1 Kaplanski's finiteness conjecture

Recall that Kaplanski's finiteness conjecture for a countable discrete group Γ asserts that if K is a field, then the group algebra $K\Gamma$ is directly finite. This means that if a, b are elements of $K\Gamma$ such that $ab = 1$ then also $ba = 1$. Equivalently if b is a multiplicative right inverse of a, then b is also a multiplicative left inverse of a. This conjecture has been confirmed when Γ is a sofic group by Elek and Szabo in [1].

Definition 1 *If R is a ring, then a rank function on R is a function $N : R \to [0,1]$ such that*

- $N(1) = 1$;
- $N(x) = 0$ iff $x = 0$;
- $N(xy) \leq \min\{N(x), N(y)\}$;
- $N(x+y) \leq N(x) + N(y)$.

If N is a rank function on R then

$$d(x,y) = N(x-y)$$

defines a metric that makes the function $x \mapsto x + a$ isometric and the functions $x \mapsto xa$ and $x \mapsto ax$ contractive for every $a \in R$. A ring endowed with rank function is called a *rank ring*.

In the context of rank rings, a *term* in the variables $x_1, \ldots, x_n$ is just a polynomial in the indeterminates $x_1, \ldots, x_n$. *Formulas*, *sentences* and their *interpretation* in a rank ring can then be defined starting from terms analogously as in the case of metric groups. Also the notion of elementary property and axiomatizable class carry over without change.

If R_n is a sequence of rank rings and $\mathcal{U}$ is an ultrafilter over $\mathbb{N}$, the ultraproduct $\prod_{\mathcal{U}} R_n$ is the quotient of the product ring $\prod_n R_n$ by the ideal

$$I_{\mathcal{U}} = \left\{ (x_n) \in \prod\nolimits_n R_n \,\middle|\, \lim_{n \to \mathcal{U}} N_n(x_n) = 0 \right\}.$$

The function

$$N_{\mathcal{U}}(x_n) = \lim_{n \to \mathcal{U}} N_n(x_n)$$

induces in the quotient a rank function, making $\prod_{\mathcal{U}} R_n$ a rank ring. Los theorem for ultraproducts can be proved in this context in a way analogous to the case of length groups.

Theorem 2 (Los for ranked rings) *If $(R_n)_{n \in \mathbb{N}}$ is a sequence of rank rings and φ is a sentence then*

$$\varphi^R = \lim_{n \to \mathcal{U}} \varphi^{R_n}.$$

A rank ring R is such that for every $x, y \in R$

$$N(xy - 1) = N(yx - 1)$$

is called a finite rank ring. Observe that clearly a finite rank ring is a directly finite ring. Moreover the class of finite rank rings is axiomatizable by the formula

$$\sup_x \sup_y |N(xy-1) - N(yx-1)|.$$

It follows that an ultraproduct of finite rank rings is a finite rank ring and in paricular a directly finite ring.

Exercise 3 *Suppose that R is a finite rank ring. Define*

$$\ell(x) = N(x-1)$$

for $x \in R$. Show that ℓ is a length funtion on the multiplicative group $R^\times$ of invertible elements of R.

Suppose in the following that K is a field.

Exercise 4 *Denote by $M_n(K)$ the ring of $n \times n$ matrices with coefficients in K. If $x \in M_n(K)$ define $\rho(x)$ to be the dimension of the range of x regarded as an operator on K^n. This is the usual notion of rank of an $n \times n$ matrix. Define*

$$N_n(x) = \frac{1}{n}\rho(x).$$

Prove that N_n is a rank function on $M_n(K)$ as in Definition 1. Show that $M_n(K)$ endowed with the rank N_n is a finite rank ring.

More generally a von Neumann algebra enowed with a faithful normalized trace τ can be regarded as a finite rank ring with respect to the rank

$$N_\tau(x) = \sup\{\tau(p) \mid p \text{ is a projection and } xp = x\}.$$

If $\mathcal{U}$ is an ultrafilter over $\mathbb{N}$, then $\prod_{\mathcal{U}} M_n(K)$ denotes the ultraproduct with respect to $\mathcal{U}$ of the matrix rings $M_n(K)$ regarded as rank rings. By Exercise 4 and Los theorem on ultraproducts $\prod_{\mathcal{U}} M_n(K)$ is a finite rank ring (and in particular a directly finite ring).

