1 Second class on sofic and hyperlinear groups

1.1 Logic for length groups

The logic for metric structures is a generalization of usual first order logic. It is a natural framework to study algebraic structures endowed with a nontrivial metric and their elementary properties (i.e. properties preserved by ultrapowers or equivalently expressible by formulas). In the following the particular instance of the logic for metric structures to describe and study groups endowed with a length functions as defined in ... is introduced.

A **term** $t(x_1, \ldots, x_n)$ in the language of length groups in the variables $x_1, \ldots, x_n$ is a **word** in the indeterminates $x_1, \ldots, x_n$, i.e. an expression of the form

$$x_{i_1}^{n_1} \ldots x_{i_l}^{n_l}$$

for $l \in \mathbb{N}$ and $n_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, l$. For example

$$xyx^{-1}y^{-1}$$

is a term in the variables x, y. The empty word will be denoted by 1. If G is a length group, $g_1, \ldots, g_m$ are elements of G, and $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a term in the variables $x_1, \ldots, x_n, y_1, \ldots, y_m$, then one can consider the term $t(x_1, \ldots, x_n, g_1, \ldots, g_m)$ with **parameters** from G, which is obtained from $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$ replacing formally y_i with g_i for $i = 1, 2, \ldots, m$. The evaluation t^G in a given length group G of a term t in the variables $x_1, \ldots, x_n$ (possibly with parameters from G) is the function from G^n to G defined by

$$(g_1, \ldots, g_n) \to t(g_1, \ldots, g_n)$$

where $t(g_1, \ldots, g_n)$ is the element of G obtained replacing in t every occurrence of x_i with g_i for $i = 1, 2, \ldots, n$. For example the evaluation in a length group G of the term $xyx^{-1}y^{-1}$ is the function from G^2 to G that associates to every pair (g, h) of elements of G their commutator $ghg^{-1}h^{-1}$. The evaluation of the empty word is the function on G constantly equal to 1_G.

A **basic formula** φ in the free variables $x_1, \ldots, x_n$ is an expression of the form

$$\ell(t(x_1, \ldots, x_n))$$

where $t(x_1, \ldots, x_n)$ is a term in the free variables $x_1, \ldots, x_n$. The evaluation φ^G of φ in a length group G is the function from G^n to $[0,1]$ defined by

$$(g_1, \ldots, g_n) \mapsto \ell_G\left(t^G(g_1, \ldots, g_n)\right)$$

where ℓ_G is the bi-invariant metric in G. For example

$$\ell\left(xyx^{-1}y^{-1}\right)$$

is a basic formula whose interpretation in a length group G is the function associating to a pair of elements of G the length of their commutator. This basic formula can be thought as measuring how much x and y commute. The evaluation at (g,h) of its interpretation in a length group G will be 0 if and only if g and h commute.

Finally a **formula** φ is any expression that can be obtained starting from basic formulas, composing with continuous functions from $[0,1]^n$ to $[0,1]$, taking infima and suprema over some variables. Continuous functions serve the role of *logical connectives* while infima and suprema are the *quantifiers*. Having this in mind, terminology from the usual first order logic carry over to this setting: A formula is quantifier free if it does not contain any quantifier; A variable x is bound if it is preced by a quantifier over x, i.e. $\sup_x$ or $\inf_x$, and free otherwise; a formula with not free variables is called a sentence. The interpretation of a formula in a length group G is defined in the obvious way by recursion on its complexity. For example

$$\sup_x \sup_y \ell\left(xyx^{-1}y^{-1}\right)$$

is a sentence, with bound variables x and y. Its evaluation in a length group G is the real number

$$\sup_{x\in G}\sup_{y\in G} \ell_G\left(xyx^{-1}y^{-1}\right) \in [0,1]\,.$$

This sentence can be thought as measuring how much the group G is abelian. Its interpretation in G is zero if and only if G is abelian. This example enlightens the fact that the possible truth values of a sentence (i.e. values of its evaluations in a length group) are all real numbers between 0 and 1. Moreover 0 can be thought as "true" while strictly postive real numbers as different degrees of "false". In this spirit we say that a sentence φ holds in G iff its interpretation in G is zero. Using this terminology we can assert for example that a length group G is abelian if and only if the formula

$$\sup_x \sup_y \ell\left(xyx^{-1}y^{-1}\right)$$

holds in G. Observe that if φ is a sentence, then $1-\varphi$ is a sentence such that φ holds in G if and only if the interpretation of $1-\varphi$ in G is 1. Thus $1-\varphi$ can be though as a sort of negation of the sentence φ. Another example of sentence is

$$\sup_x \min\left\{\left|\ell\left(x\right)-1\right|, \left|\ell\left(x\right)\right|\right\}.$$

