Local Volatility Model in Commodity Markets and Online Calibration

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IMPA

11/04/2013

Outline

Derivative Markets

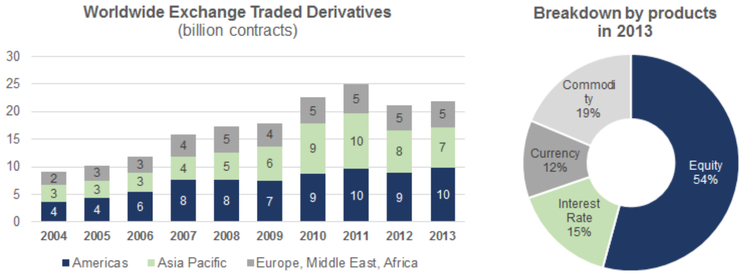

Figure: In 2013, commodities represented 19% of the total amount of traded derivatives.

Source: World Federation of Exchanges

Pricing Problem

Required properties:

- Robustness.
- Reliability.
- Simple calibration.

Desirable property: **implied smile adherence**.
Well established model: *Dupire's Local Volatility* [5].

Applications: Calendar spread options, path dependent options, ...

Challenges in Commodity Markets

Peculiarity in some futures for energy commodities:

- WTI options: three business days before the futures' maturity.
- HH natural gas options: the business day before the futures' maturity.
- Heating oil options: three business days before the futures' maturity.
- RBOB options: three business days before the futures' maturity.

Source: CME webpage.

Conclusion: We do not have a surface of option prices for each future.

Challenges in Commodity Markets (cont.)

- *Convenience yield* is an important feature.
- In general, options are *American*.
- Vol Calibration from American pricing is much harder:
  The forward problem should be solved for each strike and maturity. See Achdou-Pironneau [1].
- Then, evaluate *European prices* from the American ones.

Principal Features

- The term-structure of future prices is given by:
  1. The curve of initial future prices.
  2. The local volatility surface.
- Then, we can form a unique surface of option prices after a normalization.
- We apply usual Tikhonov regularization to calibrate local volatility.
- Use many surfaces of prices in the calibration procedure: *online* setting.

Online Approach

- Option prices change with movements of the underlying asset.
- After technical adaptations, the main underlying is the commodity spot price.
- Consider the nearest to maturity future as the spot price (proxy).
- Then, index the option price surface by this underlying.
- Reorder the underlying price in ascending order.
- Then, the forward problem associates families of local volatility surfaces to call option prices:

$$\sigma(s,T,K) \longmapsto C(s,T,K), \qquad s \in [0, S_{\text{max}} - S_{\text{max}}].$$

Reordering Future Prices

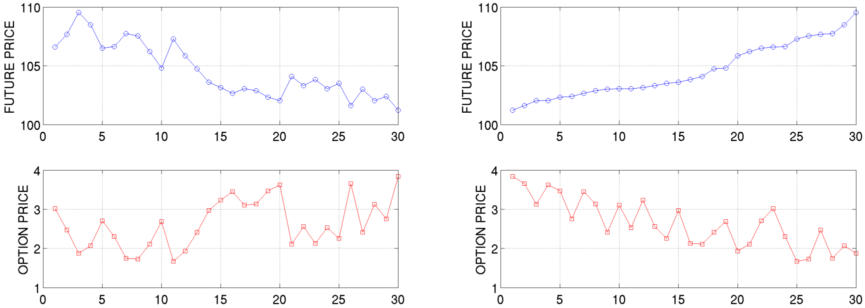

Figure: Left: 30 market future and option prices expiring at Nov. 2013. Strike price is US$ 105,00. Right: Reordered prices.

Dupire's Local Volatility Model in Commodity Markets

- $(\Omega, \mathcal{V}, \mathcal{F}, \widetilde{\mathbb{P}})$ - risk neutral filtered probability space.
- Commodity futures are the underlying assets.
- $F_{t,T}$ - future price at $t \geq 0$ with maturity $T \geq t$.
- S_t (unknown) spot price at $t \geq 0$.
- $F_{t,T} = \widetilde{\mathbb{E}}[S_T | \mathcal{F}_t]$, then $\{F_{t,T}\}_{t\in[0,T]}$ is a martingale.

