1 Higman group

Higman group is the finitely presented group with generators a_i for $i \in \mathbb{Z}/4\mathbb{Z}$ and relations

$$a_{i+1}a_ia_{i+1}^{-1} = a_i^2$$

for $i \in \mathbb{Z}/4\mathbb{Z}$.

It was defined by Higman as an example of a finitely presented not resudually finite group.

Consider the system of $\mathcal{R}$ relations $a_{i+1}a_ia_{i+1}^{-1} = a_{ii}^2$ for $i \in \mathbb{Z}/4\mathbb{Z}$.

Lemma 1 *If G is a group and $a_i \in G$ have finite order and satisfy exactly the system $\mathcal{R}$ then $a_i = 1$ for every i*

Proof. Suppose by contradiction that $a_i \neq 1$ for some i. Suppose that p is the minimum prime $p \geq 2$ dividing the order of one of the a_i. Suppose wlog that $p|ord(a_0)$. Observe that

$$a_1a_0a_1^{-1} = a_0^2$$

and by induction

$$a_1^na_0a_1^{-n} = a_0^{2^n}$$

It follows that $a_0^{2^{ord(a_1)}-1} = 1$. Thus $2^{ord(a_1)} - 1$ is a multiple of $ord(a_0)$ and hence of p. This implies that p is odd. Considering the element $2+p\mathbb{Z}$ in $\mathbb{Z}/p\mathbb{Z}^\times$ one sees that $ord(a_1)$ is a multiple of the order $2 + p\mathbb{Z}$ in $\mathbb{Z}/p\mathbb{Z}^\times$ which is a divisor of $p-1$ (different from 1). It follows that $ord(a_1)$ is divisible by a prime strictly smaller than p, agains our assumption. ■

Lemma 2 *If $n \in \mathbb{N}$ and $a_i \in GL_n(\mathbb{C})$ satisfy exactly the system $\mathcal{R}$ then $a_i = 1$ for every i*

Proof. It is enough to show that the a_i's have finite order. From

$$a_1a_0a_1^{-1} = a_0^2$$

it follows that a_0 and a_0^2 have the same eigenvalues and hence the eigenvalues of a_0 are roots of unity. Consider the Jordan normal form of a_0. The fact that the eigenvalues of a_0 are roots of unity implies that the entries of a_0^n grow at most polynomially in n. The same holds for the other a_i. The identity

$$a_1^na_0a_1^{-n} = a_0^{2^n}$$

implies that also the entries of $a_0^{2^n}$ grow at most polynomially in n. This implies that a_0 is diagonalizable and hence unitary of finite order. The same argument as before applies then. ■

The fact that the Higman group is nontrivial is nontrivial (and 4 generators are the minimal number that works).

If G_1, G_2 are groups groups, the free product $G_1 * G_2$ is the group having the elements of G_1 and G_2 as generators and having no relations beyond the ones already present in G_1 and G_2. Alternatively $G_1 * G_2$ can be described as the group of words $g_1 \dots g_n$ such that any two consecutive g_i and g_{i+1} do not belong to the same G_i. The free product is the coproduct in the category of groups. This means that if Γ is any other group and there are morphisms from G_i to Γ then there is a morphism from $G_1 * G_2$ to Γ extending the given morphisms.

We the notion of free product of groups relative to a common subgroup H. Suppose that G_1 and G_2 are groups with a common subgroup H. The free product of G_1 and G_2 amalgamated over H is denote by $G_1 *_H G_2$ and has the elements of G_1 and G_2 as generators and the relations coming from G_1 and G_2 and moreover relations identifying the copy of H in G_1 and the copy of H in G_2. The amalgamated free product has the following universal property: if morphisms from G_i to a group Γ agreeing on H are given then there is a morphism from $G_1 *_H G_2$ to Γ extending the given ones. Suppose that

$$G_1 = \bigcup_{s \in S_1} Hs$$

and

$$G_2 = \bigcup_{s \in S_2} Hs$$

are partitions into left right H cosets. Consider X to be the set of words

$$hs_1 \dots s_n$$

such that $h \in H$ and $s_i \in S_1 \cup S_2$ and no two consecutive s_i belong to the same S_i. One can define an operatation $\cdot$ on X such that $(X, \cdot) \simeq G_1 *_H G_2$.

