)s_m)y_m. \end{aligned}$$

The proof that $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ has a convex-like structure will be given in the next section. Here we prove the representation theorem, stating that convex-like structures are exactly the convex and bounded subsets of a Banach space.

Theorem 2.3. (Capraro-Fritz[3]) *Every convex-like structure is isometrically and affinely embeddable into a Banach space.*

This result already allows to avoid all technicalities in Section 2. of Brown's paper. It can be also useful in other contexts, because of its generality. For instance, it was used by Liviu Păunescu to prove that his own convex-like structure on the set of sofic embeddings embeds into a vector space (see [9]).

The proof of Theorem 2.3 will be divided in several steps.

Definition 2.4 ([5]). A *convex space* is given by a set X and a family of binary operations $\{cc_\lambda\}_{\lambda\in[0,1]}$ on X such that

(cs.1) $cc_0(x, y) = x, \quad \forall x, y \in X;$
(cs.2) $cc_\lambda(x, x) = x, \quad \forall x \in X,\ \lambda \in [0, 1];$
(cs.3) $cc_\lambda(x, y) = cc_{1-\lambda}(y, x), \quad \forall x, y \in X,\ \lambda \in [0, 1];$
(cs.4) $cc_\lambda(cc_\mu(x, y), z) = cc_{\lambda\mu}(x, cc_\nu(y, z)), \quad \forall x, y, z \in X,\ \lambda, \mu \in [0, 1]$; where ν is arbitrary if $\lambda = \mu = 1$ and $\nu = \frac{\lambda(1-\mu)}{1-\lambda\mu}$ otherwise.

If $\mu = (\lambda, 1-\lambda)$ is a measure on the 2-point set, we write $\gamma_{\lambda,1-\lambda}$ instead of γ_μ.

Lemma 2.5. *A convex-like structure is also a convex space, by setting*

$$cc_\lambda(x, y) = \gamma_{\lambda,1-\lambda}(x, y)\ . \tag{1}$$

The converse of this lemma is also true, but the proof is more involved (see [3], Theorem 3).

Proof of Lemma 2.5. We prove the lemma axiom-by-axiom.

(cs.1) We have $cc_0(x,y) = \gamma_{0,1}(x,y) = y$, thanks to Brown's axiom 3.

(cs.2) We have $cc_\lambda(x,x) = \gamma_{\lambda,1-\lambda}(x,x)$, thanks to Brown's axiom 2.

(cs.3) We have

$$cc_\lambda(x,y) = \gamma_{\lambda,1-\lambda}(x,y) = \gamma_{1-\lambda,\lambda}(y,x) = cc_{1-\lambda}(y,x),$$

thanks to Brown's axiom 1.

(cs.4) This is implied by the previous axioms when $\lambda = \mu = 1$, so it is enough to treat the case $\lambda\mu \neq 1$. We will evaluate $cc_\lambda(cc_\mu(x,y),z)$ and $cc_{\lambda\mu}(x, cc_{\frac{\lambda(1-\mu)}{1-\lambda\mu}}(y,z))$ separately and obtain two identical expressions. Using axiom 6, we have

$$cc_\lambda(cc_\mu(x,y),z) = \gamma_\eta(x,y,z),$$

where $\eta(1) = \lambda\mu, \eta(2) = \lambda(1-\mu)$ and $\eta(3) = 1-\lambda$. On the other hand, the same 6 also implies

$$cc_{\lambda\mu}(x, cc_{\frac{\lambda(1-\mu)}{1-\lambda\mu}}(y,z)) = \gamma_\eta(x,y,z),$$

with the same distribution $\eta \in \mathrm{Prob}_3$.

□

Lemma 2.6. *If the equation*

$$cc_\lambda(y,x) = cc_\lambda(z,x)$$

holds for some $x,y,z \in X$ and $\lambda \in (0,1)$, then it also holds for all $\lambda \in (0,1)$.

Proof. Let us write λ_0 for the original value for which the equation holds. Then for all $\lambda < \lambda_0$,

$$cc_\lambda(y,x) = cc_{\lambda/\lambda_0}(cc_{\lambda_0}(y,x),x) = cc_{\lambda/\lambda_0}(cc_{\lambda_0}(z,x),x) = cc_\lambda(z,x),$$

by (cs.4) and (cs.2), so that the equation is also true in that case. Hence it is enough to find a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \overset{n\to\infty}{\longrightarrow} 1$ for which the equation holds. We construct such a sequence by defining $\lambda_{n+1} = \frac{2\lambda_n}{1+\lambda_n}$, for which an inductive argument shows the validity of the equation:

$$\begin{aligned} cc_{\lambda_{n+1}}(y,x) & \\ &= cc_{\lambda_n/(1+\lambda_n)}(y, cc_{\lambda_n}(y,x)) \\ &= cc_{\lambda_n/(1+\lambda_n)}(y, cc_{\lambda_n}(z,x)) \\ &= cc_{\lambda_n/(1+\lambda_n)}(z, cc_{\lambda_n}(y,x)) \\ &= cc_{\lambda_n/(1+\lambda_n)}(z, cc_{\lambda_n}(z,x)) \\ &= cc_{\lambda_{n+1}}(z,x)\,. \end{aligned}$$

□

Proposition 2.7. *Let X be a convex-like structure. Then the following cancellative property is verified:*

$$cc_\lambda(x,y) = cc_\lambda(x,z) \;\; \textit{with} \;\; \lambda \in (0,1) \quad \Longrightarrow \quad y = z\,.$$

Proof. By the previous lemma, we know that if $\gamma_{\lambda,1-\lambda}(x,y) = \gamma_{\lambda,1-\lambda}(x,z)$ holds for some $\lambda \in (0,1)$, then it holds for all $\lambda \in (0,1)$. But then, we get from 4, for any $\lambda > 0$,

$$d(y,z) \leq d(y,\gamma_{\lambda,1-\lambda}(x,y)) + d(z,\gamma_{\lambda,1-\lambda}(x,z)) \leq \lambda d(x,y) + \lambda d(x,z) = \lambda\,[d(x,y) + d(x,z)]$$

Since λ was arbitrary, we conclude $d(y,z) = 0$, and hence $y = z$. □

Theorem 2.8. *Let X be a convex-like structure. Then there is a linear embedding of X into some vector space.*

Proof. By the previous results, we can see a convex-like structure as a convex space with verifies the cancellative property in Proposition 2.7. Such spaces are linearly embeddable into vector spaces by the Stone representation theorem [12]. □

Detailed about Stone's representation theorem can be found in [3], Theorem 4. Roughly speaking, one considers the free vector space generated by the elements of the convex space modded out by *obvious* equivalence relations. The cancellative property implies that the original convex space embeds faithfully into this quotient.

To prove that such an embedding is isometric with respect to a natural norm on this *universal* vector space we need to work a bit more.