Assume that $F_{t,T}$ satisfies:

$$\begin{cases} dF_{t,T} = \sigma(F_{0,T}, t, F_{t,T}) F_{t,T} d\widetilde{W}_t, & \text{for } 0 \leq t \leq T \\ F_{0,T} \text{ is given and } F_{T,T} = S_T. \end{cases}$$

Dupire's Equation

Fix the current time at $t = 0$, European call options satisfy, with $T \leq T'$:

$$\begin{cases} \dfrac{\partial C}{\partial T} = \dfrac{1}{2}\sigma^2(F_{0,T'}, T, K)K^2\dfrac{\partial^2 C}{\partial K^2}, \; 0 < T < T', \; K \geq 0 \\ \lim\limits_{K\to 0} C(T,K) = F_{0,T'}, \; 0 < T < T', \\ \lim\limits_{K\to +\infty} C(T,K) = 0, \; 0 < T < T', \\ C(T=0,K) = (F_{0,T'} - K)^+, \text{ for } K > 0. \end{cases} \quad (1)$$

We need some technical adaptations.

Technical Points

Perform the change of variables

$$\tau = T \quad \text{and} \quad y = \log(K/F_{0,T'}).$$

Then define:

$$V(F_{0,T'}, \tau, y) := C(F_{0,T'}, \tau, F_{0,T'}e^y) \text{ and } a(F_{0,T'}, \tau, y) := \frac{1}{2}\sigma^2(F_{0,T'}, \tau, F_{0,T'}e^y).$$

Moreover, normalize the option prices by its underlying futures:

$$V(F_{0,T'}, \tau, y) = V(F_{0,T'}, \tau, y)/F_{0,T'}.$$

Thus, from the previous PDE we have the following problem:

A Surface of Option Prices

We also assume that

$$V(F_{0,T'},\tau,y) = V(S_0,\tau,y) \quad \text{and } a(F_{0,T'},\tau,y) = a(S_0,\tau,y).$$

Then, V satisfies:

$$\left\{\begin{array}{rcl} \dfrac{\partial V}{\partial \tau}(\tau,y) & = & a(S_0,\tau,y)\left(\dfrac{\partial^2 V}{\partial y^2}(\tau,y) - \dfrac{\partial V}{\partial y}(\tau,y)\right), \; T>0,\; y\in\mathbb{R} \\ \lim\limits_{y\to-\infty} V(\tau,y) & = & 1,\; \tau>0, \\ \lim\limits_{y\to+\infty} V(\tau,y) & = & 0,\; \tau>0, \\ V(\tau,y) & = & (1-e^y)^+, \text{ for } y\in\mathbb{R}. \end{array}\right. \tag{2}$$

It is independent of $F_{0,T}$!

We present some background properties of the forward operator.

The Forward Problem

- $a_1, a_2 \in \mathbb{R}$ s.t. $0 < a_1 \leq a_2 < +\infty$.
- a_0 in $H^{1+\varepsilon}(D)$, with $\varepsilon > 0$ and $a_1 \leq a_0 \leq a_2$.
- Define the set

$$Q := \{a \in a_0 + H^{1+\varepsilon}(D) : a_1 \leq a \leq a_2\}. \tag{3}$$

Proposition

If $a \in Q$, then the Cauchy problem of Dupire's Equation is a well-posed.

See, Crepey [3], De Cezaro-Scherzer-Zubelli [4] and Egger-Engl [6].