Lemma 3 *If $g_i \in G_i$ for $i \in \{1,2\}$ generate in G_i a free subgroup disjoint from H, then $\{g_1, g_2\}$ generates in $G_1 *_H G_2$ a free group.*

Proof. One can consider as before partions

$$G_1 = \bigcup_{s \in S_1} Hs$$

and

$$G_2 = \bigcup_{s \in S_2} Hs$$

of G where S_1 and S_2 contain the free subgroups generated by g_1 and g_2. Then in the description of $G_1 *_H G_2$ as before any word in g_1 and g_2 generate a distinct element of $G_1 *_H G_2$. ■

Example 4 *Suppose that G_1 is the group generated by a_0, a_1 with relation*

$$a_1 a_0 a_1^{-1} = a_0^2$$

Observe that every element can be written uniquely as

$$a_1^{n_1} a_0^{n_0}$$

for $n_1, n_0 \in \mathbb{Z}$. In particular the cyclic subgroups generated by a_0 and a_1 respectively are free. Analogously consider the group G_2 generated by elements a_1, a_2 with relation

$$a_2 a_1 a_2^{-1} = a_1^2$$

Observe that the cyclic subgroups generated by a_1 and a_2 respectively are free. The amalgamated free product of G_1 and G_2 over $\langle a_1 \rangle$ is generated by a_0, a_1, a_2 with relations

$$\begin{aligned} a_1 a_0 a_1^{-1} &= a_0^2 \\ a_2 a_1 a_2^{-1} &= a_1^2 \end{aligned}$$

Moreover a_0 and a_2 generate a free group $\langle a_0, a_2 \rangle$ in $H_1 = G_1 *_H G_2$ (and a_1 generates a free group $\langle a_1 \rangle$).

Consider now the group H_2 generated by a_2, a_3, a_0 with relations

$$\begin{aligned} a_3 a_2 a_3^{-1} &= a_2^2 \\ a_0 a_3 a_0^{-1} &= a_3^2 \end{aligned}$$

where a_0, a_2 generate a free group $\langle a_0, a_2 \rangle$ (and a_3 genreates a free group $\langle a_3 \rangle$)

Define

$$H = H_1 *_{\langle a_0, a_2 \rangle} H_2$$

and observe that H has genrators a_i and relations

$$a_{i+1} a_i a_{i+1}^{-1} = a_i^2$$

for $i \in \mathbb{Z}/4\mathbb{Z}$.. Observe that a_1, a_3 generate a free group in H.

Define $[x, y] = x^{-1} y^{-1} x y$

Suppose that G is a finite group with length function ℓ_G. Say that ℓ is commutator contractive if

$$\ell([x, y]) \leq 4 \ell(x) \ell(y)$$

Observe that

$$d([g, h], [g, k]) \leq 4 \ell(g) d(h, k)$$

Observe that if H is a normal subgroup of G then $\ell_{G/H}$ on G/H defined by

$$\ell_{G/H}(xH) = \min_{h \in H} \ell(xh)$$

is a length function on G/H which is commutator contractive if ℓ is. Moreover the quotient function is a contraction.

Define

- $\min \ell_G = \min \ell_G [G \setminus \{1\}]$
- G_ε the (necessarily normal) subgroup generated by $\{g \in G \,|\, \ell(g) \leq \varepsilon\}$

- $gen(G) = \min\left\{\varepsilon > 0 \,\middle|\, G_{gen(G)} = G\right\}$ (G is generated by elements of length at most $gen(G)$)

Lemma 5 *If* $\min(\ell_G) < \frac{1}{4}$ *then* G *is abelian.*

Proof. Observe that if $\min(\ell_G) < \frac{1}{4}$ and $\ell(g) = \ell(h) = \min(\ell_G)$ then

$$\ell([g,h]) \leq 4\ell(g)\,\ell(h) < \min(\ell_G)$$

and hence $[g,h] = 1$. This shows that $G_{\min(\ell_G)}$ is abelian. ∎

Suppose in the following that G is a finite group with commutator contractive length function.