Exercise 5 *If $\sigma \in S_n$ denote as in by $P_\sigma \in M_n(K)$ the permutation matrix associated with σ. Prove that*

$$N_n(P_\sigma - I) \le \ell_{S_n}(\sigma)$$

where ℓ_{S_n} is the Hamming length function.

Hint. Define $c(\sigma)$ the number of cycles of σ (including fixed points). Show that
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$$N_n\left(P_\sigma - I\right) = 1 - \frac{c(\sigma)}{n}$$

by induction on the number of cycles.

Exercise 6 *Suppose that $\sigma_0, \ldots, \sigma_{l-1} \in S_n$ and $\lambda_0, \ldots, \lambda_{l-1} \in K \backslash \{0\}$. Define*

$$\varepsilon = \min\left\{1 - d\left(\sigma_\alpha, 1_{S_n}\right) | \alpha \in l\right\}.$$

Prove that

$$N_n\left(\sum_{\alpha \in l} \lambda_\alpha P_{\sigma_\alpha}\right) \geq \frac{1 - \varepsilon l}{l^2}.$$

Hint. Define X to be a maximal subset of n such that for $i, j \in X$ and $\alpha, \beta \in l$ such that $(i, \alpha) \neq (j, \beta)$ one has

$$\sigma_\alpha(i) \neq \sigma_\beta(j).$$

Denote by $\{e_i \,|\, i \in n\}$ the canonical basis of K^n. Observe that if $x \in \mathit{span}\{e_i \,|\, i \in X\}$ then

$$\sum_{\alpha \in l} \lambda_\alpha P_{\sigma_\alpha}(x) \neq 0.$$

Infer that

$$N_n\left(\sum_{\alpha \in l} \lambda_\alpha P_{\sigma_\alpha}\right) \geq \frac{|X|}{n}.$$

By maximality of X for every $i \in n$ there is $j \in X$ and $\alpha, \beta \in l$ such that

$$i = \sigma_\alpha^{-1} \sigma_\beta(j)$$

When α, β vary in l and j varies in X the expression

$$\sigma_\alpha^{-1} \sigma_\beta(j)$$

attains at most $\varepsilon n l + l^2 |X|$ values. Infer that

$$\frac{|E|}{m} \geq \frac{1 - \varepsilon l}{l^2}$$

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Exercise 7 *Define using Exercise 5 a ring morphism $\Psi : K\left(\prod_{\mathcal{U}} S_n\right) \to \prod_{\mathcal{U}} M_n(K)$. Prove using Exercise 6 that if $x_1, \ldots, x_l \in \prod_{\mathcal{U}} S_n$ are such that $d(x_i, 1) = 1$ for $i = 1, 2, \ldots, l$ and $\lambda_1, \ldots, \lambda_l \in K$ then*

$$N\left(\lambda_1 x_1 + \cdots + \lambda_l x_l\right) \geq \frac{1}{l}.$$

One can now easily prove that a sofic group Γ satisfies Kaplanski's finiteness conjecture. In fact if Γ is sofic then Γ embeds into $\prod_{\mathcal{U}} S_n$ in such a way that the distance of any element in the range of $\Gamma \backslash \{1\}$ from the identity is 1. This induces a ring morphism from $K\Gamma$ into $K\left(\prod_{\mathcal{U}} S_n\right)$. Compositing with the ring morphism described in Exercise 7 one obtains a ring morphis from $K\Gamma$ into $\prod_{\mathcal{U}} M_n(K)$ that is one to one by the second statement in Exercise 7. This shows that $K\Gamma$ is isomorphic to a subring of a directly finite ring and, in particular, directly finite.

References

[1] Elek, Szabo, Sofic groups and direct finiteness