Such sentence holds in a length group G if and only if the length function in G attains values in $\{0,1\}$, i.e. it is the trivial length function on G. It is worth at this point observing that for any sentence φ as defined in logic for length groups there is a corresponding formula φ_d in the usual first order (discrete) logic in the language of groups such that the evaluation of φ_d in a group G coincides with the evaluation of φ in G endowed with the trivial length function. For example the (metric) formula expressing that a group is abelian corresponds to the (discrete) formula

$$\forall x \forall y\ (xy = yx)\,.$$

Sentences in the language of length groups allow one to determine which properties of a length group are elementary. A property concerning length groups is elementary if there is a set Φ of sentences such that a length group has the given property if and only if G satisfies all the sentences in Φ. For example the property of being abelian is elementary, since a length group is abelian if and only if it satisfies the sentence $\sup_{x,y} \ell\left(xyx^{-1}y^{-1}\right)$. A class $\mathcal{C}$ of length groups will is *axiomatizable* if the property of belonging to $\mathcal{C}$ is elementary or, equivalently, there is a set $\Phi_{\mathcal{C}}$ of sentences such that a length group G belongs to $\mathcal{C}$ if and only if it satisfies all the sentences in $\Phi_{\mathcal{C}}$. The previous example shows that the class of abelian length groups is axiomatizable. Elementary properties and axiomatizable classes are tightly connected with the notion of ultraproduct of length groups.

Suppose that $(G_n)_{n\in\mathbb{N}}$ is a sequence of length groups and $\mathcal{U}$ is a free ultrafilter over $\mathbb{N}$. The *ultraproduct* $\prod_{\mathcal{U}} G_n$ of the sequence $(G_n)_{n\in\mathbb{N}}$ with respect to the ultrafilter $\mathcal{U}$ is by definition the length quotient of $\prod_n G_n$ with respect to the pseudo length function

$$\ell_{\mathcal{U}}\left((g_n)_{n\in\mathbb{N}}\right) = \lim_{n\to\mathcal{U}} \ell_{G_n}(g_n).$$

This is by definition the quotient of $\prod_n G_n$ with respect to the normal subgroup

$$N_{\ell_{\mathcal{U}}} = \left\{(g_n)_{n\in\mathbb{N}} \in \prod\nolimits_n G_n \,\middle|\, \lim_{n\to\mathcal{U}} \ell(g_n) = 0\right\}$$

endowed with the quotient length function. Observe that ultraproducts of the sequence $(S_n)_{n\in\mathbb{N}}$ of permutation groups endowed with the Hamming length function or of the sequence $(U_n)_{n\in\mathbb{N}}$ of unitary groups endowed with the Hilbert-Schmidt length function are particular cases of this definition. In the particular case when the sequence $(G_n)_{n\in\mathbb{N}}$ is constantly equal to a fixed length group G the ultraproduct $\prod_{\mathcal{U}} G_n$ is called *ultraprower* of G and denoted by $G^{\mathcal{U}}$. Observe that the function from G to $G^{\mathcal{U}}$ associating with $g \in G$ the element of $G^{\mathcal{U}}$ corresponding to the sequence constantly equal to g is a length presercing group homomorpism called *diagonal embeeding* of G into $G^{\mathcal{U}}$. The group G can be regarded as a subgroup of $G^{\mathcal{U}}$ via the diagonal embedding.

The ultraproduct construction behaves well with respect to interpretation of formulas. This is the content of a theorem proved in the setting of usual first order logic by Łos in 1955 (see [4]).