Lemma 6 *If* $G_\varepsilon \subsetneq G$ *then* $\min\left(\ell_{G/G_\varepsilon}\right) > \varepsilon$

Proof. Suppose by contradiction that $\min\left(\ell_{G/G_\varepsilon}\right) \leq \varepsilon$. Pick a nonidentity element hG_ε of G/G_ε such that $\ell_{G/G_\varepsilon}(hG_\varepsilon) \leq \varepsilon$. Thus for some $k \in G_\varepsilon$ one has $\ell(hk) \leq \varepsilon$ and hence $h = (hk)\,k^{-1} \in G_\varepsilon$ contradiction the fact that hG_ε is a nonidentity element of hG_ε. ∎

Lemma 7 *If* $gen(G) > \varepsilon$ *then*

$$gen(G/G_\varepsilon) = gen(G)$$

Proof. Since the quotient map is length-reducing

$$gen(G/G_\varepsilon) \leq gen(G)$$

Observe that G/G_ε is generated by elements of length at most $gen(G/G_\varepsilon)$. It follows that G is generated by elements of length at most $\max\{\varepsilon, gen(G/G_\varepsilon)\}$. This implies that

$$gen(G) \leq \max\{\varepsilon, gen(G/G_\varepsilon)\} \leq gen(G)$$

Since $\varepsilon < gen(G)$ it follows that

$$gen(G) = \max\{\varepsilon, gen(G/G_\varepsilon)\} = gen(G/G_\varepsilon)$$

∎

Observe that $gen(G_\varepsilon) \leq \varepsilon$. Moreover $\min(\ell_{G_\varepsilon}) = \min(\ell_G)$ if $\min(\ell_G) \leq \varepsilon$.

Lemma 8 *If* G *is a group with commutator contractive length function and* $a_i \in G$ *for* $i \in \mathbb{Z}/4\mathbb{Z}$ *satisfy up to* ε *such that* $4\varepsilon \leq \frac{3}{16}$ *the system* $\mathcal{R}$. *Either* $\ell(a_i) < 4\varepsilon$ *for every* $i \in \mathbb{Z}/4\mathbb{Z}$ *or* $\ell(a_i) \geq \frac{3}{16}$ *for every* i.

Proof. Suppose that $\ell(a_i) \geq 4\varepsilon$ for some i and wlog $i = 0$. Observe that since equation

$$[a_1, a_0] = a_0$$

is in $\mathcal{R}$ and the a_i satisfy $\mathcal{R}$ up to ε we have

$$\ell(a_0) - \varepsilon \leq \ell([a_1, a_0]) \leq 4\ell(a_0)\,\ell(a_1)$$

abd hence

$$\ell(a_1) \geq \frac{\ell(a_0) - \varepsilon}{4\ell(a_0)} \geq \frac{3}{16} \geq 4\varepsilon$$

The conclusion follows completing the cycle. ■

Lemma 9 *Suppose that G is a finite group with commutator contractive length function and $a_i \in G$ for $i \in \mathbb{Z}/4\mathbb{Z}$ satisfy up to ε such that $33\varepsilon < \frac{3}{16}$ (i.e. $\varepsilon < \frac{1}{176}$) the system $\mathcal{R}$. Then $\ell(a_i) < 4\varepsilon$ for every $i \in \mathbb{Z}/4\mathbb{Z}$.*

