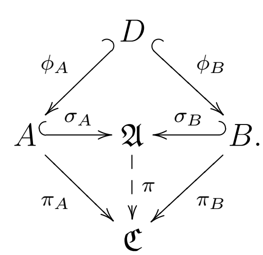
ON EMBEDDINGS OF FULL AMALGAMATED FREE PRODUCT C*–ALGEBRAS

SCOTT ARMSTRONG, KEN DYKEMA, RUY EXEL AND HANFENG LI

Abstract. We examine the question of when the $*$–homomorphism $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ of full amalgamated free product C*–algebras, arising from compatible inclusions of C*–algebras $A \subseteq \widetilde{A}$, $B \subseteq \widetilde{B}$ and $D \subseteq \widetilde{D}$, is an embedding. Results giving sufficient conditions for λ to be injective, as well of classes of examples where λ fails to be injective, are obtained. As an application, we give necessary and sufficient condition for the full amalgamated free product of finite dimensional C*–algebras to be residually finite dimensional.

1. Introduction

Given C*–algebras A, B and D with injective $*$–homomorphisms $\phi_A : D \to A$ and $\phi_B : D \to B$, the corresponding full amalgamated free product C*–algebra (see [1] or [9, Chapter 5]) is the C*–algebra $\mathfrak{A}$, equipped with injective $*$–homomorphisms $\sigma_A : A \to \mathfrak{A}$ and $\sigma_B : B \to \mathfrak{A}$ such that $\sigma_A \circ \phi_A = \sigma_B \circ \phi_B$, such that $\mathfrak{A}$ is generated by $\sigma_A(A) \cup \sigma_B(B)$ and satisfying the universal property that whenever $\mathfrak{C}$ is a C*–algebra and $\pi_A : A \to \mathfrak{C}$ and $\pi_B : B \to \mathfrak{C}$ are $*$–homomorphisms satisfying $\pi_A \circ \phi_A = \pi_B \circ \phi_B$, there is a $*$–homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi \circ \sigma_A = \pi_A$ and $\pi \circ \sigma_B = \pi_B$. This situation is illustrated by the following commuting diagram:



(1)

The full amalgamated free product C*–algebra $\mathfrak{A}$ is commonly denoted by $A *_D B$, although this notation hides the dependence of $\mathfrak{A}$ on the embeddings ϕ_A and ϕ_B.

Date: 17 March, 2003.

The first author was supported in part by an REU stipend from the NSF. The second author was supported in part by NSF grant DMS–0070558. The third author was supported in part by CNPq grant 303968/85–0. The fourth author was supported jointly by the Mathematics Department of the University of Toronto and NSERC Grant 8864-02 of George A. Elliott.

Question 1.1. *Let D, A, B, $\widetilde{D}$, $\widetilde{A}$ and $\widetilde{B}$ be C*–algebras and suppose there are injective $*$–homomorphisms making the following diagram commute:*

$$\begin{array}{ccccc} \widetilde{A} & \xleftarrow{\phi_{\widetilde{A}}} & \widetilde{D} & \xrightarrow{\phi_{\widetilde{B}}} & \widetilde{B}. \\ \uparrow{\scriptstyle \lambda_A} & & \uparrow{\scriptstyle \lambda_D} & & \uparrow{\scriptstyle \lambda_B} \\ A & \xleftarrow{\phi_A} & D & \xrightarrow{\phi_B} & B \end{array}$$

Let $A*_D B$ and $\widetilde{A}*_{\widetilde{D}}\widetilde{B}$ be the corresponding full amalgamated free product C–algebras and let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the $*$–homomorphism arising from λ_A and λ_B via the universal property. When is λ injective?*

We prove in §2 that λ is injective when either (i) $D = \widetilde{D}$, (or more precisely, when the $*$–homomorphism λ_D is surjective), or (ii) there are conditional expectations $E_A : \widetilde{A} \to A$ and $E_B : \widetilde{B} \to B$ that send $\widetilde{D}$ onto D and agree on $\widetilde{D}$. Injectivity in the case $D = \widetilde{D}$ was previously proved by G.K. Pedersen [10]. (Moreover, earlier results of F. Boca [4] imply that the map λ is injective when $D = \widetilde{D}$ and when there are conditional expectations

$$\widetilde{A} \overset{E_A^{\widetilde{A}}}{\to} A \overset{E_D^A}{\to} D \overset{E_D^B}{\leftarrow} B \overset{E_B^{\widetilde{B}}}{\leftarrow} \widetilde{B};$$

an argument for the case $D = \widetilde{D} = \mathbf{C}$, which uses Boca's results, is outlined in [3, 4.7].) However, we include our proof because it is different from that found in [10] and because it contains the main idea of our proof of injectivity in case (ii). In §3, we consider some general conditions and give some concrete examples when λ fails to be injective. Finally, in §4, we apply this embedding result to extend a result from [5] about residual finite dimensionality of full amalgamated free products of finite dimensional C*–algebras.

2. Embeddings of full free products

The following result is of course well known. We include a proof for completeness.

