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Regional Stability of a Class of Nonlinear Hybrid Systems: An LMI Approach [1]

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Abstract

This paper presents sufficient conditions to the regional stability problem of a class of nonlinear hybrid systems in the piecewise nonlinear form. The nonlinear local models are defined by a differential equation of the type $\dot{x} = A_i(x)x + b_i(x)$, where $A_i(x)$ and $b_i(x)$ are affine functions of x. This class of systems is equivalently represented by $\dot{x} = A(x,\delta)x + b(x,\delta)$ with δ denoting a vector of logical variables that modifies the local model of the system in accordance with the continuous dynamics. Using a single polynomial Lyapunov function, $v(x) = x'\mathcal{P}(x)x$, we present LMI conditions that assure the local stability of the nonlinear system with a guaranteed domain of attraction.

1 Introduction

Piecewise affine systems, also called piecewise linear systems, have been investigated during the past years because of their wide applicability in a large class of nonlinear systems, such as linear systems with saturation, variable structure control, and systems described by fuzzy and ARMAX techniques [1]. Despite the existence of powerful methods to cope with this class of systems, the generalization to the nonlinear case is a difficult task that has been recently studied by several researchers. For example: *Agrachev & Liberzon* in [2] use Lie-algebraic conditions to analyze the local stability of nonlinear switched systems by using a common quadratic Lyapunov function; *Beldiman & Bushnell* in [3] extends the Lyapunov's indirect method to deal with this class of system; *Mancilla-Aguilar* in [4] establishes sufficient conditions for global asymptotic stability for switched systems where the local models are Lipschitz vector fields, *Li et al.* in [5] consider the robust stability problem for a general class of hybrid systems, and *El-Farra & Christofides* in [6] uses multiple Lyapunov functions to the control of switched nonlinear systems.

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Since the development of *interior-point methods* for solving semi-definite programming problems and the publication of the book *Linear matrix inequalities in systems and control theory* [7], the LMI framework has been widely used to solve many control problems using quadratic Lyapunov functions. More recently, several nonlinear problems such as regional stability, synthesis and performance have been addressed by means of polynomial Lyapunov functions and convex optimization problems [8, 9]. In particular, we have proposed in [10] an LMI based approach for the stability analysis of switched linear systems by considering a single polynomial Lyapunov function.

As an extension of [10], this paper focuses on the regional analysis of a class of nonlinear switched systems where the local system matrices are affine functions of the state. This class of switched systems is modelled by a differential equation of the form $\dot{x} = A(x,\delta)x + b(x,\delta)$ where x is the continuous state vector and δ denotes a vector of logical variables that modifies the local model of the system. Then, to analyze the local stability of this class of systems and estimate its domain of attraction, we use a common polynomial Lyapunov function, $v(x) = x'\mathcal{P}(x)x$ where the Lyapunov matrix $\mathcal{P}(x)$ is computed via a convex optimization problem in terms of LMIs.

The notation used in this paper is standard. $\mathcal{J}_n$ denotes the set of n integers $\{1, 2, \ldots, n\}$. $\mathbb{J}^n$ is the set of n-dimensional integer vectors. $\mathbb{R}^n$ denotes the set of n-dimensional real vectors; $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices; I_n is the $n \times n$ identity matrix; $0_{n\times m}$ is the $n \times m$ matrix of zeros and 0_n is the $n \times n$ matrix of zeros). For a real matrix S, the notation $S > 0$ means that S is symmetric and positive definite and S' is its transpose. Let X and Y be normed spaces and T a function defined from X into Y, then, we say

T is a bounded function if exists some real number M_T such that $||T(x)|| \leq M_T||x||$, $\forall\ x \in X$. For two polytopes $\Pi_1 \subseteq \mathbb{R}^{n_1}$ and $\Pi_2 \subseteq \mathbb{R}^{n_2}$ the notation $\Pi_1 \times \Pi_2$ represents a meta-polytope of dimension $n_1 + n_2$ obtained by the cartesian product. For simplicity of notation, the arguments t (time) and x (state) will be often omitted as well as matrix and vector dimensions whenever they can be determined from the context.