Lemma 2.9. *Let (X,d) be a metric space which is a convex subset $X \subseteq E$ of some vector space E such that*

$$d(\lambda y + (1-\lambda)x, \lambda z + (1-\lambda)x) \leq \lambda d(y,z), \quad \forall x,y \in X,\ \lambda \in [0,1], \tag{2}$$

holds. Then there is a norm $||\cdot||$ on E such that for all $x,y \in X$,

$$d(x,y) = ||x-y||\,.$$

Proof. As a special case, (2) gives for $z = x$,

$$d(\lambda y + (1-\lambda)x, x) \leq \lambda d(y,x),$$

which yields, in combination with the triangle inequality,

$$\begin{aligned} &d(y,x) \\ &\qquad \leq d(y, \lambda y + (1-\lambda)x) + d(\lambda y + (1-\lambda)x, x) \\ &\qquad \leq (1-\lambda)d(y,x) + \lambda d(y,x)\,. \end{aligned}$$

Since the term on the left-hand side equals the term on the right-hand side, we deduce that both inequalities are actually equalities. In particular, the metric is "uniform on lines" in the sense that

$$d(x,(1-\lambda)x + \lambda y) = \lambda d(x,y) \quad \forall x,y \in X,\ \lambda \in [0,1]\,.$$

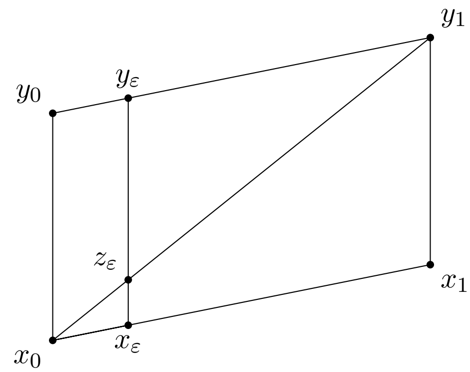

FIGURE 1. Illustration of the proof of lemma 2.9.

Now in order to prove the assertion, it needs to be shown that d is translation-invariant in the following sense: suppose that $x_0, x_1, y_0, y_1 \in X$ are such that

$$y_1 - x_1 = y_0 - x_0 ,$$

then $d(x_1, y_1) = d(x_0, y_0)$. See figure 2 for an illustration. For $\varepsilon \in (0, 1)$, we will also consider the points

$$x_\varepsilon = \varepsilon x_1 + (1-\varepsilon)x_0 , \qquad y_\varepsilon = \varepsilon y_1 + (1-\varepsilon)y_0 , \qquad z_\varepsilon = (1-\varepsilon)x_\varepsilon + \varepsilon y_\varepsilon = \varepsilon y_1 + (1-\varepsilon)x_0 .$$

Then by the assumption (2),

$$d(x_\varepsilon, z_\varepsilon) = d\left(\varepsilon x_1 + (1-\varepsilon)x_0, \varepsilon y_1 + (1-\varepsilon)x_0\right) \leq \varepsilon d(x_1, y_1) .$$

By the definition of z_ε and the uniformity of d on the line connecting z_ε with x_ε and y_ε, we have

$$d(x_\varepsilon, y_\varepsilon) = \varepsilon^{-1} d(x_\varepsilon, z_\varepsilon) \leq d(x_1, y_1) .$$

Upon taking the limit $\varepsilon \to 0$ we therefore arrive at

$$d(x_0, y_0) \leq d(x_1, y_1) ,$$

and the other inequality direction is then clear by symmetry, so that d is indeed translation invariant.

Now d can be uniquely extended to a translation-invariant metric on the affine hull of X. Assuming $0 \in X$ without loss of generality, this affine hull equals the linear hull, $\mathrm{lin}(X)$, and then the translation-invariant metric on $\mathrm{lin}(X)$ comes from a norm. If necessary, this norm can be extended from the subspace $\mathrm{lin}(X)$ to all of E. □

Now we have assembled all the ingredients for the main theorem of this section.

Theorem 2.10. *Every convex-like structure is affinely and isometrically isomorphic to a closed convex subset of a Banach space.*

Proof. Since the inequality (2) is an instance of the metric compatibility axiom 4, this is a direct consequence of corollary 2.8 and lemma 2.9 and the fact that every norm space embeds into its completion, which is a Banach space. Closedness then follows from the requirement that a convex-like structure is assumed to be complete. □

Theorem 2.10 tells us that we can regard $\mathbb{H}\text{om}(M, R^{\mathcal{U}})$ as a convex, closed and bounded subset of a Banach space. Nevertheless, the construction passes through Stone's representation theorem, that is very abstract. If one wants to use this embedding to find out new properties of $\mathbb{H}\text{om}(M, R^{\mathcal{U}})$, one needs a more concrete realization. Such concrete realization will be given in the next section.

3. Concrete embedding of $\mathbb{H}\text{om}(M, R^{\mathcal{U}})$ into a Banach space

Nate Brown proved that $\mathbb{H}\text{om}(M, R^{\mathcal{U}})$ has a convex-like structure given by the convex combinations defined in Definition 1.2. His proof is quite involved and does not solve the problem of finding a concrete linear and isometric embedding of $\mathbb{H}\text{om}(M, R^{\mathcal{U}})$ into a Banach space. In this section we build directly this embedding, using a construction appeared in [2].

First of all we need to recall the definition of tensor product of von Neumann algebras. Let $H_1, \ldots, H_n$ be Hilbert spaces with inner products $\ll \cdot, \cdot \gg_1, \ldots, \ll \cdot, \cdot \gg_n$, respectively. We can define an inner product on the algebraic tensor product of the H_i's as follows

$$\ll \xi_1 \otimes \ldots \otimes \xi_n, \eta_1 \otimes \ldots \otimes \eta_n \gg = \ll \xi_1, \eta_1 \gg_1 \cdot \ldots \cdot \ll \xi_n, \eta_n \gg_n$$

for all $\xi_i, \eta_i \in H_i$. The completion of the algebraic tensor product of the H_i's with respect to this inner product is a Hilbert space, called tensor product of the H_i's and denoted by $H_1 \otimes \ldots \otimes H_n$. Now, let $M_1, \ldots, M_n$ be von Neumann algebras acting on $H_1, \ldots, H_n$, respectively. Let M_0 be the *algebra acting on $H_1 \otimes \ldots \otimes H_n$ of all finite sums of operators of the form $x_1 \otimes \ldots \otimes x_n$, with $x_i \in M_i$. This is a *subalgebra of $B(H_1 \otimes \ldots \otimes H_n)$ and therefore its strong closure is a von Neumann algebra. This von Neumann algebra is called *tensor product* of the M_i's and it is denoted by $M_1 \bar{\otimes} \ldots \bar{\otimes} M_n$.