Lemma 2.1. *Let A be a C*–subalgebra of a C*–algebra $\widetilde{A}$ and let $\pi : A \to B(\mathcal{H})$ be a $*$–representation. Then there is a Hilbert space $\mathcal{K}$ and a $*$–representation $\tilde{\pi} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ such that*

$$\tilde{\pi}(a)(h \oplus 0) = (\pi(a)h) \oplus 0, \qquad (a \in A,\, h \in \mathcal{H}). \tag{2}$$

Proof. Since in general π is a direct sum of cyclic representations, we may without loss of generality assume π is a cyclic representation with cyclic vector ξ. Let ϕ be the vector state $\phi(\cdot) = \langle \pi(\cdot)\xi, \xi \rangle$ of A. Then $\mathcal{H}$ is identified with $L^2(A, \phi)$ and π is the associated GNS representation. Let $\tilde{\phi}$ be an extension of ϕ to a state of $\widetilde{A}$ and let $\widetilde{\mathcal{H}} = L^2(\widetilde{A}, \tilde{\phi})$. Then the inclusion $A \hookrightarrow \widetilde{A}$ gives rise to an isometry $\mathcal{H} \to \widetilde{\mathcal{H}}$, and we may thus write $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{K}$ for a Hilbert space $\mathcal{K}$. If $\tilde{\pi} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ is the GNS representation associated to $\tilde{\phi}$, then (2) holds. ∎

The following result was first proved by G.K. Pedersen [10, Thm. 4.2]. We offer a new proof, which is perhaps more elementary. This proof contains essentially the same idea as our proof of Proposition 2.4 below.

Proposition 2.2. *Let*

$$\widetilde{A} \supseteq A \supseteq D \subseteq B \subseteq \widetilde{B}$$

be inclusions of C^*–algebras and let $A *_D B$ and $\widetilde{A} *_D \widetilde{B}$ be the corresponding full amalgamated free product C^*–algebras. Let $\lambda : A *_D B \to \widetilde{A} *_D \widetilde{B}$ be the $*$–homomorphism arising via the universal property from the inclusions $A \hookrightarrow \widetilde{A}$ and $B \hookrightarrow \widetilde{B}$. Then λ is injective.

Proof. Let $\pi : A *_D B \to B(\mathcal{H})$ be a faithful $*$–homomorphism. We will find a Hilbert space $\mathcal{K}$ and a $*$–homomorphism $\tilde{\pi} : \widetilde{A} *_D \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$ such that

$$\tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0, \qquad (x \in A *_D B,\ h \in \mathcal{H}). \tag{3}$$

This will imply λ is injective.

Let $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ be the $*$–representations obtained by composing π with the inclusions $A \hookrightarrow A *_D B$ and $B \hookrightarrow A *_D B$. Let

$$\sigma_{A,0} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K}_{A,0}),$$
$$\sigma_{B,0} : \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K}_{B,0})$$

be $*$–representations obtained from Lemma 2.1 such that

$$\sigma_{A,0}(a)(h \oplus 0) = (\pi_A(a)h) \oplus 0, \qquad (a \in A,\ h \in \mathcal{H}),$$

and similarly with A replaced by B. Note that $0 \oplus \mathcal{K}_{A,0}$ is reducing for $\sigma_{A,0}(D)$. Using Lemma 2.1, we find Hilbert spaces $\mathcal{K}_{B,1}$ and $\mathcal{K}_{A,1}$ and $*$–representations

$$\sigma_{B,1} : \widetilde{B} \to B(\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1})$$
$$\sigma_{A,1} : \widetilde{A} \to B(\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1})$$

such that

$$\sigma_{B,1}(d)(k \oplus 0) = \sigma_{A,0}(d)(0 \oplus k), \qquad (d \in D,\ k \in \mathcal{K}_{A,0}),$$
$$\sigma_{A,1}(d)(k \oplus 0) = \sigma_{B,0}(d)(0 \oplus k), \qquad (d \in D,\ k \in \mathcal{K}_{B,0}).$$

Proceeding recursively, for every integer $n \geq 2$ we find $*$–representations

$$\sigma_{B,n} : \widetilde{B} \to B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}),$$
$$\sigma_{A,n} : \widetilde{A} \to B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n})$$

such that

$$\sigma_{B,n}(d)(k \oplus 0) = \sigma_{A,n-1}(d)(0 \oplus k), \qquad (d \in D,\ k \in \mathcal{K}_{A,n-1}),$$
$$\sigma_{A,n}(d)(k \oplus 0) = \sigma_{B,n-1}(d)(0 \oplus k), \qquad (d \in D,\ k \in \mathcal{K}_{B,n-1}).$$

We now define the Hilbert spaces

$$\begin{aligned} \widetilde{\mathcal{H}}_A &= \overbrace{\mathcal{H}\oplus\mathcal{K}_{A,0}}^{\sigma_{A,0}}\oplus\overbrace{\mathcal{K}_{B,0}\oplus\mathcal{K}_{A,1}}^{\sigma_{A,1}}\oplus\overbrace{\mathcal{K}_{B,1}\oplus\mathcal{K}_{A,2}}^{\sigma_{A,2}}\oplus\cdots, \\ \widetilde{\mathcal{H}}_B &= \underbrace{\mathcal{H}\oplus\mathcal{K}_{B,0}}_{\sigma_{B,0}}\oplus\underbrace{\mathcal{K}_{A,0}\oplus\mathcal{K}_{B,1}}_{\sigma_{B,1}}\oplus\underbrace{\mathcal{K}_{A,1}\oplus\mathcal{K}_{B,2}}_{\sigma_{B,2}}\oplus\cdots, \end{aligned} \tag{4}$$

where the brackets indicate where the constructed representations act, and we let $\sigma_{\widetilde{A}} : \widetilde{A} \to B(\widetilde{\mathcal{H}}_A)$ and $\sigma_{\widetilde{B}} : \widetilde{B} \to B(\widetilde{\mathcal{H}}_B)$ be the $*$–representations

$$\begin{aligned} \sigma_{\widetilde{A}} &= \sigma_{A,0}\oplus\sigma_{A,1}\oplus\sigma_{A,2}\oplus\cdots, \\ \sigma_{\widetilde{B}} &= \sigma_{B,0}\oplus\sigma_{B,1}\oplus\sigma_{B,2}\oplus\cdots, \end{aligned}$$