2 Problem Statement

Consider the following class of systems

$$\dot{x}(t) = A_i(x(t))x(t)+b_i(x(t)),\ x(t) \in X_i,\ i \in \mathcal{J}_n,\ t > 0 \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the continuous component of the state taking values in the subsets $X_i \subset \mathbb{R}^{n_x}$; $i \in \mathcal{J}_n$ denotes the discrete variable; $A_i(x(t)) \in \mathbb{R}^{n_x \times n_x}$ and $b_i(x(t)) \in \mathbb{R}^{n_x}$, for $i = 1, \ldots, n$, are given functions of $x(t)$ for $t > 0$.

Notice that systems with switching vector fields require special attention in their mathematical description [11]. To avoid undesirable behavior in our system model, we will assume with respect to system (1) that:

A1 The origin, $x = 0$, is an equilibrium point.

A2 The analysis is performed on a polytopic region $\mathcal{B}_x$ of the state space containing the origin.

A3 The subsets X_i have non-empty disjoint interiors and satisfy $\mathbb{X} = X_1 \cup \ldots \cup X_n \subseteq \mathbb{R}^{n_x}$, i.e., there are non-overlapping among the regions X_i.

A4 The matrices $A_i(x)$ and the vectors $b_i(x)$ are affine functions of x for all $i \in \mathcal{J}_n$.

A5 The changes in the discrete variable i are only governed by the continuous state component. In other words, the system (1) can be characterized as an autonomous switched system.

A6 There is a finite number of transitions in any finite interval of time.

A7 On the boundaries among l_k adjacent regions of X_k, $k = i, \ldots, i + l_k$, there is no sliding behavior, i.e. if $x(t^-) \in X_i \cap X_j$ then $x(t) \notin X_i \cap X_j$, where $x(t^-)$ denotes the trajectory position before time t.

In this paper, we will represent the subsets X_i as follows:

$$X_i = \{x\ :\ \psi_{ik}(x) \geq 0,\ k = 1, \ldots, m_i\},\ i \in \mathcal{J}_n \tag{2}$$

where $\psi_{ik}(x) \in \mathbb{R}$ are given affine functions of x.

Now, consider that the Lyapunov function has the form $v(x) = x'\mathcal{P}(x)x$, where $\mathcal{P}(x)$ is a quadratic function of x, and is single for all sub-sets X_i. Then, the problem of concern in this paper is to analyze the local stability of system (1) and estimate its domain of attraction using a convex optimization problem for determining the matrix $\mathcal{P}(x)$.

3 Preliminaries

In this section, we will introduce some preliminary results before the statement of our main result. Firstly, we present a continuous model that describes the nonlinear switched system. Secondly, we define the class of Lyapunov functions to be considered in this paper and, subsequently, we define a level surface of this Lyapunov function to estimate the domain of attraction.

3.1 The System Model

Consider system (1) with assumptions **A1-7**. For convenience, the discrete variable $i \in \mathcal{J}_n$ will be associated to a set of logical variables $\{\delta_1, \ldots, \delta_n\}$ with $\delta_i \in \{0, 1\}$ as follows. Let c_i be the i-th column of the identity matrix I_n and define the logical vector $\delta \in \mathbb{J}^n$ as

$$\delta := \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix} = c_i,\ \text{if } x(t^-) \text{ and } x(t) \text{ belong to } X_i \tag{3}$$

The above relation states that if the discrete state assumes a given value $i \in \mathcal{J}_n$ the i-th logical component of the vector δ assumes a unitary value and all the remaining ones are zeros. To represent the fact that δ may take any value in the set $\{c_1, \ldots, c_n\}$, we use the notation $\delta \in \Delta$ where

$$\Delta := \{\ c_1, \ldots, c_n\ \} \subset \mathbb{J}^n \tag{4}$$

In order to simplify our system representation, notice from above that the elements δ_i of the vector δ satisfy $\sum_{i=1}^{n} \delta_i = 1$ and thus $\delta_n = 1 - \sum_{i=1}^{n-1} \delta_i$. Also, without loss of generality, consider the following assumption on the dynamics corresponding to the sub-set X_n:

A8 The origin is an equilibrium point of the dynamics associated to the region X_n, i.e. $b_n(x) = 0$.