In the sequel, we will mostly interested in a particular case, namely, $B(H)\bar{\otimes}R^{\mathcal{U}}$, where H is a separable Hilbert space. Let $e_{ii} \in B(H)$ be a countable set of pairwise orthogonal one-dimensional projections such that $\sum e_{ii} = 1$ and let v_{jk} be partial isometries mapping e_{jj} to e_{kk}. Define $f_{jk} = e_{jk} \otimes 1 \in B(H)\bar{\otimes}R^{\mathcal{U}}$. Such a system $\{f_{ij}\}$ is called system of matrix units for $B(H)\bar{\otimes}R^{\mathcal{U}}$. Now, $B(H)\bar{\otimes}R^{\mathcal{U}}$ is a II_∞-factor[1] and then we can consider a faithful, semi-finite, weakly-continuous non-zero trace on $B(H)\bar{\otimes}R^{\mathcal{U}}$. It is in fact a very basic result in Operator Algebras that II_∞-factors have, up to multiplication by a positive scalar, a unique faithful, semi-finite, weakly-continuous non-zero trace. Therefore, we may choose a the trace τ_∞ in such a way that $\tau(f_{ii}) = 1$, for all i. Now let M be a separable II_1-factor

[1] II_∞-factors can be defined intrinsecally. However, it is a basic result in Operator Theory that they can be always written as a tensor product between $B(H)$ and some II_1-factor.

which embeds into $R^{\mathcal{U}}$ and denote by $\mathbb{H}om_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ the set of all morphisms (necessarily non-unital) $\phi : M \to B(H)\bar{\otimes}R^{\mathcal{U}}$ such that $\tau_\infty(\phi(1)) < \infty$ modulo unitary equivalence. Observe that $\mathbb{H}om(M, R^{\mathcal{U}})$ can be regarded as a subset of $\mathbb{H}om_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ just identifying $[\pi] \in \mathbb{H}om(M, R^{\mathcal{U}})$ with $[\tilde{\pi}] \in \mathbb{H}om_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$, where $\tilde{\pi}$ is defined by the following conditions:

$$f_{ij}\tilde{\pi}f_{ij} = \begin{cases} \pi, & \text{if } (i,j) = (1,1); \\ 0, & \text{otherwise.} \end{cases}$$

Basically, we are defining $\tilde{\pi}$ to be the morphism which is equal to π in the first block of the infinite matrix representing $B(H)\bar{\otimes}R^{\mathcal{U}}$ and zero elsewhere.

So we have an embedding $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}}) \hookrightarrow \mathbb{H}\mathrm{om}_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$. We want to construct a concrete embedding of $\mathbb{H}\mathrm{om}_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ into a Banach space and show that this embedding agrees with the embedding $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}}) \hookrightarrow \mathbb{H}\mathrm{om}_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$.

The sum on $\mathbb{H}\mathrm{om}_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ is pretty easy to define and reflects one of the reasons why it is important to work inside a II_∞-factor. In such factors, indeed, the following operation is possible: given two projections $p, q \in B(H)\bar{\otimes}R^{\mathcal{U}}$ such that $\tau_\infty(p) < \infty$ and $\tau_\infty(q) < \infty$, there is a unitary u such that upu^* is orthogonal to q. This allows to define the sum in $\mathbb{H}om_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ as follows.

Definition 3.1. Let $[\phi], [\psi] \in \mathbb{H}om_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$, then $\phi(1)$ and $\psi(1)$ are finite projections and thus there exists a unitary $u \in U(B(H)\bar{\otimes}R^{\mathcal{U}})$ such that $u\phi(1)u^* \perp \psi(1)$. We define $[\phi] + [\psi] := [u\phi u^* + \psi]$.

Exercise 3.2. Prove that the sum is well-defined; that is,

1. $u\phi u^* + \psi$ is still a morphism from M to $B(H)\bar{\otimes}R^{\mathcal{U}}$ with finite trace.
2. The class $[u\phi u^* + \psi]$ does not depend on u with the property that $u\phi(1)u^* \perp \psi(1)$.
3. The class $[u\phi u^* + \psi]$ does not depend on ϕ and ψ taken in their own classes.

Exercise 3.3. Show that $\mathbb{H}om_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ is a commutative monoid.

Definition 3.4. A monoid $(M, \cdot)$ is called *left-cancellative* if the condition $a \cdot b = a \cdot c$, implies $b = c$. Analogously one defines *right-cancelative* monoids. A monoid is called *cancelative* if it is both right- and left-cancelative.

To prove that $\mathbb{H}\mathrm{om}_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ is cancellative, we need a preliminary result. Let M^∞ denote the subset of $B(H)\bar{\otimes}R^{\mathcal{U}}$ of those elements x of finite trace.

Lemma 3.5. *Given a morphism $\phi : N \to M^\infty$ and projections $p, q \in \phi(N)' \cap M^\infty$, with $p, q \leq \phi(1)$. The following are equivalent:*

(1) There exists a partial isometry $v \in \phi(1)M^\infty\phi(1)$ such that $vv^* = q$, $v^*v = p$ and $v\phi(x)v^* = q\phi(x)$, for all $x \in N$.
(2) $p \sim q$ in $\phi(N)' \cap \phi(1)M^\infty\phi(1)$.
(3) $[p\phi] = [q\phi]$, where $p\phi : N \to M$ is defined by $x \to p\phi(x)$.

Proof. 1) $\Rightarrow$ 2). It suffices to show that v commutes with $\phi(x)$, for all $x \in N$. Indeed

$$\begin{aligned} v^*\phi(x) \\ &= v^*q\phi(x) \\ &= v^*v\phi(x)v^* \\ &= p\phi(x)v^* \\ &= \phi(x)v^* \end{aligned}$$

2) $\Rightarrow$ 3). Choose partial isometries $v \in \phi(N)' \cap \phi(1)M^\infty\phi(1)$ and $w \in \phi(N)' \cap \phi(1)M^\infty\phi(1)$ such that $v^*v = p, vv^* = q$, $w^*w = p^\perp$ and $ww^* = q^\perp$. (It is possible to find w since $\phi(N)' \cap \phi(1)M^\infty\phi(1)$ is a finite von Neumann algebra.) Hence $u = v + w \in \phi(N)' \cap \phi(1)M^\infty\phi(1)$ is a unitary and

$$up\phi(x)u^* = upu^*\phi(x) = q\phi(x).$$

Extending u to a unitary in $B(H)\bar{\otimes}M$ we see $[p\phi] = [q\phi]$.