Theorem 1 (Łos for metric groups) *Suppose that $\varphi(x_1, \ldots, x_n)$ is a formula with free variables $x_1, \ldots, x_n$, $(G_n)_{n\in\mathbb{N}}$ is a sequence of length groups, and $\left(g_1^{(n)}\right)_{n\in\mathbb{N}}, \ldots, \left(g_k^{(n)}\right)_{n\in\mathbb{N}}$ are elements of $\prod_n G_n$. If $\mathcal{U}$ is a free ultrafilter over $\mathbb{N}$ and $g_1, \ldots, g_k$ are the elements of $\prod_{\mathcal{U}} G_n$ having $\left(g_1^{(n)}\right)_{n\in\mathbb{N}}, \ldots, \left(g_k^{(n)}\right)_{n\in\mathbb{N}}$ as representative sequences, then*

$$\varphi^{\prod_{\mathcal{U}} G_n}(g_1, \ldots, g_k) = \lim_{n\to\mathcal{U}} \varphi^{G_n}\left(g_1^{(n)}, \ldots, g_k^{(n)}\right).$$

In particular if φ is a sentence then

$$\varphi^{\prod_{\mathcal{U}} G_n} = \lim_{n \to \mathcal{U}} \varphi^{G_n}.$$

In particular a sentence φ has the same evaluation in a length group G and in any length ultrapower $G^{\mathcal{U}}$ of G.

Proof. This first statement is proved by induction on the complexity of the formula φ. ■

The ultraproduct construction in fact allows one to characterize elementary properties or, equivalently, axiomatizable classes of length groups. It is obvious that if $\mathcal{C}$ is an axiomatizable class then any length group isometrically isomorphic to an element of $\mathcal{C}$ belongs to $\mathcal{C}$. Moreover by Theorem 1 $\mathcal{C}$ is closed with respect to ultraproduct and ultraroot, i.e. ultraproducts of elements of $\mathcal{C}$ belong to $\mathcal{C}$ and if some ultrapower of a length group G is in $\mathcal{C}$ then G belongs to $\mathcal{C}$. It is a theorem due to Keisler in the setting of usual first order logic that these conditions are in fact sufficient to characterize axiomatizable classes (see [3]).

Theorem 2 (Keisler for metric groups) *A class $\mathcal{C}$ of length groups is axiomatizable if and only if it is closed with respect to isomorphisms, ultraproducts, and ultraroots.*

1.2 Model theoretic characterization of sofic and hyperlinear groups

In order to formulate the result of this sections, it is convenient to introduce the following terminology. Suppose that G, H are length groups, F is a subset of H, and ε is a nonnegative real number. A function $\Phi : H \to G$ is an (F, ε)-*approximate morphism* if for every $h_1, h_2 \in F$ the following holds:

- $\ell_G \left(\Phi(h_1 h_2) \Phi(h_2)^{-1} \Phi(h_1)^{-1} \right) < \varepsilon$;
- $\ell_G (\Phi(1_H)) < \varepsilon$;
- $|\ell_H(h_1) - \ell_G(\Phi(h_1))| < \varepsilon$.

An (H, ε)-approximate morphism will be simply called an ε-approximate morphism. A sequence $(G_n)_{n \in \mathbb{N}}$ is *approximately directed* if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ for all but finitely many $m \in \mathbb{N}$ there is an ε-approximate morphism from G_n to G_m. A *universal sentence* is a sentence of the form

$$\sup_{x_1} \sup_{x_2} \ldots \sup_{x_n} \varphi(x_1, \ldots, x_n)$$

where $\varphi(x_1, \ldots, x_n)$ is a quantifier-free formula. Analogously an existential sentence is a sentence of the form

$$\inf_{x_1} \inf_{x_2} \ldots \inf_{x_n} \varphi(x_1, \ldots, x_n)$$

where $\varphi(x_1, \ldots, x_n)$ is a quantifier-free formula. Interpretations of universal and existential formulas in approximately directed sequences have asymptotic values, as shown in Exercise 3.

Exercise 3 *Suppose that $(G_n)_{n\in\mathbb{N}}$ is an approximately directed sequence. Show that if φ is an existential sequence, then the sequence*

$$\left(\varphi^{G_n}\right)_{n\in\mathbb{N}}$$

of interpretations of φ in the elements of such sequence is convergent. Infer that the same is true if φ is a universal sequence.