Thus, we can recast the nonlinear switched system (1) as follows:

$$\dot{x} = A_n(x)x+\sum_{i=1}^{n-1}(A_i(x)-A_n(x))\delta_i x+\sum_{i=1}^{n-1} b_i(x)\delta_i, \tag{5}$$

for all $(x, \delta) \in \mathcal{B}_x \times \Delta$. Observe that δ_n was removed from the expression through the equality $\delta_n =$

$1-\sum_{i=1}^{n-1}\delta_i$. Now, let us introduce the following auxiliary vector $\pi \in \mathbb{R}^{n_\pi}$ where $n_\pi = (n-1)n_x + n - 1$.

$$\pi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_{n-1} \\ \pi_n \\ \vdots \\ \pi_{2n-2} \end{bmatrix}, \text{ with } \begin{array}{rcl} \pi_i & = & \delta_i x, \ \forall\, i \in \mathcal{J}_{n-1} \\ \pi_{i+n-1} & = & \delta_i, \ \forall\, i \in \mathcal{J}_{n-1} \end{array}. \tag{6}$$

To simplify the notation the above relations between x and π will be hereafter represented in a more compact form by the notation $(x,\pi) \in \mathcal{D}$ as indicated bellow:

$$\mathcal{D} = \left\{ (x,\pi) : \begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = 0 \right\} \tag{7}$$

where $\Omega_1 \in \mathbb{R}^{n_\Omega \times n_x}$ and $\Omega_2 \in \mathbb{R}^{n_\Omega \times n_\pi}$ are affine matrix functions of $(x, \delta_1, \ldots, \delta_{n-1})$.

With the above notation, we can represent system (5) as follows:

$$\dot{x} = A_n(x)x + \mathbf{A}(x)\pi, \ x \in \mathcal{B}_x, \ (x,\pi) \in \mathcal{D} \tag{8}$$

where the matrix $\mathbf{A}(x)$ is given by

$$\begin{bmatrix} (A_1 - A_n) & \cdots & (A_{n-1} - A_n) & b_1 & \cdots & b_{n-1} \end{bmatrix}.$$

As defined above, the matrices Ω_1 and Ω_2 are used to represent the whole set of relations between x and π. In particular, these matrices will represent the fact that δ is a logical variable. To this end, note from (3) that the following relation holds $\delta\delta' = diag(\delta_i)$ or, equivalently:

$$\delta_i\delta_j = 0 \quad \text{and} \quad \delta_i(\delta_i - 1) = 0, \forall i \neq j \in \mathcal{J}_n.$$

From (6) and above, we get the following:

$$\begin{array}{l} (\delta_i - 1)\delta_i x = (\delta_i - 1)\pi_i = 0, \forall i \in \mathcal{J}_{n-1} \\ (\delta_i - 1)\delta_i = (\delta_i - 1)\pi_{i+n-1} = 0, \forall i \in \mathcal{J}_{n-1} \\ \delta_i\delta_j x = \delta_i\pi_j = 0, \forall i \neq j \in \mathcal{J}_{n-1} \\ \delta_i\delta_j = \delta_i\pi_{j+n-1} = 0, \forall i \neq j \in \mathcal{J}_{n-1} \end{array} \tag{9}$$

Similarly, the following identities must hold:

$$\delta_i x - \pi_i = 0, \ \pi_i - x\pi_{i+n-1} = 0, \forall i \in \mathcal{J}_{n-1}. \tag{10}$$

As a consequence, the set of identities in (9) and (10) can be incorporated into (7) by rewriting them as new lines of Ω_1 and Ω_2.