3) $\Rightarrow$ 1). Choose a unitary $u \in B(H)\bar{\otimes}M$ such that $up\phi(x)u^* = q\phi(x)$, for all $x \in N$. Define $v = up$ and, using the assumption that $p, q \leq \phi(1)$, one can check this does the trick. □

Proposition 3.6. *$\mathbb{H}\mathrm{om}(N, B(H)\bar{\otimes}R^{\mathcal{U}})$ is a cancellative monoid.*

Proof. We prove that $\mathbb{H}\mathrm{om}(N, B(H)\bar{\otimes}R^{\mathcal{U}})$ is left-cancellative. The proof of right-cancellation is similar. Let ρ, ϕ, ψ such that

$$[\rho] + [\phi] = [\rho] + [\psi].$$

We may assume that $\phi(1) = \psi(1)$ (since they have the same trace) and $\phi(1) \perp \rho(1)$. Let $u \in M\bar{\otimes}B(H)$ be a unitary such that $\rho + \phi = u(\rho + \psi)u^*$ and set $p = \rho(1)$ and $q = u\rho(1)u^*$. Then $p(\rho + \phi) = \rho$ and $q(\rho + \phi) = q(u(\rho + \psi)u^*) = u\rho u^*$. It follows that $[p(\rho+\phi)] = [q(\rho+\phi)]$ and so, by Lemma 3.5, p and q are Murray-von Neumann equivalent inside $((\rho + \phi)(N))' \cap (\rho + \phi)(1)M(\rho + \phi)(1)$; hence, so are $(\rho + \phi)(1) - p = \phi(1)$ and $(\rho + \phi)(1) - q = u\psi(1)u^*$. Therefore, using once again Lemma 3.5, we get

$$[\phi] = [\phi(1)(\rho + \phi)] = [u\psi(1)u^*(u(\rho + \psi)u^*)] = [u\psi u^*] = [\psi].$$

□

We now recall the construction of the Grothendieck group of a commutative monoid $(M, \cdot)$. Consider in $M \times M$ the equivalence relation $\sim$ defined by

$$(m_1, n_1) \sim (m_2, n_2) \qquad \text{iff there is } m \in M \text{ such that } m_1 \cdot n_2 \cdot m = m_2 \cdot n_1 \cdot m$$

In the quotient set $(M \times M)/\sim$ define the operation, still denoted by $\cdot$, setting

$$[(m_1, n_1)]_\sim \cdot [(m_2, n_2)]_\sim := [(m_1 \cdot m_2, n_1 \cdot n_2))]_\sim$$

Exercise 3.7. (1) Prove that $(M \times M)/\sim$ is a group. It will be denoted by $\mathcal{G}(M)$.

(2) Suppose M is cancelative and denote by 0 its neutral element. Prove that the mapping $M \to \mathcal{G}(M)$ defined by $m \to [(m,0)]_\sim$ is a monoidal embedding.
(3) Give an example of a monoid which does not embed into its Grothendiek group.

The use of the subscript $+$ in the notation $\mathbb{H}om_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ should now be clearer: Proposition 3.6 and Exercise 3.7 tell that this space embeds into its Grothendieck group in some sense as the positive part. In fact, we now define in $\mathbb{H}om_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ the scalar product by a positive scalar and then we will extend everything in the Grothendieck group.

The construction is quite involved and here we give only a detailed sketch. The first thing to do is to consider only a subclass of standard isomorphisms. This restriction does not create any problem since we have seen that the convex-like structure on $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ is independent of the choice of the standard isomorphisms.

To this end, first recall that there is an isomorphism $\Phi : R\bar{\otimes}R \to R$. Given a free ultrafilter $\mathcal{U}$ on the natural numbers, let $\Phi_{\mathcal{U}}$ be the component-wise isomorphism $(R\bar{\otimes}R)^{\mathcal{U}} \to R^{\mathcal{U}}$ induced by Φ.

Definition 3.8. Let $p \in R^{\mathcal{U}}$ be a projection such that $\Phi_{\mathcal{U}}^{-1}(p)$ has the form $\tilde{p} \otimes 1 = (\tilde{p}_n \otimes 1)_n \in (R \otimes R)^{\mathcal{U}}$, with $\tau(\tilde{p}_n) = \tau(\tilde{p}) = \tau(p)$. Only throughout this section, a standard isomorphism $\theta : R^{\mathcal{U}} \to pR^{\mathcal{U}}p$ will be any isomorphism gotten in the following way. Fix isomorphisms $\alpha_n : R \to \tilde{p}_n R \tilde{p}_n$ and let $\theta_n := \alpha_n \otimes Id : R\bar{\otimes}R \to \tilde{p}_n R \tilde{p}_n \bar{\otimes} R$. Define θ to be the isomorphism on the right hand side of the following diagram

$$\begin{tikzcd}
\ell^\infty(R\bar{\otimes}R) \arrow[r] \arrow[d, "\oplus_{\mathbb{N}}\theta_n"] & (R\bar{\otimes}R)^{\mathcal{U}} \arrow[r] \arrow[d, "{}_{\mathcal{U}}\theta"] & R^{\mathcal{U}} \arrow[d, "\theta"] \\
\ell^\infty((\tilde{p}_n \otimes 1)(R\bar{\otimes}R)(\tilde{p}_n \otimes 1)) \arrow[r] & (\tilde{p} \otimes 1)(R\bar{\otimes}R)^{\mathcal{U}}(\tilde{p} \otimes 1) \arrow[r] & pR^{\mathcal{U}}p
\end{tikzcd}$$

where the horizontal left-hand side arrows are the projections onto the quotient, the horizontal right-hand side arrows are the ultrapower isomorphisms $\Phi_{\mathcal{U}}$, and the isomorphism ${}_{\mathcal{U}}\theta$ is the one obtained by imposing commutativity on the left-half of the diagram.

The following Lemma is very similar to Proposition 1.5 and it is one of the main technical tools that we need.

Lemma 3.9. *Let $p, q \in R^{\mathcal{U}}$ be projections of the same trace as needed to define standard isomorphisms θ_p, θ_q. For all separable von Neumann subalgebras $M_1 \subseteq R^{\mathcal{U}}$, there is a partial isometry $v_1 \in R^{\mathcal{U}}$ such that $v_1^* v_1 = p$, $v_1 v_1^* = q$ and*

$$v_1 \theta_p(x) v_1^* = \theta_q(x) \qquad \text{for all } x \in M_1$$

Proof. With the obvious notation, consider the following commutative diagram

$$\begin{tikzcd}
\ell^\infty((\tilde{p}_n \otimes 1)(R\bar{\otimes}R)(\tilde{p}_n \otimes 1))) \arrow[r] \arrow[d, "\oplus\theta_{p_n}^{-1}"] & (\tilde{p}\otimes 1)(R\bar{\otimes}R)^{\mathcal{U}}(\tilde{p}\otimes 1) \arrow[r] \arrow[d, "({}_{\mathcal{U}}\theta_p)^{-1}"] & pR^{\mathcal{U}}p \arrow[d, "\theta_p^{-1}"] \\
\ell^\infty(R\bar{\otimes}R) \arrow[r] \arrow[d, "\oplus_{\mathbb{N}}\theta_{q_n}"] & (R\bar{\otimes}R)^{\mathcal{U}} \arrow[r] \arrow[d, "{}_{\mathcal{U}}\theta_q"] & R^{\mathcal{U}} \arrow[d, "\theta_q"] \\
\ell^\infty((\tilde{q}_n \otimes 1)(R\bar{\otimes}R)(\tilde{q}_n \otimes 1)) \arrow[r] & (\tilde{q}\otimes 1)(R\bar{\otimes}R)^{\mathcal{U}}(\tilde{q}\otimes 1) \arrow[r] & qR^{\mathcal{U}}q
\end{tikzcd}$$