Proof. If $\min(\ell_G) \geq \varepsilon$ then the a_i satisfy $\mathcal{R}$ exactly and hence $a_i = 1$ since the system $\mathcal{R}$ has no nontrivial exact realization in a finite group. Suppose that $\min(\ell_G) < \varepsilon$. Assume by contradiction that $\ell(a_i) \geq 4\varepsilon$ for some i. Define $n = |G|$. Suppose wlog that $\min(\ell_G) < \varepsilon$ is maximal among all groups G of order at most n having elements a_i satisfying $\mathcal{R}$ up to ε such that $\ell(a_i) \geq 4\varepsilon$ for some i. Assume first that $G_{\min(\ell_G)} \neq G$. Thus

$$\min\left(\ell_{G/G_{\min(\ell_G)}}\right) > \min(\ell_G)$$

Observe that the $a_i G_{\min(\ell_G)}$ in $G/G_{\min(\ell_G)}$ satisfy $\mathcal{R}$ up to ε. By maximality of G one of the conditions must to be violated, and hence $\ell\left(a_i G_{\delta(G)}\right) < 4\varepsilon$ for every i. Thus there are $\widetilde{a}_i \in G_{\min(\ell_G)}$ for every i such that

$$d(a_i, \widetilde{a}_i) = \ell\left(a_i \widetilde{a}_i^{-1}\right) < 4\varepsilon$$

Obviously the same is true if $G_{\min(\ell_G)} = G$. Since $\min(\ell_G) \leq \varepsilon < \frac{1}{4}$ we know that $G_{\min(\ell_G)}$ is abelian. We have

$$\begin{aligned} d([a_{i+1}, a_i], [a_{i+1}, \widetilde{a}_i]) &\leq 4\ell(a_{i+1})\, d(a_i, \widetilde{a}_i) \\ &< 16\varepsilon \ell(a_{i+1}) \end{aligned}$$

and

$$\begin{aligned} \ell([a_{i+1}, \widetilde{a}_i]) &= d([\widetilde{a}_{i+1}, \widetilde{a}_i], [a_{i+1}, \widetilde{a}_i]) \\ &\leq 4d(\widetilde{a}_{i+1}, a_{i+1})\, \ell(\widetilde{a}_i) \\ &< 16\varepsilon \ell(\widetilde{a}_i) \end{aligned}$$

Therefore

$$\begin{array}{rcl} \ell(a_i) & \leq & \ell([a_{i+1}, a_i]) + \varepsilon \\ & < & \ell([a_{i+1}, \widetilde{a_i}]) + 16\varepsilon\ell(a_{i+1}) + \varepsilon \\ & < & (16\ell(\widetilde{a}_i) + 16\ell(a_{i+1}) + 1)\,\varepsilon \\ & \leq & 33\varepsilon \\ & < & \dfrac{3}{16} \end{array}$$

It follows from the previous lemma that $\ell(a_i) < 4\varepsilon$. ■

Suppose that a_i for $i \in \mathbb{Z}/4\mathbb{Z}$ belong to a finite group with commutator contractive length function. I claim that

$$\max_{i\in\mathbb{Z}/4\mathbb{Z}} d(a_i, 1) \leq 176 \max_{i\in\mathbb{Z}/4\mathbb{Z}} d([a_{i+1}, a_i], a_i)$$

In fact this is clear if $\max_{i\in\mathbb{Z}/4\mathbb{Z}} d([a_{i+1}, a_i], a_i) \geq \frac{1}{176}$. Suppose that $\max_{i\in\mathbb{Z}/4\mathbb{Z}} d([a_{i+1}, a_i], a_i) < \frac{1}{176}$. It follows from the previous lemma that

$$\max_{i\in\mathbb{Z}/4\mathbb{Z}} d(a_i, 1) \leq 4 \max_{i\in\mathbb{Z}/4\mathbb{Z}} d([a_{i+1}, a_i], a_i) \leq 176 \max_{i\in\mathbb{Z}/4\mathbb{Z}} d([a_{i+1}, a_i], a_i)$$

Consider the metric formula φ defined by

$$\sup_{x_0,x_1,x_2,x_3} \max\left\{\max_{i\in\mathbb{Z}/4\mathbb{Z}} d(x_i, 1) - 176 \max_{i\in\mathbb{Z}/4\mathbb{Z}} d([a_{i+1}, a_i], a_i), 0\right\}$$

and observe that $\varphi^G = 0$ for every finite group G endwed with commutator contractive length function. On the other hand $\varphi^H = 1$ where H is the Higman group.