3.2 Lyapunov function candidate

In this paper, we consider that the Lyapunov function candidate, $v(x) : \mathcal{B}_x \mapsto \mathbb{R}$, is common for all subset X_i, and has the following structure:

$$v(x) = x'\mathcal{P}(x)x, \ \mathcal{P}(x) = \begin{bmatrix} I_{n_x} \\ \Theta(x) \end{bmatrix}' P \begin{bmatrix} I_{n_x} \\ \Theta(x) \end{bmatrix} \tag{11}$$

where $\Theta(x) \in \mathbb{R}^{n_1 \times n_x}$ is a given affine matrix function of x and P is a constant matrix to be determined.

From the Lyapunov theory, [12], the nonlinear switched system (1) is locally stable if there are positive scalars ε_a, ε_b and ε_c that satisfy the following inequalities for all $x \in \mathcal{B}_x$:

$$\varepsilon_a x'x \leq v(x) = x'\mathcal{P}(x)x \leq \varepsilon_b x'x \tag{12}$$
$$\dot{v}(x) = 2\dot{x}'\mathcal{P}(x)x + x'\dot{\mathcal{P}}(x)x \leq -\varepsilon_c x'x \tag{13}$$

As a result, we have to compute the time derivative of the Lyapunov matrix $\mathcal{P}(x)$. To this end, note that $\Theta(x)$ is an affine function of x. Therefore, we can rewrite it with the following structure:

$$\Theta(x) = \sum_{i=1}^{n_x} T_i x_i + U \tag{14}$$

where x_i is the i-th entry of the vector x and the matrices T_i, for $i = 1, \ldots, n_x$, and U are constant with the same dimensions of $\Theta(x)$.

In addition, the term $\dot{\Theta}(x)x$ that appears in $x'\dot{\mathcal{P}}(x)x$ can be rewritten as follows:

$$\dot{\Theta}(x)x = \sum_{i=1}^{n_x} T_i \dot{x}_i x = \Theta_x(x)\dot{x}, \ \Theta_x(x) = \sum_{i=1}^{n_x} T_i x r_i \tag{15}$$

where r_i is the i-th row of the identity matrix I_{n_x}.

Therefore, we can use (14) and (15) in order to obtain a convex characterization of the Lyapunov inequalities (12) and (13).

3.3 Domain of Attraction

The Lyapunov inequalities (12) and (13) assure the local stability of system (1). As a consequence, we can estimate its domain of attraction.

In this paper, we estimate the domain of attraction by computing the region that contains the largest level surface of $v(x)$ belonging to the polytope $\mathcal{B}_x$.

Hence, without lost of generality, we use the level surface of the Lyapunov function candidate $v(x) = 1$ and define the region:

$$\Upsilon = \left\{ x : v(x) = x'\mathcal{P}(x)x \leq 1 \right\} \tag{16}$$

as an estimation of the domain of attraction. Observe that this set is not an usual ellipsoid because $v(x)$ is a polynomial function of x.

Also, for convenience, we will represent $\mathcal{B}_x$ by the set of inequalities:

$$\mathcal{B}_x = \left\{ x : a_l' x \leq 1,\ l = 1, \ldots, n_e \right\} \tag{17}$$

where the vectors $a_l \in \mathbb{R}^{n_x}$ are associated with the n_e edges of the polytope $\mathcal{B}_x$. Keep in mind that $\mathcal{B}_x$ may be equivalently represented by its vertices.