Online Setting

- Denote the index by $s \in [0, \bar{s}]$.
- The family of local volatility surfaces:

$$\mathcal{A} : s \in [0, \bar{s}] \longmapsto a(s; \tau, y) \in Q.$$

- The family of call prices given by Dupire's equation:

$$\mathcal{V} : (s, a(s)) \longmapsto V(a(s))$$

- Then define the *forward operator*:

$$\mathcal{F} : \mathcal{A} \longmapsto \mathcal{V}.$$

Properties of The Forward Operator[1]

Proposition

Under some regularity assumptions on the index, the forward operator $\mathcal{F}$ satisfies:

(i) *It is continuous and compact.*

(ii) *It is weakly continuous and weakly closed.*

(iii) *It is Frechét differentiable*

(iv) *It is injective.*

We now start the analysis of the inverse problem.

[1] V.A. & J.P. Zubelli, *Online Local Vol. Calib. by Convex Regularization with Morozov's Principle and Conv. Rates. Available on SSRN*

Local Volatility Calibration

Problem

- *Let $\widetilde{V}$ be a surface of European call option prices.*
- *Assume also that it is a solution of Dupire's equation.*
- *Then, find its correspondent local volatility surface $a^\dagger$, i.e., solve the equation*

$$\widetilde{V} = V(a^\dagger). \tag{4}$$

Dupire's formula:

$$a^\dagger = \frac{\widetilde{V}_\tau}{\widetilde{V}_{yy} - \widetilde{V}_y}.$$

The uncorrupted data should be known, at least, with continuous precision.

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$$a^\dagger = \frac{\widetilde{V}_\tau}{\widetilde{V}_{yy} - \widetilde{V}_y}.$$

The uncorrupted data should be known, at least, with continuous precision. That is unreasonable!

Option Price Data

Data Issues:

- Time-to-maturity $\times$ Strike - mesh is sparse.
- Missing prices for some strikes.
- Noise introduced by trading.
- Noise level varies with strike and maturity.

Let $\widetilde{V}$ denote the noiseless data, given by Dupire's eq.

The observed prices are denoted by V^δ, where

$$V^\delta = P(\widetilde{V} + E) \qquad \text{and} \qquad \delta := \|\widetilde{V} - V^\delta\|,$$

where E is the noise and
P is the observation operator.

Tikhonov-type Regularization

- The calibration problem is ill-posed.
- Recall that

$$Q := \{a \in a_0 + H^{1+\varepsilon}(D) : a_1 \leq a \leq a_2\}.$$

  Then, finding a solution to

$$\min\{\|V(a) - V^\delta\|^2 \ : \ \text{subjected to } a \in Q\}$$

  is not possible.

We then regularize it and solve:

Problem

Find an element of

$$\textit{argmin}\{\|V(a) - V^\delta\|^2 + \alpha f_{a_0}(a) \ : \ \textit{subjected to } a \in Q\}. \tag{5}$$

Tikhonov-type Regularization (cont.)

Under the *online* setting, it becomes:

Problem

Find an element of

$$\operatorname{argmin}\left\{\int_0^{\overline{s}} \|V(a(s)) - V^\delta(s)\|^2 ds + \alpha f_{\mathcal{A}_0}(\mathcal{A}) \ : \ \text{subjected to } \mathcal{A} \in \mathfrak{Q}\right\}, \quad (6)$$

where $\mathfrak{Q}$ *is the set of continuous trajectories*

$$\mathcal{A} : s \in [0, \overline{s}] \longmapsto a(s) \in Q.$$

The penalization functional $f_{\mathcal{A}_0}$ should be convex and coercive.

The regularization parameter α should be appropriately chosen.

Morozov's Discrepancy Principle

The choice of α is based on the discrepancy principle:

Definition

For $1 < \tau_1 \leq \tau_2$ *we choose* $\alpha = \alpha(\delta, u^\delta) > 0$ *such that*

$$\tau_1\delta \leq \|V(a^\delta_\alpha) - V^\delta\| \leq \tau_2\delta \tag{7}$$

holds for some a^δ_α *minimizer of the Tikhonov Functional.*

The same principle works under the *online* setting, i.e.,

$$\tau_1\delta \leq \frac{1}{\overline{s}} \int_0^{\overline{s}} \|V(a^\delta_\alpha(s)) - V^\delta(s)\| \leq \tau_2\delta$$

Some Canonical Examples of f_{a_0}

1. Quadratic Regularization:

$$f_{a_0}(a) = \|a - a_0\|^2_{L^2(D)}.$$

2. Smoothing Regularization:

$$f_{a_0}(a) = \beta_1\|a - a_0\|^2_{L^2(D)} + \beta_2\|\partial_x a - \partial_x a_0\|^2_{L^2(D)} + \beta_3\|\partial_\tau a - \partial_\tau a_0\|^2_{L^2(D)}.$$

   β_j should account discretization levels.