It is obvious that if a length group H isometrically embeds in a length group G, then the interpretation of a universal sentence in H is smaller than or equal to the interpretation in H, and the interpretation of an existential sentence in H is greater than or equal to the interpetation in G. In particular any universal sentence that holds in G also holds in H, while an existential sentence that holds in H also holds in G. By Theorem 1 the same is true if H just embeds in an ultrapower $G^{\mathcal{U}}$ of G. Exercise 4 shows that if H is separable (i.e. it has a countable dense subset) this condition is also necessary.

Exercise 4 *Suppose that H and G are length groups, where H is separable. Show that the following statements are equivalent:*

1. *H isometrically embeds in the ultrapower $G^{\mathcal{U}}$ for some free ultrafilter $\mathcal{U}$ over $\mathbb{N}$;*
2. *H isometrically embeds in the ultrapower $G^{\mathcal{U}}$ for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$;*
3. *every universal sentence holding in G holds in H;*
4. *for every $F \subset H$ finite and every $\varepsilon > 0$ there is an (F, ε)-approximate morphism from H to G.*

Hint. To prove the equivalence of 3 and 4 and the equivalence of 4 and 5 oserve that if φ is a universal formula then $1 - \varphi$ is an existential formula (or more precisely it has the same interepretation as a suitable existential formula) and viceversa. To prove that 5 implies 6 write a suitable existential formula to build the map Φ. Finally to prove that 6 implies 1 use the hypothesis to build a sequence of approximate isometric embeddings from H to G that induce an isometric embedding from H to G. ■

The characterization of length groups isometrically embeddable into an ultrapower of a length group G can be generalized to a characterization of length groups embeddable in a ultraproduct of an approximately directed sequence of length groups $(G_n)_{n\in\mathbb{N}}$.

Exercise 5 *Suppose that H is a separable length group and that $(G_n)_{n\in\mathbb{N}}$ is an approximately directed sequence of length groups. Show that the following statements are equivalent:*

1. *H isometrically embeds in the ultraproduct $\prod_{\mathcal{U}} G_n$ for some free ultrafilter $\mathcal{U}$ over $\mathbb{N}$;*

2. *H isometrically embeds in the ultraproduct $\prod_{\mathcal{U}} G_n$ for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$;*

3. *every universal sentence φ such that $\lim_{n\to+\infty} \varphi^{G_n} = 0$ holds in H;*

4. *for every $F \subset H$ finite and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and an (F, ε)-approximate morphism from H to G_n.*

Recall that a countable discrete group Γ is sofic if, regarded as a length group endowed with the trivial discrete metric, it isometrically embeds into an ultraproduct $\prod_{\mathcal{U}} S_n$ of the sequence of permutation groups endowed with the Hamming length. The following characterization of countable sofic groups can be inferred from Exercise 5.

Proposition 6 *Suppose that Γ is a countable discrete group. The following statements are equivalent:*

1. *Γ is sofic;*

2. *if $q(x_1, \ldots, x_l)$ is a quantifier free formula such that*

$$\lim_{n\to+\infty} \left(\sup_{\sigma_1,\ldots,\sigma_l \in S_n} q(\sigma_1, \ldots, \sigma_n) \right) = 0$$

   then for every. $\gamma_1, \ldots, \gamma_l \in \Gamma$

$$q^{\Gamma}(\gamma_1, \ldots, \gamma_l) = 0.$$

An analgous characterization for hyperlinear groups can be obtained replacing the permutation groups S_n with the unitary groups U_n.

1.3 Classes of sofic and hyperlinear groups

A classicaly theorem of Cayley (see [1]) asserts that any finite group is isomorphic to a group of permutations with no fixed points on a finite set. To see this just let the group act on itself by left translation. This observation implies in particular that finite groups are sofic. This argument can be generalized to prove that amenable groups are sofic.