Through the $\mathcal{S}$-Procedure [7], the relation $\Upsilon \subset \mathcal{B}_x$ holds if the following conditions are satisfied:

$$2(1 - a_k' x) + x' \mathcal{P}(x) x - 1 \geq 0,\ k = 1, \ldots, n_e \tag{18}$$

Observe that the problem of maximizing the size of Υ is related with the trace of $\mathcal{P}(x)$. Taking into account the structure of $\mathcal{P}(x)$, we get that

$$\begin{bmatrix} 1 \\ x \\ \Theta(x) \end{bmatrix}' \begin{bmatrix} 1 & \begin{bmatrix} a_k' & 0 \end{bmatrix} \\ \begin{bmatrix} a_k \\ 0 \end{bmatrix} & P \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \Theta(x) \end{bmatrix} \geq 0, \qquad k = 1, \ldots, n_e \tag{19}$$

As in [10], the above expression may be used to derive a convex optimization problem to optimize the size of Υ.

4 Stability Analysis

In this section, we present a sufficient convex condition to the regional stability problem of the class of nonlinear switched systems defined in section 2. Since the switching rules of the model are described by (2) and the expressions defining the auxiliary vector π are expressed in (7), for analyzing the system stability we need to take into account the following inequality and equality constraints:

$$\begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = 0,\ \psi_{ik}(x) \geq 0, i \in \mathcal{J}_n,\ k \in \mathcal{J}_{m_i}$$

Moreover, from the definition of the Lyapunov function in (11) the following equality constraint will be of interest:

$$\begin{bmatrix} \Theta(x) & -I_{n_1} \end{bmatrix} \begin{bmatrix} I_{n_1} \\ \Theta(x) \end{bmatrix} x = 0$$

In deriving the stability conditions, we take into account the above equality and inequality constraints through the Finsler's lemma and the S-procedure [7], respectively. When these techniques are applied they introduce a set of scaling variables that play an important role in reducing the conservativeness of the final result.

Before we state the main result of this paper, we need the following auxiliary notation:

$$\Phi = \sum_{i=1}^{2n-1} E_i' \phi_i E_i + \sum_{i=1}^{2n-2} E_{2n-1}' \Gamma_i E_i + E_i' \Gamma_i' E_{2n-1} \tag{20}$$

where Γ_i ($i \in \mathcal{J}_{2n-2}$) are free matrices with appropriate dimensions and ϕ_i are affine matrix functions of x given by

$$\begin{aligned} \phi_i &= \sum_{k=1}^{m_i} R_{ik} \psi_{ik}, R_{ik} \in \mathbb{R}^{n_x \times n_x}, i \in \mathcal{J}_{n-1} \\ \phi_{i+n-1} &= \sum_{k=1}^{m_i} R_{(i+n-1)k} \psi_{ik}, R_{(i+n-1)k} \in \mathbb{R}, i \in \mathcal{J}_{n-1} \\ \phi_{2n-1} &= \sum_{k=1}^{m_n} R_{(2n-1)k} \psi_{nk}, R_{(2n-1)k} \in \mathbb{R}^{n_x \times n_x} \end{aligned}$$

with E_i denoting constant matrices of structure given in the sequel and $R_{ik} > 0$ representing scaling variables to be determined.

The matrices E_i are defined as follows. Let $\tilde{E}_i$ and $\tilde{E}_{2n-1}$ be constant matrices such that $\tilde{E}_i \pi = \pi_i$ ($i \in \mathcal{J}_{2n-2}$) and $\tilde{E}_{2n-1} \pi = \sum_{i=1}^{n-1} \pi_i$. Then, define:

$$\begin{aligned} E_i &= \begin{bmatrix} 0_{n_x} & 0_{n_x \times n_1} & \tilde{E}_i \end{bmatrix}, i \in \mathcal{J}_{n-1} \\ E_{i+n-1} &= \begin{bmatrix} 0_{1 \times n_x} & 0_{1 \times n_1} & \tilde{E}_{i+n-1} \end{bmatrix}, i \in \mathcal{J}_{n-1} \\ E_{2n-1} &= \begin{bmatrix} I_{n_x} & 0_{n_x \times n_1} & -\tilde{E}_{2n-1} \end{bmatrix} \end{aligned}$$