Consider $\Phi_{\mathcal{U}}^{-1}(M_1) \subseteq (R\bar{\otimes}R)^{\mathcal{U}}$. In the left-half of the previous diagram, we may apply Proposition 1.5 to $\Theta ={}_{\mathcal{U}} \theta_q \circ ({}_{\mathcal{U}}\theta_p)^{-1}$ and $M = \Phi_{\mathcal{U}}^{-1}(M_1)$, since all isomorphisms act only on the hyperfinite II_1-factor R. Thus, there is a partial isometry $v \in (R\bar{\otimes}R)^{\mathcal{U}}$ such that $v^*v = \tilde{p}\otimes 1$, $vv^* = \tilde{q}\otimes 1$ and

$$v({}_{\mathcal{U}}\theta_p(x))v^* ={}_{\mathcal{U}} \theta_q(x), \qquad \text{for all } x \in \Phi_{\mathcal{U}}^{-1}(M_1). \tag{3}$$

Define $v_1 = \Phi_{\mathcal{U}}(v)$ and one can verify that it works. □

Let $t \in (0,1)$ and let $p_t \in R^{\mathcal{U}}$ be a projection of trace t as needed to define a standard isomorphism $\theta_t : R^{\mathcal{U}} \to p_t R^{\mathcal{U}} p_t$. Let us recall the construction of a trace-scaling automorphism Θ_t of $B(H)\bar{\otimes}R^{\mathcal{U}}$, since it will be helpful in the proof of Proposition 3.12. More details can be found in [7], Proposition 13.1.10.

Let $\{e_{jj}\} \subseteq B(H)$ be a countable family of orthogonal one-dimensional projections such that $\sum e_{jj} = 1$ and let e_{jk} be partial isometries mapping e_{jj} to e_{kk}. Define $f_{jk} = e_{jk}\otimes 1 \in B(H)\bar{\otimes}R^{\mathcal{U}}$. We know that $f_{11}(B(H)\bar{\otimes}R^{\mathcal{U}})f_{11}$ is *isomorphic to $R^{\mathcal{U}}$ and that τ_∞ is normalized in such a way that $\tau_\infty(f_{11}) = 1$. Thus we can look at p_t as a projection in $f_{11}(B(H)\bar{\otimes}R^{\mathcal{U}})f_{11}$ with trace t and, for simplicity, let us denote it by g_{11}. Let g_{jj} be a countable family of orthogonal projections, each of which is equivalent to g_{11}, such that $\sum g_{jj} = 1 \in B(H)\bar{\otimes}R^{\mathcal{U}}$ and extend the family $\{g_{jj}\}$ to a system of matrix units $\{g_{jk}\}$ of $B(H)\bar{\otimes}R^{\mathcal{U}}$ adding appropriate partial isometries. Now, for any algebra $A \subset B(K)$, denote by $\aleph_0 \otimes A$ the algebra of countably infinite matrices with entries in A that define bounded operators on $\oplus_{\mathbb{N}} K \cong H\otimes K$. The isomorphism $\theta_t : R^{\mathcal{U}} \to p_t R^{\mathcal{U}} p_t$ can be seen as an isomorphism $\theta_t : f_{11}(B(H)\bar{\otimes}R^{\mathcal{U}})f_{11} \to p_t(B(H)\bar{\otimes}X^{\mathcal{U}})p_t$ and then it gives rise to an isomorphism

$$\aleph_0 \otimes \theta_t : \aleph_0 \otimes (f_{11}(B(H)\bar{\otimes}R^{\mathcal{U}})f_{11}) \to \aleph_0 \otimes (p_t(B(H)\bar{\otimes}R^{\mathcal{U}})p_t).$$

Now, let G be the matrix in $\aleph_0 \otimes (f_{11}(B(H)\bar{\otimes}R^{\mathcal{U}})f_{11})$ having the unit in the position $(1,1)$ and zeros elsewhere. Then $(\aleph_0 \otimes \theta_t)(G)$ is the matrix in $\aleph_0 \otimes (p_t(B(H)\bar{\otimes}R^{\mathcal{U}})p_t)$ having the unit in the position $(1,1)$ and zeros elsewhere. Now, take isomorphisms

$$\phi_1 : B(H)\bar{\otimes}R^{\mathcal{U}} \to \aleph_0\otimes(f_{11}(B(H)\bar{\otimes}R^{\mathcal{U}})f_{11}), \qquad \phi_2 : B(H)\bar{\otimes}R^{\mathcal{U}} \to \aleph_0\otimes(p_t(B(H)\bar{\otimes}R^{\mathcal{U}})p_t),$$

such that $\phi_1(f_{11}) = G$ and $\phi_2(g_{11}) = (\aleph\otimes\theta_t)(G)$. Define

$$\Theta_t = \phi_2^{-1} \circ (\aleph_0 \otimes \theta_t) \circ \phi_1.$$

It is easily checked that $\tau_\infty(\Theta_t(x)) = t\tau_\infty(x)$, for all x.

Remark 3.10. For the sequel, it is important to stress the fact that Θ_t is nothing but the isomorphism obtained by writing $B(H)\bar{\otimes}R^{\mathcal{U}}$ as an algebra of countably infinite matrices and letting θ_t act on each component. Therefore, if we want to prove that two isomorphisms $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$ constructed in such a fashion are unitarily equivalent, it suffices to find unitaries mapping $\theta_t^{(1)}$ to $\theta_t^{(2)}$ and the matrix units used in the first representation of $B(H)\bar{\otimes}R^{\mathcal{U}}$ as a matrix algebra to the matrix units used in the second representation.

Definition 3.11. *Let $t \in (0,1]$ and $[\phi] \in \mathbb{H}om_+(N, B(H)\bar{\otimes}R^{\mathcal{U}})$. We define*

$$t[\phi] = [\Theta_t \circ \phi].$$

Remark 3.10 is important because now we need to prove that the definition of $t[\phi]$ depends only on t and $[\phi]$ and is independent of Θ_t.