where the summands act as indicated by brackets in (4). Consider the unitary $U : \widetilde{\mathcal{H}}_A \to \widetilde{\mathcal{H}}_B$ mapping the summands in $\widetilde{\mathcal{H}}_A$ identically to the corresponding summands in $\widetilde{\mathcal{H}}_B$ as indicated by the arrows below:

$$\begin{array}{ccccccccccccccc} \widetilde{\mathcal{H}}_A & = & \mathcal{H} & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,2} & \oplus & \cdots \\ {\scriptstyle U}\downarrow & & \downarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow \\ \widetilde{\mathcal{H}}_B & = & \mathcal{H} & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,2} & \oplus & \cdots. \end{array}$$

Let $\mathcal{K} = \mathcal{K}_{A,0}\oplus\mathcal{K}_{B,0}\oplus\mathcal{K}_{A,1}\oplus\mathcal{K}_{B,1}\oplus\cdots$ and identify $\mathcal{H}\oplus\mathcal{K}$ with $\widetilde{\mathcal{H}}_A$. Then we have the $*$–representations $\tilde{\pi}_{\widetilde{A}} = \sigma_{\widetilde{A}} : \widetilde{A} \to B(\mathcal{H}\oplus\mathcal{K})$ and $\tilde{\pi}_{\widetilde{B}} : \widetilde{B} \to B(\mathcal{H}\oplus\mathcal{K})$, the latter defined by $\tilde{\pi}_{\widetilde{B}}(\cdot) = U^*\sigma_{\widetilde{B}}(\cdot)U$. By construction, the restrictions of $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ to D agree, and we have

$$\begin{aligned} \tilde{\pi}_{\widetilde{A}}(a)(h\oplus 0) &= (\pi_A(a)h)\oplus 0, \qquad (a\in A,\ h\in\mathcal{H}), \\ \tilde{\pi}_{\widetilde{B}}(b)(h\oplus 0) &= (\pi_B(b)h)\oplus 0, \qquad (b\in B,\ h\in\mathcal{H}). \end{aligned}$$

Letting $\tilde{\pi} : \widetilde{A} *_D \widetilde{B} \to B(\mathcal{H}\oplus\mathcal{K})$ be the $*$–homomorphism obtained from $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ via the universal property, we have that (3) holds. □

For a C*–algebra A, unital or not, let A^u denote the unitization of A. Thus, as a vector space, $A^u = A \oplus \mathbf{C}$ with multiplication defined by $(a,\mu)\cdot(a',\mu') = (aa'+\mu a+\mu' a, \mu\mu')$. We identify A with the ideal $A\oplus 0$ of A^u, which has codimension 1.

Lemma 2.3. *Let $A \supseteq D \subseteq B$ be inclusions of C*–algebras. Consider the unitizations and corresponding inclusions*

$$\begin{array}{ccccc} A^u & \hookleftarrow & D^u & \hookrightarrow & B^u \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B. \end{array}$$

Let $\lambda : A *_D B \to A^u *_{D^u} B^u$ be the resulting $*$–homomorphism between full amalgamated free products. Then there is an isomorphism $\pi : A^u *_{D^u} B^u \to (A *_D B)^u$ such *that* $\pi \circ \lambda : A *_D B \to (A *_D B)^u$ *is the canonical embedding arising in the definition of the unitization.*

Proof. Since any $*$–representations of A and B that agree on D extend to $*$–representations of A^u and B^u that agree on D^u, the $*$–homomorphism λ is injective. Let $e \in A^u *_{D^u} B^u$ be the unit of A^u, which is of course identified with the units of B^u and D^u. Clearly, $A^u *_{D^u} B^u$ is generated by the image of λ together with e. One easily sees

$$(\lambda(x) + \mu e)(\lambda(x') + \mu' e) = \lambda(xx') + \mu\lambda(x') + \mu'\lambda(x) + \mu\mu' e.$$

Moreover, if $\rho : A^u *_{D^u} B^u \to \mathbf{C}$ is the $*$–homomorphism arising from the unital $*$–homomorphisms $A^u \to \mathbf{C}$ and $B^u \to \mathbf{C}$, then $\rho(e) = 1$ and $\lambda(A *_D B) \subseteq \ker \rho$. Hence $\lambda(A *_D B)$ has codimension 1 in $A^u *_{D^u} B^u$. Now π can be defined by $\pi(\lambda(x) + \mu e) = (x, \mu)$. □

Proposition 2.4. *Suppose*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array} \tag{5}$$

is a commuting diagram of inclusions of C^**–algebras. Let* $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ *be the resulting* $*$*–homomorphism of full free product* C^**–algebras. Suppose there are conditional expectations* $E_A : \widetilde{A} \to A$*,* $E_D : \widetilde{D} \to D$ *and* $E_B : \widetilde{B} \to B$ *onto* A*,* D *and* B*, respectively, such that the diagram*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \downarrow E_A & & \downarrow E_D & & \downarrow E_B \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array} \tag{6}$$

commutes. Then λ *is injective.*

Proof. By appealing to Lemma 2.3, we may without loss of generality assume all the algebras and $*$–homomorphisms in (5) are unital. Let $\pi : A *_D B \to B(\mathcal{H})$ be a faithful, unital $*$–representation. As in the proof of Proposition 2.2, in order to show λ is injective, we will find a Hilbert space $\mathcal{K}$ and a $*$–homomorphism $\tilde{\pi} : \widetilde{A} *_{\widetilde{D}} \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$ such that

$$\tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0, \qquad (x \in A *_D B,\ h \in \mathcal{H}). \tag{7}$$