Also, in order to apply the Finsler's lemma, we introduce the following notation:

$$\begin{aligned} C_x &= \begin{bmatrix} x_2 & -x_1 & 0 & \cdots & \cdots & 0 \\ 0 & x_3 & -x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{n_x} & -x_{n_x-1} \end{bmatrix}, \\ N &= \begin{bmatrix} I_{n_x+n_1} & 0 \end{bmatrix} \in \mathbb{R}^{(n_x+n_1)\times(n_x+n_1+n_\pi)}, \\ C &= \begin{bmatrix} -\Theta(x) & I_{n_1} \\ C_x & 0 \end{bmatrix}, \\ D &= \begin{bmatrix} I_{n_x} & 0 \\ -(\Theta(x) + \Theta_x(x)) & I_{n_1} \end{bmatrix}, \\ F &= \begin{bmatrix} A_n(x) & 0 & \mathbf{A}(x) \\ 0 & 0 & 0 \end{bmatrix}, \\ \Omega &= \begin{bmatrix} -D & F \\ 0 & \begin{bmatrix} C & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} \Omega_1 & 0 & \Omega_2 \end{bmatrix} \end{bmatrix}. \end{aligned}$$

From above, we state the following theorem which proposes a sufficient condition for the regional stability problem introduced in Section 2.

Theorem 1 *Consider system (1) with assumptions* **A1-A8***, its associated system in (5) and the above notation. Let* $\Theta(x)$ *and* $\Theta_x(x)$ *be affine matrix functions of* x *as defined in (14) and (15), respectively. Suppose that* $P = P'$, L, M, R_{ik} *and* Γ_i *are a solution of the following optimization problem where the LMIs are* *constructed at all vertices of* $\mathcal{B}_x \times \Delta$.

$$\min \quad \mathsf{trace}\ (P + LC + C^{'}L^{'}) \quad \textit{subject to:}$$

$$\begin{bmatrix} 1 & \begin{bmatrix} a_l^{'} & 0 \end{bmatrix} \\ \begin{bmatrix} a_l \\ 0 \end{bmatrix} & (P + LC + C^{'}L^{'}) \end{bmatrix} > 0, l \in \mathcal{J}_{n_e} \quad (21)$$

$$\begin{bmatrix} 0 & PN \\ N^{'}P & \Phi \end{bmatrix} + M\Omega + \Omega^{'}M^{'} < 0 \qquad (22)$$

Then, system (1) is asymptotically locally stable and $v(x)$ *given by (11) is a Lyapunov function in* $\mathcal{B}_x$*. Also,* $\Upsilon = \{x : v(x) \leq 1\}$ *is an invariant set, i.e., for all* $x(0) \in \Upsilon$*, the system trajectory* $x(t) \in \Upsilon$ *and approaches the origin as* $t \to \infty$.

Remark 1 It was assumed that the system cannot have sliding modes. However, the proposed approach can be modified to tackle with sliding modes. The idea is to use the *Filippov's* approach [11] for stability and the Finsler's Lemma to take the sliding mode dynamics into account. In fact, more work on this topic is being carried out by the authors. □

5 Numerical Result

In this section, we present a numerical example to illustrate the potential of our approach. To this end, we analyze the regional stability of the chemical reactor introduced in [13, Example 7.2] with input saturation.