3. Kullback-Leibler:

$$f_{a_0}(a) = \int_{\mathbb{R}_+}\int_{\mathbb{R}}[\log(a(\tau,y)/a_0(\tau,y)) - (a_0(\tau,y) - a(\tau,y))]dyd\tau.$$

4. Total Variation:

$$f_{a_0}(a) = \|\partial_y a - \partial_y a_0\|_{L^1(D)} + \|\partial_\tau a - \partial_\tau a_0\|_{L^1(D)}.$$

Well-Posedness

Proposition

The level sets

$$\mu_\alpha(M) = \left\{ \mathcal{A} \in \mathfrak{Q} \; : \; \int_0^{\bar{s}} \|V(a(s)) - V^\delta(s)\|^2 ds + \alpha f_{\mathcal{A}_0}(\mathcal{A}) \leq M \right\}$$

are weakly pre-compact. The restriction of the forward operator $\mathcal{F}$ onto $\mu_\alpha(M)$ is weakly continuous.

Theorem (Existence)

Let $\alpha > 0$ and $\mathcal{A}_0$ be fixed. Then, the Tikhonov functional

$$\int_0^{\bar{s}} \|V(a(s)) - V^\delta(s)\|^2 ds + \alpha f_{\mathcal{A}_0}(\mathcal{A})$$

has a minimizer in $\mathfrak{Q}$.

Stability of Minimizers

Definition

A minimizer of the Tikhonov functional is stable if, for small perturbations on the data, there exists a minimizer correspondent to the perturbed data in its weak neighborhood.

Theorem (Stability)

Every minimizer of the Tikhonov functional is stable.

Convergence of Minimizers[1]

Theorem

The regularization parameter $\alpha = \alpha(\delta, \mathcal{U}^\delta)$ *obtained through Morozov's discrepancy principle satisfies:*

$$\lim_{\delta \to 0+} \alpha(\delta, \mathcal{U}^\delta) = 0 \quad \textit{and} \quad \lim_{\delta \to 0+} \frac{\delta^2}{\alpha(\delta, \mathcal{U}^\delta)} = 0.$$

Theorem

Let $\{\delta_k\}_{k\in\mathbb{N}}$ *be s.t.* $\delta_k \to 0$.
Let $\{V^{\delta_k}\}_{k\in\mathbb{N}}$ *be the sequence of noisy data, satisfying* $V^{\delta_k} \to \widetilde{V}$.
Then,

$$\mathcal{A}^{\delta_k}_{\alpha_k} \overrightarrow{w} \mathcal{A}^\dagger,$$

where $\mathcal{A}^\dagger$ *is the family of true local volatility surfaces.*

[1] V.A. & J.P. Zubelli, *Online Local Vol. Calib. by Convex Regularization with Morozov's Principle and Conv. Rates. Available on SSRN*

Convergence Rates[1]

Theorem (Convergence Rates)

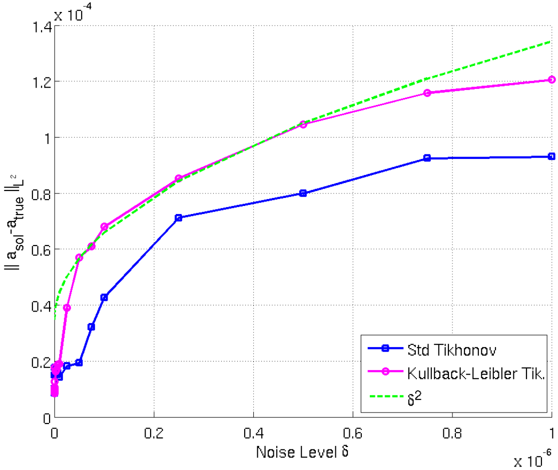
Assume that $\alpha = \alpha(\delta, u^\delta)$ *is chosen through the Morozov's discrepancy principle.*
Furthermore, assume that $f_{\mathcal{A}_0}(a) = \|\mathcal{A} - \mathcal{A}_0\|^2$.
Then