Let $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ be the $*$–representations obtained by composing π with the inclusions $A \hookrightarrow A *_D B$ and $B \hookrightarrow A *_D B$, and let $\pi_D : D \to B(\mathcal{H})$ be their common restriction to D. Consider the canonical left action of $\widetilde{D}$ on the right Hilbert D–module $L^2(\widetilde{D}, E_D)$, which is obtained from $\widetilde{D}$ by separation and completion with respect to the D–valued inner product $\langle \tilde{d}_1, \tilde{d}_2 \rangle = E_D(\tilde{d}_1^* \tilde{d}_2)$. Consider the Hilbert space $L^2(\widetilde{D}, E_D) \otimes_D \mathcal{H}$, where the left action of D on $\mathcal{H}$ is via π_D. Since π_D is unital, $\mathcal{H}$ embeds as a subspace, and we can write

$$L^2(\widetilde{D}, E_D) \otimes_D \mathcal{H} = \mathcal{H} \oplus \mathcal{K}_D. \tag{8}$$

Consider the left action of $\widetilde{D}$ on the Hilbert space $\mathcal{H} \oplus \mathcal{K}_D$. The subspace $\mathcal{H}$ is reducing for the restriction of σ_D to D, and we have $\sigma_D(d)(h \oplus 0) = (\pi_D(d)h) \oplus 0$ for every $d \in D$ and $h \in \mathcal{H}$.

In a similar way, consider the Hilbert spaces

$$L^2(\widetilde{A}, E_A) \otimes_A \mathcal{H}, \qquad L^2(\widetilde{B}, E_B) \otimes_B \mathcal{H} \tag{9}$$

and the associated left actions $\sigma_{A,0}$ of $\widetilde{A}$, respectively $\sigma_{B,0}$ of $\widetilde{B}$. As the diagram (6) commutes, the Hilbert space (8) embeds canonically as a subspace of both spaces (9). We may thus write

$$\begin{aligned} L^2(\widetilde{A}, E_A) \otimes_A \mathcal{H} &= \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{A,0} \\ L^2(\widetilde{B}, E_B) \otimes_B \mathcal{H} &= \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{B,0}, \end{aligned}$$

the subspace $\mathcal{H} \oplus \mathcal{K}_D \oplus 0$ is reducing for the restrictions of $\sigma_{A,0}$ and $\sigma_{B,0}$ to $\widetilde{D}$, and we have $\sigma_{A,0}(\tilde{d})(\eta \oplus 0) = (\sigma_D(\tilde{d})\eta) \oplus 0 = \sigma_{B,0}(\tilde{d})(\eta \oplus 0)$ for every $\tilde{d} \in \widetilde{D}$ and $\eta \in \mathcal{H} \oplus \mathcal{K}_D$. Moreover, $\mathcal{H} \oplus 0 \oplus 0$ is reducing for the restrictions of $\sigma_{A,0}$ to A and $\sigma_{B,0}$ to B, and we have

$$\begin{aligned} \sigma_{A,0}(a)(h \oplus 0 \oplus 0) &= (\pi_A(a)h) \oplus 0 \oplus 0 \qquad (a \in A,\, h \in \mathcal{H}) \\ \sigma_{B,0}(b)(h \oplus 0 \oplus 0) &= (\pi_B(b)h) \oplus 0 \oplus 0 \qquad (b \in B,\, h \in \mathcal{H}). \end{aligned}$$

Let $\sigma_{A,0,\widetilde{D}}$ denote the action of $\widetilde{D}$ on $\mathcal{K}_{A,0}$ obtained by restricting $\sigma_{A,0}$ to $\widetilde{D}$ and compressing, and similarly for $\sigma_{B,0,\widetilde{D}}$.

We now proceed recursively as in the proof of Proposition 2.2. If Hilbert spaces $\mathcal{K}_{A,n-1}$ and $\mathcal{K}_{B,n-1}$ have been constructed with actions $\sigma_{A,n-1,\widetilde{D}}$ and $\sigma_{B,n-1,\widetilde{D}}$, respectively, of $\widetilde{D}$, use Lemma 2.1 to construct Hilbert spaces $\mathcal{K}_{B,n}$ and $\mathcal{K}_{A,n}$ and $*$–homomorphisms

$$\begin{aligned} \sigma_{B,n} &: \widetilde{B} \to B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}) \\ \sigma_{A,n} &: \widetilde{A} \to B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n}), \end{aligned}$$

such that

$$\begin{aligned} \sigma_{B,n}(\tilde{d})(k \oplus 0) &= (\sigma_{A,n-1,\widetilde{D}}(\tilde{d})k) \oplus 0 \qquad (\tilde{d} \in \widetilde{D},\, k \in \mathcal{K}_{A,n-1}) \\ \sigma_{A,n}(\tilde{d})(k \oplus 0) &= (\sigma_{B,n-1,\widetilde{D}}(\tilde{d})k) \oplus 0 \qquad (\tilde{d} \in \widetilde{D},\, k \in \mathcal{K}_{B,n-1}). \end{aligned}$$

Then let $\sigma_{B,n,\widetilde{D}}$ be the action of $\widetilde{D}$ on $\mathcal{K}_{B,n}$ obtained from the restriction of $\sigma_{B,n}$ to $\widetilde{D}$ by compressing, and similarly define the action $\sigma_{A,n,\widetilde{D}}$ of $\widetilde{D}$ on $\mathcal{K}_{A,n}$.