Proposition 3.12. *Let $t \in (0,1]$, $p_t^{(i)} \in R^{\mathcal{U}}$, $i = 1,2$, be two projections of trace t and $\theta_t^{(i)} : R^{\mathcal{U}} \to p_t^{(i)} R^{\mathcal{U}} p_t^{(i)}$ be two standard isomorphisms. Then $\Theta_t^{(1)} \circ \phi$ is unitarily equivalent to $\Theta_t^{(2)} \circ \phi$.*

Proof. Let us start with an observation. The image $\phi(N)$ a priori belongs to $B(H)\bar{\otimes}R^{\mathcal{U}}$, but since $\tau_\infty(\phi(1)) < \infty$, we can twist it by a unitary and suppose that $\phi(N) \subseteq M_n(\mathbb{C}) \otimes R^{\mathcal{U}}$, for some $n > \tau_\infty(\phi(1))$. Now, for all $j = 1, \ldots, n$, let

$$M_j = (e_{jj} \otimes 1)\phi(N)(e_{jj} \otimes 1) \subseteq (e_{jj} \otimes 1)(B(H)\bar{\otimes}R^{\mathcal{U}})(e_{jj} \otimes 1) \cong R^{\mathcal{U}}.$$

Since $p_t^{(1)}$ is equivalent to $p_t^{(2)}$ and $(p_t^{(1)})^\perp$ is equivalent to $(p_t^{(2)})^\perp$, in Lemma 3.9 we may find a unitary $u_i \in X^{\mathcal{U}}$ such that

$$(e_{jj} \otimes u_j)((e_{jj} \otimes \theta_t^{(1)})(x))(e_{jj} \otimes u_j) = (e_{jj} \otimes \theta_t^{(2)})(x), \qquad \text{for all } x \in M_j,$$

where $e_{jj} \otimes \theta_t^{(1)}$ stands for the endomorphism obtained letting $\theta_t^{(1)}$ act only on $f_{jj}(B(H)\bar{\otimes}R^{\mathcal{U}})f_{jj}$. Since the partial isometries $e_{jj} \otimes u_j$ act on orthogonal subspaces, we may extend them *all together* to a unitary $u \in B(H)\bar{\otimes}R^{\mathcal{U}}$ such that

$$u((e_{jj} \otimes \theta_t^{(1)})(x))u^* = (e_{jj} \otimes \theta_t^{(2)})(x), \qquad \text{for all } j = 1, \ldots, n \text{ and for all } x \in M_j.$$

Set $e_n = \sum_{j=1}^n e_{jj}$. We have

$$u((e_n \otimes \theta_t^{(1)})(x))u^* = (e_n \otimes \theta_t^{(2)})(x), \qquad \text{for all } x \in (e_n \otimes 1)\phi(N)(e_n \otimes 1) = \phi(N).$$

Now observe that the matrix units $\left\{f_{jk}^{(1)}\right\}$ and $\left\{f_{jk}^{(2)}\right\}$ used to construct $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$ are unitarily equivalent, since the projections on the diagonal have the same trace. Therefore, also the matrix units $\left\{uf_{jk}^{(1)}u^*\right\}$ and $\left\{f_{jk}^{(2)}\right\}$ are unitarily equivalent. Let $w \in B(H)\bar{\otimes}R^{\mathcal{U}}$ be a unitary such that

$$w(uf_{jk}^{(1)}u^*)w^* = f_{jk}^{(2)}, \qquad \text{for all } j,k \in \mathbb{N}.$$

The unitary w then twists the matrix units $uf_{jk}^{(1)}u^*$ into the matrix units $f_{jk}^{(2)}$ and it twists $u((e_n \otimes \theta_t^{(1)})(x))u^*$ to $(e_n \otimes \theta_t^{(2)})(x)$, for all $x \in \phi(N)$. Therefore, by Remark 3.10,

$$wu\Theta_t^{(1)}(x)u^*w^* = \Theta_t^{(2)}(x), \qquad \text{for all } x \in \phi(N),$$

as required. □

Recall that we have already fixed a *isomorphism $\Phi : R\bar{\otimes}R \to R$ and we have denoted by $\Phi_{\mathcal{U}} : (R\bar{\otimes}R)^{\mathcal{U}} \to R^{\mathcal{U}}$ the induced component-wise *isomorphism.

Definition 3.13. Let $\phi : N \to (R\bar{\otimes}R)^{\mathcal{U}}$. For each $x \in N$, let $(X_i^{\phi}) \in \ell^\infty(R\bar{\otimes}R)$ be a lift of $\phi(x)$. Define $1 \otimes \phi$ through the following diagram

$$\begin{tikzcd}
(1 \otimes X_n^{\phi})_n \in \ell^\infty(R\bar{\otimes}R\bar{\otimes}R) \arrow[r] \arrow[d, "\oplus_{\mathbb{N}}(1\otimes\Phi)"] & (R\bar{\otimes}R\bar{\otimes}R)^{\mathcal{U}} \arrow[d, "(1\otimes\Phi)_{\mathcal{U}}"] \\
\ell^\infty(R\bar{\otimes}R) \arrow[r] & (R\bar{\otimes}R)^{\mathcal{U}}
\end{tikzcd}$$

i.e. $(1 \otimes \phi)(x)$ is the image of the element $(1 \otimes X_n^{\phi})_n \in \ell^\infty(R\bar{\otimes}R\bar{\otimes}R)$ down in $(R\bar{\otimes}R)^{\mathcal{U}}$.

Exactly as in Lemma 3.2.3 in [1], one gets the following

Lemma 3.14. *For all $\phi : N \to (R\bar{\otimes}R)^{\mathcal{U}}$, one has $[1 \otimes \phi] = [\phi]$.*

Lemma 3.15. *Let θ_s, θ_t be two standard isomorphisms. Then*

$$\theta_s \circ \theta_t : R^{\mathcal{U}} \to \theta_s(p_t)R^{\mathcal{U}}\theta_s(p_t)$$

is still a standard isomorphism.

Proposition 3.16. *For all $s, t > 0$ and $[\phi], [\psi] \in \mathbb{H}om(N, (X^{\mathcal{U}})^\infty)$, the following properties are satisfied:*

(1) $0[\phi] = 0$,
(2) $1[\phi] = [\phi]$,
(3) $s(t[\phi]) = (st)[\phi]$,
(4) $s([\phi] + [\psi]) = s[\phi] + s[\psi]$,
(5) *if* $s + t \leq 1$, *then* $(s + t)[\phi] = s[\phi] + t[\phi]$.