$$\|\mathcal{A}^\delta_\alpha - \mathcal{A}^\dagger\| = \mathcal{O}(\delta^{1/2}) \quad \textit{and} \quad \|V(a^\delta_\alpha) - V^\delta\| = \mathcal{O}(\delta),$$

where $a^\delta_\alpha \in Q$ *is the regularized solution.*

[1] V.A. & J.P. Zubelli, *Online Local Vol. Calib. by Convex Regularization with Morozov's Principle and Conv. Rates. Available on SSRN*

Convergence Rates[4]



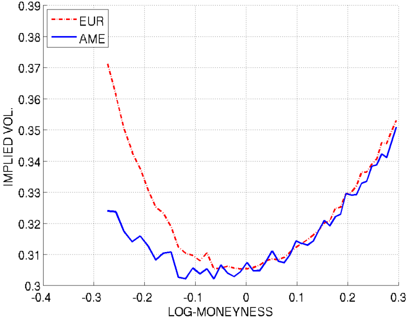
[4] V.A., A. De Cezaro & J.P. Zubelli, *Convex Regularization of Local Volatility Estimation in a Discrete Setting. Available on SSRN.*

European Prices from American Ones

- Assume Black's model: constant coefficients [2].
- Evaluate American implied volatilities from market prices.
- Then, evaluate European call prices, with such implied volatilities.

$$C_{\text{AME}} \xrightarrow{\text{Black AME Pricing}} \sigma_{\text{AME}} \xrightarrow{\text{B-S Formula}} C_{\text{EUR.}}$$

American and European Implied Volatilities: HH Nat. Gas



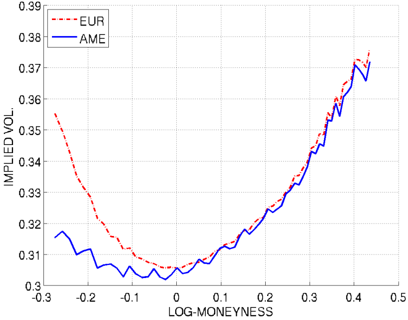
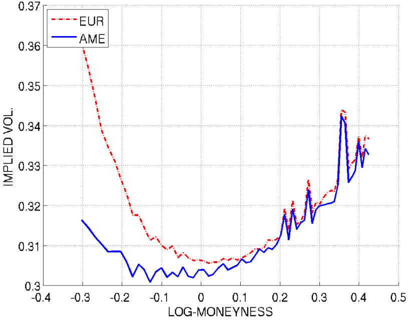


Figure: Left: Mat.:12/26/2013. Right: Mat.:01/28/2014

American and European Implied Volatilities: HH Nat. Gas



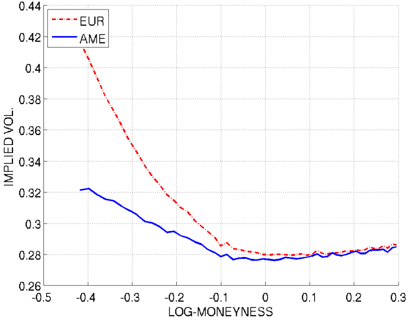
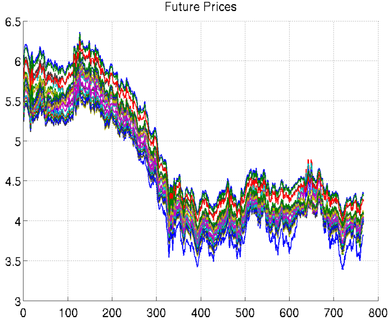


Figure: Left: Mat.:02/25/2014. Right: Mat.: 03/26/2014

Correlations

Futures on the same commodity for different maturities are highly correlated.