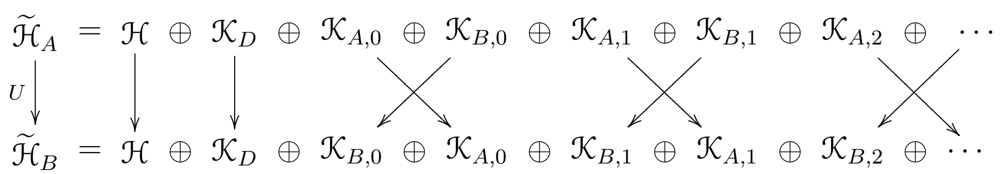
We may now define the Hilbert spaces

$$\begin{aligned} \widetilde{\mathcal{H}}_A &= \overbrace{\overbrace{\mathcal{H} \oplus \mathcal{K}_D}^{\sigma_D} \oplus \mathcal{K}_{A,0}}^{\sigma_{A,0}} \oplus \overbrace{\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}}^{\sigma_{A,1}} \oplus \overbrace{\mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2}}^{\sigma_{A,2}} \oplus \cdots , \\ \widetilde{\mathcal{H}}_B &= \underbrace{\underbrace{\mathcal{H} \oplus \mathcal{K}_D}_{\sigma_D} \oplus \mathcal{K}_{B,0}}_{\sigma_{B,0}} \oplus \underbrace{\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}}_{\sigma_{B,1}} \oplus \underbrace{\mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2}}_{\sigma_{B,2}} \oplus \cdots , \end{aligned} \tag{10}$$

where the brackets indicate where the constructed representations act. We let $\sigma_{\widetilde{A}} : \widetilde{A} \to B(\widetilde{\mathcal{H}}_A)$ and $\sigma_{\widetilde{B}} : \widetilde{B} \to B(\widetilde{\mathcal{H}}_B)$ be the $*$–representations

$$\begin{aligned} \sigma_{\widetilde{A}} &= \sigma_{A,0} \oplus \sigma_{A,1} \oplus \sigma_{A,2} \oplus \cdots , \\ \sigma_{\widetilde{B}} &= \sigma_{B,0} \oplus \sigma_{B,1} \oplus \sigma_{B,2} \oplus \cdots , \end{aligned}$$

where the summands act as indicated by brackets in (10). Consider the unitary $U : \widetilde{\mathcal{H}}_A \to \widetilde{\mathcal{H}}_B$ mapping the summands in $\widetilde{\mathcal{H}}_A$ identically to the corresponding summands in $\widetilde{\mathcal{H}}_B$ as indicated by the arrows below:



Let $\mathcal{K} = \mathcal{K}_D \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \cdots$ and identify $\mathcal{H} \oplus \mathcal{K}$ with $\widetilde{\mathcal{H}}_A$. Then we have the $*$–representations $\tilde{\pi}_{\widetilde{A}} = \sigma_{\widetilde{A}} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ and $\tilde{\pi}_{\widetilde{B}} : \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$, the latter defined by $\tilde{\pi}_{\widetilde{B}}(\cdot) = U^* \sigma_{\widetilde{B}}(\cdot) U$. By construction, the restrictions of $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ to $\widetilde{D}$ agree, and we have

$$\begin{aligned} \tilde{\pi}_{\widetilde{A}}(a)(h \oplus 0) &= (\pi_A(a)h) \oplus 0 \qquad (a \in A,\ h \in \mathcal{H}) \\ \tilde{\pi}_{\widetilde{B}}(b)(h \oplus 0) &= (\pi_B(b)h) \oplus 0 \qquad (b \in B,\ h \in \mathcal{H}). \end{aligned}$$

Letting $\tilde{\pi} : \widetilde{A} *_{\widetilde{D}} \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$ be the $*$–homomorphism obtained from $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ via the universal property, we have that (7) holds. □

3. Examples of non–embedding

In this section, we give some examples when the map λ of Question 1.1 fails to be injective. (In contrast, it is known [2] that in the more stringent situation of *reduced* amalgamated free products, the map analogous to λ is always injective.)

We begin with a trivial class of examples.

Examples 3.1. Let A and B be C*-subalgebras of a C*-algebra E with $A \not\subseteq B$ and $B \not\subseteq A$. Let $D = A \cap B$, $\widetilde{A} = E$ and $\widetilde{D} = \widetilde{B} = B$, equipped with the natural inclusions. Then the map $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B} = E$ is injective if and only if $A *_D B$ is exactly the C*-subalgebra of E generated by A and B. This doesn't hold in general. Notice that in these examples, $B \cap \widetilde{D} = B \supsetneq D$.

Proposition 3.2. *Suppose*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array}$$

is a commuting diagram of inclusions of C**–algebras and let* $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ *be the resulting* $*$*–homomorphism of full free product* C**–algebras. Suppose there are conditional expectations* $E_D^A : A \to D$ *and* $E_D^B : B \to D$ *with* E_D^B *faithful. Suppose there are* $\tilde{d} \in \widetilde{D}$*,* $a \in A$ *and* $b \in B$ *satisfying* $a\tilde{d} \in A$*,* $\tilde{d}b \in B$*,*

$$D(\tilde{d}b) \cap Db = \{0\} \tag{11}$$

$$E_D^A(\tilde{d}^* a^* a \tilde{d}) b \neq 0. \tag{12}$$

Then λ *is not injective.*

Proof. Letting

$$\begin{aligned} \sigma_A &: A \hookrightarrow A *_D B, & \sigma_B &: B \hookrightarrow A *_D B, \\ \sigma_{\widetilde{A}} &: \widetilde{A} \hookrightarrow \widetilde{A} *_{\widetilde{D}} \widetilde{B}, & \sigma_{\widetilde{B}} &: \widetilde{B} \hookrightarrow \widetilde{A} *_{\widetilde{D}} \widetilde{B} \end{aligned} \tag{13}$$

be the embeddings as in (1), we have

$$\lambda(\sigma_A(a\tilde{d})\sigma_B(b)) = \sigma_{\widetilde{A}}(a\tilde{d})\sigma_{\widetilde{B}}(b) = \sigma_{\widetilde{A}}(a)\sigma_{\widetilde{B}}(\tilde{d}b) = \lambda(\sigma_A(a)\sigma_B(\tilde{d}b)).$$