Proof. The first two properties are trivial. The third property follows by Lemma 3.15 and Proposition 3.12. The fourth property can be easily proved by direct computation. Let us prove the fifth property. Fix $n > (s + t)\tau_\infty(\phi(1))$ and twist ϕ by a unitary in such a way that $\phi(N) \subseteq M_n(\mathbb{C}) \otimes R^{\mathcal{U}} = (M_n(\mathbb{C}) \otimes X)^{\mathcal{U}}$, since $M_n(\mathbb{C})$ is finite dimensional. Now, $M_n(\mathbb{C})$ has a unique unital embedding into R up to unitary equivalence and therefore we may suppose that $\phi(N) \subseteq (R\bar{\otimes}R)^{\mathcal{U}}$ and we may apply the construction in Definition 3.13 and Lemma 3.14 to replace $[\phi]$ with $[1 \otimes \phi]$. Now we have the freedom to choose orthogonal projections of the form

$$p_s \otimes 1 \otimes 1, \quad p_t \otimes 1 \otimes 1, \quad (p_s + p_t) \otimes 1 \otimes 1 \in (R\bar{\otimes}R\bar{\otimes}R)^{\mathcal{U}},$$

and use these projections to define standard isomorphisms. It is then clear that

$$\Theta_s \circ (1 \otimes \phi) + \Theta_t \circ (1 \otimes \phi) = \Theta_{t+s} \circ (1 \otimes \phi),$$

which implies that $[\Theta_s \circ \phi] + [\Theta_t \circ \phi] = [\Theta_{s+t} \circ \phi]$; i.e. $s[\phi] + t[\phi] = (s+t)[\phi]$. □

Therefore, we are in the following situation. We have a commutative cancelative monoid G_+ equipped with an action $[0,1] \curvearrowright G_+$ that verifies the properties of the proposition above. It is now easy to check that the Grothendieck group has then a canonical structure of a vector space (see Appendix in [2]). So we get a vector space that we denote by $\mathbb{H}om(M, B(H)\bar{\otimes}R^{\mathcal{U}})$. Moreover, the multiplication by a scalar is defined in such a way that the canonical embedding of $\mathbb{H}om(M, R^{\mathcal{U}})$ into $\mathbb{H}om(M, B(H)\bar{\otimes}R^{\mathcal{U}})$ is affine, concluding our sketch of the construction of an explicit embedding of $\mathbb{H}om(M, R^{\mathcal{U}})$ into a vector space.

Theorem 3.17. *$\mathbb{H}om(M, R^{\mathcal{U}})$ embeds affinely into the vector space $\mathbb{H}om_+(M, B(H)\bar{\otimes}R^{\mathcal{U}})$.*

4. Extreme points of $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$

In this section we move back to the study of the convex set $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$. Given a separable II_1-factor M that embeds into $R^{\mathcal{U}}$, Sorin Popa asked whether there is always another representation π such that $\pi(M)' \cap R^{\mathcal{U}}$ is a factor. It turns out that this problem is equivalent to a geometric problem on $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$.

Theorem 4.1. (Nate Brown) *Let $\pi : M \to R^{\mathcal{U}}$ be a representation. Then $\pi(M)' \cap R^{\mathcal{U}}$ is a factor if and only if $[\pi]$ is an extreme point of $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$.*

In this section we prove only the "only if": if $[\pi]$ is an extreme point, then $\pi(M)' \cap R^{\mathcal{U}}$ is a factor.

Definition 4.2. *We define the cutdown of a representation $\pi : M \to R^{\mathcal{U}}$ by a projection $p \in \pi(M)' \cap R^{\mathcal{U}}$ to be the map $M \to R^{\mathcal{U}}$ defined by $x \to \theta_p(p\pi(x))$, where $\theta_p : pR^{\mathcal{U}}p \to R^{\mathcal{U}}$ is a standard isomorphism.*

Lemma 3.3.3 in [1] shows that this definition is independent by the standard isomorphism, hence we can denote it by $[\pi_p]$.

Lemma 4.3. Given a representation $\pi : M \to R^{\mathcal{U}}$ and projections $p, q \in \pi(M)' \to R^{\mathcal{U}}$ of the same trace, the following are equivalent:

(1) $[\pi_p] = [\pi_q]$,
(2) p and q are Murray-von Neumann equivalent inside $\pi(M)' \cap R^{\mathcal{U}}$, that is, there is a partial isometry $v \in \pi(M)' \cap R^{\mathcal{U}}$ such that $v^*v = p$ and $vv^* = q$.
(3) there exists $v \in R^{\mathcal{U}}$ such that $v^*v = p$, $vv^* = q$ and $v\pi(x)v^* = q\pi(x)$, for all $x \in M$.

Exercise 4.4. Prove the equivalence between (2) and (3) in Lemma 4.3.

Exercise 4.5. Given projections p, q and a partial isometry v such that $v^*v = p$ and $vv^* = q$, show that there exist lifts $(p_n), (q_n), (v_n) \in \ell^\infty(R)$ such that p_n, q_n are projections of the same trace as p and $v_n^*v_n = p_n$ and $v_nv_n^* = q_n$, for all $n \in \mathbb{N}$.

Proof of (3) $\Rightarrow$ (1). Let p_n, q_n, v_n as in Exercise 4.5 and fix isomorphisms $\theta_n : p_nRp_n \to R$ and $\gamma_n : q_nRq_n \to R$ and use them to define standard isomorphisms $\theta : pR^{\mathcal{U}}p \to R^{\mathcal{U}}$ and $\gamma : qR^{\mathcal{U}}q \to R^{\mathcal{U}}$ and use them to define π_p and π_q. The isomorphism on the right hand side of the following diagram[2] is liftable by construction and so Proposition 1.5 can be applied to it, giving unitary equivalence between π_p and π_q.

$$\begin{tikzcd}
\ell^\infty(R) \arrow[r] \arrow[d, "\oplus\theta_n^{-1}"] & R^{\mathcal{U}} \arrow[d, "\theta^{-1}"] \\
\ell^\infty(p_nRp_n) \arrow[r] \arrow[d, "\oplus Adv_n"] & pR^{\mathcal{U}}p \arrow[d, "Adv"] \\
\ell^\infty(q_nRq_n) \arrow[r] \arrow[d, "\oplus\gamma_n"] & qR^{\mathcal{U}}q \arrow[d, "\gamma"] \\
\ell^\infty(R) \arrow[r] & R^{\mathcal{U}}
\end{tikzcd}$$

$\square$

Exercise 4.6. Use a similar idea to prove the implication (1) $\Rightarrow$ (3).

We recall that a projection $p \in M$ is called *minimal* if $pMp = \mathbb{C}1$. A von Neumann algebra without minimal projections is called *diffuse*.

Exercise 4.7. Let M be a diffuse von Neumann algebra with the following property: every pair of projections with the same trace are Murray-von Neumann equivalent. Show that M is factor. (Hint: show that every central projection is minimal).

Proof of the "only if" of Theorem 4.1. Let $[\pi]$ be an extreme point of $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ and $p \in \pi(M)' \cap R^{\mathcal{U}}$. Since

$$[\pi] = \tau(p)[\pi_p] + \tau(p^\perp)[\pi_{p^\perp}],$$

it follows that $[\pi_p] = [\pi]$, for all $p \in \pi(M)' \cap R^{\mathcal{U}}$, $p \neq 0$. By Lemma 4.3, it follows that two projections in $\pi(M)' \cap R^{\mathcal{U}}$ are Murray-von Neumann equivalent into $\pi(M)' \cap R^{\mathcal{U}}$ if and only if they have the same trace. Since $\pi(M)' \cap R^{\mathcal{U}}$ is diffuse, Exercise 4.7 completes the proof. $\square$

From Theorem 4.1 we obtain the following geometric reformulation of Popa's question.