Thus, consider the following bilinear system with saturation:

$$\dot{x} = Ax + b(x)\ \mathsf{sat}(u), \ u = Kx \qquad (23)$$

where $K = \begin{bmatrix} 0.3133 & -3.5561 \end{bmatrix}$, $\mathsf{sat}(\cdot)$ is the unit saturation function and

$$A = \begin{bmatrix} 0 & 5/12 \\ -50/3 & 8/3 \end{bmatrix}, \ b(x) = \begin{bmatrix} 0 \\ 2 + x_1/8 \end{bmatrix}.$$

The above system can be equivalently represented by the following nonlinear switched system:

$$\dot{x} = A_i(x)x + b_i(x), \ x \in X_i, \ i \in \mathcal{J}_3$$

where

$$A_1(x) = A_2(x) = A, \ A_3(x) = A + b(x)K,$$
$$b_1(x) = -b(x) \quad \text{and} \quad b_2(x) = b(x).$$

The subsets X_1, X_2 and X_3 represent the positive saturation, the negative saturation and non-saturation regions, respectively. From (2), they can be defined as follows:

$$\begin{aligned} \psi_1(x) &= -(1 + Kx), \\ \psi_2(x) &= -1 + Kx, \\ \psi_{31}(x) &= 1 + Kx, \ \psi_{32}(x) \ = \ 1 - Kx \end{aligned}$$

Define the Lyapunov function candidate by choosing:

$$\Theta(x) = \begin{bmatrix} x_1 I_2 \\ x_2 I_2 \end{bmatrix}$$

Consider that the polytope $\mathcal{B}_x$ is defined by the following vertices:

$$\left\{ \begin{bmatrix} -0.15 \\ 0.2 \end{bmatrix}, \begin{bmatrix} -0.15 \\ -0.4 \end{bmatrix}, \begin{bmatrix} -0.02 \\ -0.9 \end{bmatrix}, \begin{bmatrix} 0.15 \\ -0.2 \end{bmatrix}, \begin{bmatrix} 0.15 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 0.02 \\ 0.9 \end{bmatrix} \right\}$$

Figure 1 shows the estimated domain of attraction, obtained from Theorem 1, and the phase portrait of system (23) for comparison purposes.

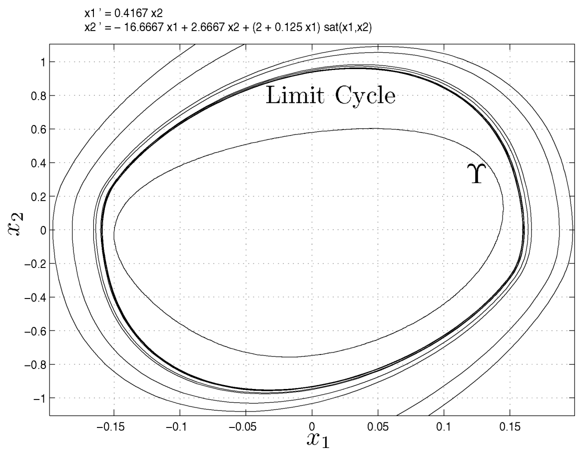

Figure 1: Theoretical and estimated domain of attraction for system (23).

6 Concluding Remarks

In this paper, we have addressed the regional stability problem of nonlinear switched systems with the form $\dot{x} = A_i(x)x + b_i(x), \ x \in X_i, i \in \mathcal{J}_n$, where $A_i(x)$ and $b_i(x)$ are affine functions of x. For this class of systems, we have proposed a sufficient condition in terms of LMIs that assures the local stability of the system while providing an estimate of its domain of attraction. To this end, we have used: (i) a representation $\dot{x} = A(x, \delta)x + b(x, \delta)$ for the switched system in which δ denotes a logic variable associated with the switching logic; and (ii) a single polynomial Lyapunov function $v(x) = x^{'}\mathcal{P}(x)x$ for all sub-set X_i, where the Lyapunov matrix $\mathcal{P}(x)$ is a quadratic function of x. As future research, we intend to extend these results for more complex systems, e.g., those with nonlinear (not only affine in x) local matrices and whose switching dynamics depend on continuous and logical variables.