Thus we need only show

$$\sigma_A(a\tilde{d})\sigma_B(b) \neq \sigma_A(a)\sigma_B(\tilde{d}b). \tag{14}$$

We consider the reduced amalgamated free product of C*–algebras (see [11] or [12]),

$$(A *_D^{\mathrm{red}} B, E_D) = (A, E_D^A) *_D (B, E_D^B)$$

and the natural quotient $*$–homomorphism $A *_D B \to A *_D^{\mathrm{red}} B$. Let $L^2(A *_D^{\mathrm{red}} B, E_D)$ be the right Hilbert D–module obtained by separation and completion from $A *_D^{\mathrm{red}} B$ with respect to the D–valued inner product $\langle x, y\rangle = E_D(x^* y)$, and given $x \in A *_D^{\mathrm{red}} B$, let $\hat{x}$ denote the corresponding element in $L^2(A *_D^{\mathrm{red}} B, E_D)$. Let $\mathcal{H}_A = L^2(A, E_D^A)$ and $\mathcal{H}_B = L^2(B, E_D^B)$ be similarly defined. Then in $L^2(A *_D^{\mathrm{red}} B, E_D)$, the closure of the subspace spanned by elements of the form $(ab)\hat{}$ for $a \in A$ and $b \in B$ is isomorphic to the tensor product $\mathcal{H}_A \otimes_D \mathcal{H}_B$ of Hilbert D–modules. In order to show (14), it will suffice to show

$$(a\tilde{d})\hat{} \otimes \hat{b} \neq \hat{a} \otimes (\tilde{d}b)\hat{}$$

in $\mathcal{H}_A \otimes_D \mathcal{H}_B$. Let $\zeta_B \in \mathcal{H}_B$. Then

$$\langle (a\tilde{d})\hat{} \otimes \zeta_B, (a\tilde{d})\hat{} \otimes \hat{b}\rangle = \langle \zeta_B, (E_D^A(\tilde{d}^* a^* a \tilde{d})b)\hat{}\rangle \tag{15}$$

$$\langle (a\tilde{d})\hat{} \otimes \zeta_B, \hat{a} \otimes (\tilde{d}b)\hat{}\rangle = \langle \zeta_B, (E_D^A(\tilde{d}^* a^* a)\tilde{d}b)\hat{}\rangle. \tag{16}$$

From assumptions (11) and (12), we obtain $E_D^A(\tilde{d}^* a^* a \tilde{d})b \neq E_D^A(\tilde{d}^* a^* a)\tilde{d}b$. Since E_D^B is faithful, there is $\zeta_B \in \mathcal{H}_B$ such that the right–hand–sides of (15) and (16) are not equal. $\square$

Remark 3.3. From the above proof, one sees that the hypotheses of Proposition 3.2 can be weakened as follows: Assumptions (11) and (12) can be dropped, and E_D^B need not be assumed faithful, but instead one must assume

$$E_D^B\big(b^*\big(E_D^A(\tilde{d}^*a^*a\tilde{d}) - E_D^A(\tilde{d}^*a^*a)\tilde{d} - \tilde{d}^*E_D^A(a^*a\tilde{d}) + \tilde{d}^*E_D^A(a^*a)\tilde{d}\big)b^*\big) \neq 0. \tag{17}$$

Note that the LHS of (17) is nothing other than

$$\langle (a\tilde{d})\hat{} \otimes \hat{b} - \hat{a} \otimes (\tilde{d}b)\hat{}, (a\tilde{d})\hat{} \otimes \hat{b} - \hat{a} \otimes (\tilde{d}b)\hat{} \rangle.$$

Corollary 3.4. *Suppose*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array} \tag{18}$$

is a commuting diagram of inclusions of C^*–algebras and let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the resulting $*$–homomorphism of full free product C^*–algebras. Suppose one of the following holds:

(i) $D = 0$
(ii) $D = \mathbf{C}$, *A and B are unital and the inclusions $D \hookrightarrow A$ and $D \hookrightarrow B$ are unital.*

Suppose there are $\tilde{d} \in \widetilde{D}$, $a \in A$ and $b \in B$ such that $a\tilde{d} \in A\backslash\{0\}$, $\tilde{d}b \in B$ and $\tilde{d}b \notin \mathbf{C}b$. Then λ is not injective.

Proof. We can reduce to the case in which (ii) holds by application of Lemma 2.3. We may without loss of generality assume A and B are separable. Letting $E_D^A : A \to \mathbf{C}$ and $E_D^B : B \to \mathbf{C}$ be faithful states, we find the hypotheses of Proposition 3.2 are satisfied. □

From this corollary, we have the following class of concrete examples, which shows that λ may be non–injective even if

$$B \cap \widetilde{D} = D = A \cap \widetilde{D}. \tag{19}$$

Example 3.5. Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space. Inside $B(\mathcal{H})$, let $D = \mathbf{C}1$ and let $A = B = D + K(\mathcal{H})$, where $K(\mathcal{H})$ is the compact operators. Let $u \in B(\mathcal{H})$ be a unitary operator that does not belong to D and let $\widetilde{D} = C^*(u)$, $\widetilde{A} = \widetilde{B} = \widetilde{D} + K(\mathcal{H})$. Let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the $*$–homomorphism arising from the inclusions (18). Then λ is not injective.