Recall that a countable discrete group Γ is *amenable* if for every finite subset F of Γ and for every $\varepsilon > 0$ there is a finite subset K of Γ such that K is (H, ε)-invariant, i.e.

$$|hK \bigtriangleup K| < \varepsilon |K|$$

for every $h \in H$. Suppose that Γ is amenable, F is a finite subset of Γ, and ε is a positive real number. Fix a finite $\left(H \cup H^{-1}, \varepsilon\right)$-invariant subset K of Γ. If $\gamma \in F$ then define

$$\sigma_\gamma(x) = \gamma x$$

when $x \in \gamma^{-1}K \cap K$ and extended σ_γ arbitrarily to a permutation of K. Observe that this defines an $(F, 2\varepsilon)$-approximate morphism from G to the group S_K of permutations of K endowed with the Hamming length (which is isomorphic to the S_n where $n = |K|$). This concludes the proof that Γ is sofic.

The classes of sofic and hyperlinear groups have nice closure properties. For example direct product of sofic groups are sofic. Recall that if σ_0, σ_1 are elements of S_n and S_m respectively then $\sigma_0 \otimes \sigma_1 \in S_{nm}$ is defined by

$$(\sigma_0 \otimes \sigma_1)(im + j) = \sigma_0(i)\, m + \sigma_1(j)$$

for $i \in n$ and $j \in m$. Suppose that Γ_0, Γ_1 are sofic groups and F_0, F_1 are finite subsets of Γ_0 and Γ_1 respectively. If Φ_0 is an (F_0, ε)-approximate morphism from Γ_0 to S_n and Φ_1 is an (F_1, ε)-approximate morphism from Γ_1 to S_m then the map

$$\Phi_0 \otimes \Phi_1 : \Gamma_0 \times \Gamma_1 \to S_{nm}$$

defined by

$$(\Phi_0 \otimes \Phi_1)(\gamma_0, \gamma_1) = \Phi_0(\gamma_0) \otimes \Phi_1(\gamma_1)$$

is an $(F_0 \times F_1, 2\varepsilon)$-approximate morphism. This observation is sufficient to conclude that a direct product of sofic groups is sofic.

If $\mathcal{C}$ is a class of (countable, discrete) groups, then a group Γ is locally embeddable into elements of $\mathcal{C}$ if for every finite subset F of $\Gamma \backslash \{1\}$ there is a function Φ from F to a group T that belongs to $\mathcal{C}$ such that Φ is nontrivial and preserves the operation on F, i.e. for every $g, h \in F$

$$\Phi(gh) = \Phi(g)\Phi(h)$$

and

$$\Phi(g) \neq 1.$$

The group Γ is residually in $\mathcal{C}$ if moreover Φ is required to be a surjective homomorphism.

It is clear from the very definition that soficity and hyperlineratity are local properties. This is made precise in Exercise 7.

Exercise 7 *Show that a group that is locally embeddable into sofic groups is sofic. The same is true for hyperlinear groups.*

Groups that are locally embeddable into finite groups were introduced and studied in [2] under the name of LEF groups. Since finite groups are sofic, Exercise 7 implies that LEF groups are sofic. In particular residually finite groups are sofic. More generally groups that are locally embeddable into amenable groups (and in particular residually amenable groups) are sofic.

It is a standard result in group theory that free groups are residually finite (see [6]). Therefore the previous discussion implies that free groups are sofic.

References

[1] A. Cayley, On the theory of groups as depending on the symbolic equation, Philosophical Magazine 7 (1854)

[2] E.I. Gordon and A.M. Vershik, Groups that are locally embeddable in the class of finite groups, St. Petersburg Math. J. 9 (1998), 49–67

[3] H. J. Keisler, Ultraproducts and elementary classes, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen (Indagationes mathematicae), Series A, 23, 1961, 477–495.

[4] J. Łoś, Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres. Mathematical interpretation of formal systems, pp. 98–113. North-Holland Publishing Co., Amsterdam.

[5] S. Shelah, Every two elementarily equivalent models have isomorphic ultrapowers, Israel Journal of Mathematics 10, 1971, 224–233.

[6] A. Kwiatkowska and V. Pestov, An introduction to hyperlinear and sofic groups, "Appalachian Set Theory 2006-2012", Edited by James Cummings, and Ernest Schimmerling (2012)