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Proof of Theorem 1

Define the following auxiliary vectors:

$$\xi = \begin{bmatrix} x \\ \Theta(x)x \end{bmatrix}, \ \zeta = \begin{bmatrix} \xi \\ \pi \end{bmatrix}. \tag{24}$$

Let D be a matrix such that $D\xi = 0$, e.g.

$$D = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_1} \end{bmatrix}$$

Consider the set of LMIs in (21). Applying the Schur complement to it leads to $P+LC+C^{'}L^{'} > 0$, $\forall\, x \in \mathcal{B}_x$. Since this LMI is strict, there exists a sufficient small scalar $\epsilon_1 > 0$ such that the following is still satisfied:

$$P + LC + C^{'}L^{'} - \epsilon_1 D^{'}D \geq 0, \ \forall\, x \in \mathcal{B}_x$$

Pre- and post-multiplying the above LMI by $\xi^{'}$ and ξ, respectively, yields $x^{'}\mathcal{P}(x)x \geq \epsilon_1 x^{'}x$, $\forall\, x \in \mathcal{B}_x$ (notice that $C\xi = 0$ by construction). As the elements of $\mathcal{P}(x)$ are bounded on $\mathcal{B}_x$, there exists a sufficient large scalar $\epsilon_2 > 0$ such that $\mathcal{P}(x) \leq \epsilon_2 I_{n_x}$. Then, we get:

$$\epsilon_1 x^{'}x \leq v(x) = x^{'}\mathcal{P}(x)x \leq \epsilon_2 x^{'}x$$

Now consider the LMI (21). Pre- and post multiplying it by $[\ \dot{\xi}^{'} \quad \zeta^{'}\ ]$ and its transpose, we then get:

$$\dot{\xi}^{'}P\xi + \xi^{'}P\dot{\xi} + \zeta^{'}\Phi\zeta < 0, \ \forall\, (x,\delta) \in \mathcal{B}_x \times \Delta \tag{25}$$

Note that $\Omega[\ \dot{\xi}^{'} \quad \zeta^{'}\ ]^{'} = 0$, since by construction we have the following identities:

$$-D\dot{\xi} + F\zeta = 0, \ C\xi = 0 \quad \text{and} \quad \Omega_1 x + \Omega_2 \pi = 0.$$

Also, we can rewrite the term $\zeta^{'}\Phi\zeta$ as follows:

$$\begin{aligned} \zeta^{'}\Phi\zeta &= \sum_{i=1}^{n-1}\left(x^{'}\phi_i x\delta_i^2 + \phi_{(i+n-1)}\delta_i^2\right) + x^{'}\phi_n x\delta_n^2 \\ &+ \sum_{i=1}^{n-1} 2x^{'}\left(\Gamma_i x\delta_n\delta_i + \Gamma_{(i+n-1)}\delta_n\delta_i\right) \end{aligned}$$

From (2), (3) and (20), it follows that $x^{'}\phi_i x \geq 0$ whenever $x \in X_i$. Moreover, in this case we have $\delta_i = 1$ and $\delta_j = 0$, $i \neq j$, and thus $\delta_n\delta_i = 0$ for $i \in \mathcal{J}_{n-1}$. Since $\mathcal{B}_x \subseteq \bigcup X_i$, we get $\zeta^{'}\Phi\zeta \geq 0$ for all $x \in \mathcal{B}_x$ and $\delta \in \Delta$.

From above and (25), we get the following:

$$\dot{v}(x) = \dot{\xi}^{'}P\xi + \xi^{'}P\dot{\xi} \leq -\epsilon_3 x^{'}x, \ \forall\, (x,\delta) \in \mathcal{B}_x \times \Delta$$

for some sufficient small $\epsilon_3 > 0$.

Finally, pre- and post-multiplying the set of LMIs in (21) by $[\ 1 \quad \xi^{'}\ ]$ and its transpose we get (19). Then, $\{x : v(x) \leq 1\} \subset \mathcal{B}_x$, i.e. Υ is an invariant set which completes the proof.