Proof. Take $\tilde{d} = u$ and $a \in K(\mathcal{H})\backslash\{0\}$. Since $u \notin \mathbf{C}1$, there is $b \in K(\mathcal{H})$ such that $ub \notin \mathbf{C}b$. Now apply Corollary 3.4. One can choose u so that $C^*(u) \cap (\mathbf{C}1 + K(\mathcal{H})) = \mathbf{C}1$, in order to get (19). □

Proposition 3.6. *Suppose*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array} \tag{20}$$

is a commuting diagram of inclusions of C–algebras and let* $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ *be the resulting* $*$*–homomorphism of full free product C*–algebras. Suppose one of the following holds:*

(i) $D = 0$

(ii) $D = \mathbf{C}$*,* A *and* B *are unital and the inclusions* $D \hookrightarrow A$ *and* $D \hookrightarrow B$ *are unital.*

Suppose there are $\tilde{d} \in \widetilde{D}$*,* $a_1, a_2 \in A$ *and* $b \in B \backslash D$ *such that* $a_1\tilde{d}, \tilde{d}a_2 \in A$*,* $a_1\tilde{d} \notin \mathbf{C}$ *and* $\tilde{d}b = b\tilde{d}$*. Then* λ *is not injective.*

Proof. We can reduce to the case in which (ii) holds by application of Lemma 2.3. We use the same notation as in (13). We have

$$\begin{aligned}\lambda(\sigma_A(a_1\tilde{d})\sigma_B(b)\sigma_A(a_2)) &= \sigma_{\widetilde{A}}(a_1\tilde{d})\sigma_{\widetilde{B}}(b)\sigma_{\widetilde{A}}(a_2) \\ &= \sigma_{\widetilde{A}}(a_1)\sigma_{\widetilde{B}}(b)\sigma_{\widetilde{A}}(\tilde{d}a_2) = \lambda(\sigma_A(a_1)\sigma_B(b)\sigma_A(\tilde{d}a_2)),\end{aligned}$$

and we must only show

$$\sigma_A(a_1\tilde{d})\sigma_B(b)\sigma_A(a_2) \neq \sigma_A(a_1)\sigma_B(b)\sigma_A(\tilde{d}a_2). \tag{21}$$

Without loss of generality, assume A and B are separable. Let $\phi_A : A \to \mathbf{C}$ and $\phi_B : B \to \mathbf{C}$ be faithful states. By adding a scalar multiple of the identity, if necessary, we may without loss of generality assume $\phi_B(b) = 0$. Let

$$(A *_{\mathbf{C}}^{\mathrm{red}} B, \phi) = (A, \phi_A) *_{\mathbf{C}} (B, \phi_B)$$

be the reduced free product of C*–algebras. Using arguments and notation as in the proof of Proposition 3.2, the closure of the subspace of $L^2(A *_{\mathbf{C}}^{\mathrm{red}} B, \phi)$ spanned by elements of the form $(aba')\hat{}$ for $a, a' \in A$ is isomorphic to $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$. To show (21), it will suffice to show

$$(a_1\tilde{d})\hat{} \otimes \hat{b} \otimes \hat{a}_2 \neq \hat{a}_1 \otimes \hat{b} \otimes (\tilde{d}a_2)\hat{}$$

in $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$. However, this follows from the assumptions. □

From the above proposition, we get the following example, which requires only "bad" relations between A and $\widetilde{D}$, not between B and $\widetilde{D}$.

Example 3.7. Let D, $\widetilde{D}$, A and $\widetilde{A}$ be as in Example 3.5. Let B be any unital C*–algebra of dimension greater than 1 and let $\widetilde{B} = B \otimes \widetilde{D}$, (for the unique C*–tensor norm). Then the $*$–homomorphism $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ arising from the inclusions (20) is not injective.

Remark 3.8. The problem with injectivity of λ in Examples 3.5 and 3.7 arises already at the algebraic level

$$A *_D^{\mathrm{alg}} B \to \widetilde{A} *_{\widetilde{D}}^{\mathrm{alg}} \widetilde{B}. \tag{22}$$

On the other hand, in Examples 3.1, we can arrange that the map between algebras (22) is injective, while λ fails to be injective, e.g. by taking E to be a reduced free product. However, we do not know of an example where λ fails to be injective and where the algebraic map (22) is injective, but where $A \cap \widetilde{D} = D = B \cap \widetilde{D}$.

4. An application to residual finite dimensionality

A C*–algebra is said to be residually finite dimensional (r.f.d.) if it has a separating family of finite dimensional ∗–representations. The first result linking full free products and residual finite dimensionality was M.-D. Choi's proof [6] that the full group C*–algebras of nonabelian free groups are r.f.d. In [7], Exel and Loring proved that the full free product of any two r.f.d. C*–algebras A and B with amalgamation over either the zero C*–algebra or over the scalar multiples of the identity (if A and B are unital) is r.f.d. In [5], N. Brown and Dykema proved that a full amalgamated free product of matrix algebras $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$ over a unital subalgebra D is r.f.d. provided that the normalized traces on $M_k(\mathbf{C})$ and $M_\ell(\mathbf{C})$ restrict to the same trace on D. In this section, we observe that by applying Proposition 2.2, one obtains (as a corollary of the result from [5]) the analogous result for full amalgamated free products of finite dimensional algebras.