Problem 4.8. (Geometric reformulation of Popa's question) Does $\mathbb{H}om(M, R^{\mathcal{U}})$ have extreme points?

This problem is still open. There are two obvious ways to try to attack it, leading to two related problems, whose positive solution would imply a positive solution of Popa's question.

[2]The notation Adu in the diagram stands for the conjugation by the unitary operator u.

(1) Since $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ is a bounded, closed and convex subset of a Banach space one cannot apply Krein-Milman's theorem and conclude existence of extreme points. Nevertheless, one can ask the question whether the Banach space into which $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ embeds is a dual Banach space. In this case, $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ would be compact in the weak*-topology and one could apply Krein-Milman's theorem.

Problem 4.9. Does $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ embed into a dual Banach space?

(2) The second approach is through a simple observation about geometry of Banach spaces. Recall that a Banach space B is called *strictly convex* if $b_1 \neq b_2$ and $||b_1|| = ||b_2|| = 1$ together imply that $||b_1 + b_2|| < 2$.

Exercise 4.10. Let B be a strictly convex Banach space and $C \subseteq B$ be a convex subset. Fix $c_0 \in C$ and assume that there is $c \in C$ such that $d(c_0, c)$ is maximized in C. Show that c is an extreme point of C.

Exercise 4.11. Let $M = W^*(X)$ be a singly generated II_1-factor which embeds into $R^{\mathcal{U}}$. Fix $[\pi_0] \in \mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$. Show that the function $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}}) \ni [\pi] \to d([\pi_0], [\pi])$ attains its maximum.

With a little bit more effort one can extend Exercise 4.11. Consequently, Exercises 4.10 and 4.11 together imply that if the convex-like structure on $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ were strictly convex, then Popa's question qould have affirmative answer.

Problem 4.12. *Is the convex-like structure on $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ strictly convex?*

The study of the extreme points of $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ is not interesting only in light of Popa's question, but also because it provides a method to distinguish II_1-factors. For instance, Brown proved in [1], Corollary 5.4, that rigidity of an $R^{\mathcal{U}}$-embeddable II_1-factor M with property (T) reflects on the rigidity of the set of the extreme points of its $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$, that turns out to be discrete. Property (T) for von Neumann algebras is a form of rigidity introduced by Connes and Jones in [4] and inspired to Kazhdan's property (T) for groups [8]. A simple way to define property (T) for von Neumann algebras is through the following definition.

Definition 4.13. A II_1-factor M with trace τ has property (T) if for all $\varepsilon > 0$, there exist $\delta > 0$ and a finite subset F of M such that for all τ-preserving unital completely positive maps $\phi : M \to M$, one has

$$\sup_{x \in F} ||\phi(x) - x||_2 \leq \delta \Rightarrow \sup_{Ball(M)} ||\phi(x) - x||_2 \leq \varepsilon.$$

The interpretation of property (T) as a form of rigidity should be clear: if a trace-preserving ucp map is closed to the identity on a finite set, then it is actually close to the identity on the whole von Neumann algebra.

Classical examples of factors with property (T) are the ones associated to $SL(n, \mathbb{Z})$, with $n \geq 3$.

Theorem 4.14. (N.P. Brown [1]**, Corollary 5.4)** *If M has property (T), then the set of extreme points of $\mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ is discrete.*

Proof. Popa proved in [10], Section 4.5, that for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $[\pi], [\rho] \in \mathbb{H}\mathrm{om}(M, R^{\mathcal{U}})$ are at distance $\leq \delta$, then there are projections $p \in \pi(M)' \cap R^{\mathcal{U}}$ and $q \in \rho(M) \cap R^{\mathcal{U}}$ and a partial isometry v such that $v^*v = p$, $vv^* = q$, $\tau(p) > 1 - \varepsilon$, and $v\pi(x)v^* = q\rho(x)$, for all $x \in M$. This implies that $p\pi$ and $q\rho$ are approximatively unitarily equivalent and consequently, by countable saturation, $[\pi_p] = [\rho_q]$.

Now fix $\varepsilon > 0$ assume that $[\pi]$ and $[\rho]$ are δ-close extreme points and take projections $p \in \pi(M)' \cap R^{\mathcal{U}}$ and $q \in \rho(M)' \cap R^{\mathcal{U}}$ such that $[\pi_p] = [\rho_q]$. Since $[\pi]$ and $[\rho]$ are extreme points, we can apply Theorem 4.1] and conclude that $[\pi] = [\pi_p]$ and $[\rho] = [\rho_q]$, that is, $[\pi] = [\rho]$. $\square$

Remark 4.15. Observe that Theorem 4.14 tells that the set of extreme points is discrete but, as far as we know, it might be empty. The problem of proving that extreme points actually exist for $R^{\mathcal{U}}$-embeddable II_1-factors with property (T) is open.

We conclude mentioning that also examples of factors with a continuous non-empty set of extreme points are also known in (see [1] Corollaries 6.10 and 6.11).

References

[1] Brown N.P., *Topological Dynamical Systems Associated to II_1-factors*, Adv. Math. 227 (4) (2011), 1665-1699

[2] Brown N.P., Capraro, V. *Groups associated to II_1-factors*, Journal of Functional Analysis 264 (2013) 493-507.

[3] Capraro V., Fritz T. *On the axiomatization of convex subsets of a Banach space*, Proceedings of the American Mathematical Society 141 (6) (2013), 2127-2135.

[4] Connes A, Jones VFR *Property (T) for von Neumann algebras.* Bull. London Math. Soc. 17 (1985), 57-62.

[5] T. Fritz, *Convex Spaces I: Definition and Examples*, Preprint, arXiv:0903.5522.

[6] Kadison V.R., Ringrose, J.R. *Fundamentals of the Theory of Operator Algebras, vol. 1*, in Pure and Applied Mathematics; Academic Press Inc., New York, New York, 1983.

[7] Kadison V.R., Ringrose, J.R. *Fundamentals of the Theory of Operator Algebras, vol. 2*, in Pure and Applied Mathematics; Academic Press Inc., Orlando, Florida, 1986.

[8] Kazhdan D. *Connection of the dual space of a group with the structure of its closed subgroup*, Functional Anal. and its Applic. 1 (1967), 63-65.

[9] Păunescu, L. *A convex structure on sofic embeddings*, Preprint (2011). Available at arXiv:1112.2195.

[10] Popa S. *Correspondences.* INCRES Preprint 1986.

[11] Sherman D. *Notes on automorphisms of ultrapowers of II_1-factors*, Studia Mathematica 195 (2009) 201-217.

[12] M.H. Stone, *Postulates of barycentric calculus*, Ann. Mat. Pura. Appl. 4, 29 (1949) 25-30.