Lemma 4.1. *Let $S = \{x \in \mathbf{R}^n \mid Ax = 0\}$, where A is an $m \times n$ matrix having only rational entries. Then vectors having only rational entries are dense in S.*

Proof. By considering the reduced row–echelon form of A, we see that there is a basis for S consisting of rational vectors. □

Theorem 4.2. *Consider unital inclusions of C*–algebras $A \supseteq D \subseteq B$ with A and B finite dimensional. Let $A *_D B$ be the corresponding full amalgamated free product. Then $A *_D B$ is residually finite dimensional if and only if there are faithful tracial states τ_A on A and τ_B on B whose restrictions to D agree.*

Proof. Since every separable r.f.d. C*–algebra has a faithful tracial state, the necessity of the existence of τ_A and τ_B is clear.

Let us recall some well known facts about a unital inclusion $D \subseteq A$ of finite dimensional C*–algebras (see e.g. Chapter 2 of [8]). Let $p_1, \dots, p_m$ be the minimal central projections of A and $q_1, \dots, q_n$ the minimal central projections of D. Then the inclusion matrix Λ_D^A is a $m \times n$ integer matrix whose (i, j)th entry is $\mathrm{rank}\,(q_j p_i A q_j)/\mathrm{rank}\,(q_j D)$, where the rank of a matrix algebra $M_k(\mathbf{C})$ is k. To a trace τ on A, we associate the column vector s of length m whose ith entry is the trace of a minimal projection in $p_i A$. Then the restriction of τ to D has associated column vector $(\Lambda_D^A)^t s$, where the superscript t indicates transpose.

Thus, given $A \supseteq D \subseteq B$ as in the statement of the theorem, the existence of faithful tracial states τ_A and τ_B agreeing on D is equivalent to the existence of column vectors s_A and s_B, none of whose components are zero, such that $(\Lambda_D^A)^t s_A = (\Lambda_D^B)^t s_B$, i.e.

$$\begin{bmatrix} (\Lambda_D^A)^t, & -(\Lambda_D^B)^t \end{bmatrix} \begin{bmatrix} s_A \\ s_B \end{bmatrix} = 0. \tag{23}$$

Supposing now that such traces τ_A and τ_B exist, by Lemma 4.1 there is a solution $\left[\begin{smallmatrix} s_A \\ s_B \end{smallmatrix}\right]$ to (23) whose entries are all strictly positive and rational. Therefore, the traces τ_A and τ_B agreeing on D can be chosen to take only rational values on minimal projections of A and, respectively, B. Hence there are unital inclusions into matrix algebras,

$$M_k(\mathbf{C}) \supseteq A \supseteq D \subseteq B \subseteq M_\ell(\mathbf{C}),$$

so that τ_A is the restriction of the tracial state on $M_k(\mathbf{C})$ to A and τ_B is the restriction of the tracial state on $M_\ell(\mathbf{C})$ to B. By Proposition 2.2, $A *_D B$ is a subalgebra of $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$. By Theorem 2.3 of [5], $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$ is r.f.d. Therefore, $A *_D B$ is r.f.d. □

References

[1] B. Blackadar, *Weak expectations and nuclear C^*–algebras,* Indiana Univ. Math. J. **27** (1978), 1021-1026.

[2] E. Blanchard and K. Dykema, *Embeddings of reduced free products of operator algebras,* Pacific J. Math. **199** (2001), 1-19.

[3] D. Blecher and V. Paulsen, *Explicit construction of universal operator algebras and applications to polynomial factorization,* Proc. Amer. Math. Soc. **112** (1991), 839-850.

[4] F. Boca, *Free products of completely positive maps and spectral sets,* J. Funct. Anal. **97** (1991), 251-263.

[5] N.P. Brown and K. Dykema, *Popa algebras in free group factors,* preprint.

[6] M.-D. Choi, *The full C^*-algebra of the free group on two generators,* Pacific J. Math. **87** (1980), 41-48.

[7] R. Exel and T. Loring, *Finite-dimensional representations of free product C^*-algebras*, Internat. J. Math. **3** (1992), 469-476.

[8] F.M. Goodman, P. de la Harpe and V.F.R. Jones, *Coxeter graphs and towers of algebras,* MSRI Publications, vol. 14, Springer–Verlag, New York, 1987.

[9] T. Loring, *Lifting solutions to perturbing problems in C^*–algebras,* Fields Institute Monographs vol. 8, American Mathematical Society, Providence, 1997.

[10] G.K. Pedersen, *Pullback and pushout constructions in C^*-algebra theory,* J. Funct. Anal. **167** (1999), 243-344.

[11] D. Voiculescu, Symmetries of some reduced free product C^*-algebras, *Operator Algebras and Their Connections with Topology and Ergodic Theory*, Lecture Notes in Mathematics, Volume 1132, Springer-Verlag, 1985, 556–588.

[12] D.V. Voiculescu, K.J. Dykema, A. Nica, *Free Random Variables*, CRM Monograph Series **1**, American Mathematical Society, 1992.

Scott Armstrong, Department of Mathematics, University of California, Berkeley CA 94720, USA

E-mail address: sarm@math.berkeley.edu

Ken Dykema, Department of Mathematics, Texas A&M University, College Station TX 77843–3368, USA

E-mail address: Ken.Dykema@math.tamu.edu

Ruy Exel, Departamento de Matematica, Universidade Federal de Santa Catarina, 88040-900 Florianopolis SC, BRAZIL

E-mail address: exel@mtm.ufsc.br

Hanfeng Li, Department of Mathematics, University of Toronto, Toronto ON M5S 3G3, CANADA

E-mail address: hli@fields.toronto.edu