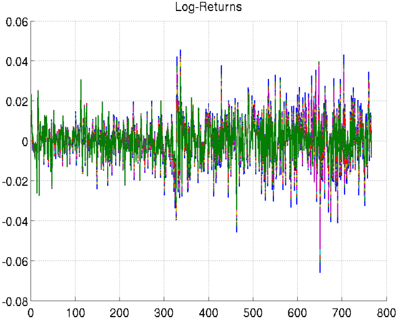


Figure: Example: Future prices and daily log-returns of Henry Hub nat. gas.

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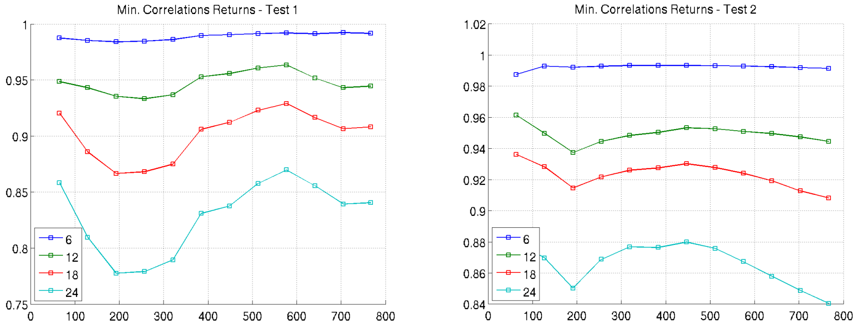


Figure: Minimum of correlations between daily log-returns - first and second tests.

Correlations

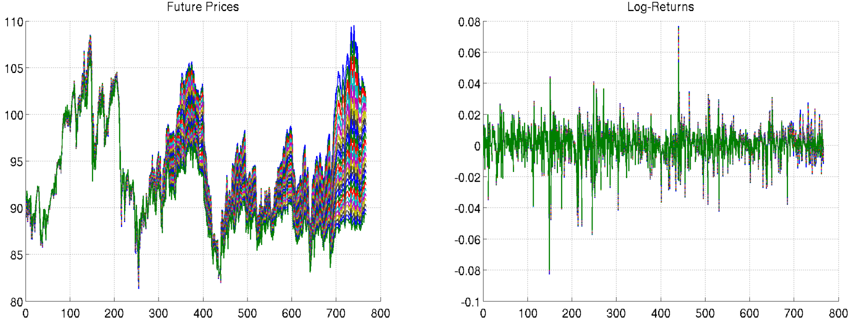


Figure: Example: Future prices and daily log-returns of WTI oil.

Correlations

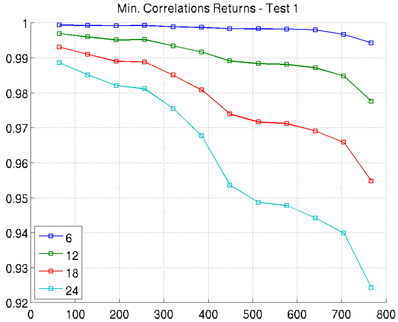


Figure: Minimum of correlations between daily log-returns - first and second tests.

Numerical Solution

- The forward problem is solved by a Crank-Nicolson scheme.
- The minimization of the online Tikhonov functional

$$\sum_{l,n,m} \Big(u(\mathcal{A}; s_l, \tau_n, y_m) - u^{\delta}(s_l, \tau_n, y_m) \Big)^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}), \tag{8}$$

  is solved by the Conjugate-Gradient method.
- The steps in iterations are chosen by the Wolfe rules.
- The stopping criteria is the Morozov discrepancy:

$$\tau_1 \delta \leq \| u(a) - u^{\delta} \| \leq \tau_2 \delta.$$

- We assume that the noise level is equal to half of the mean of the bid-ask length.

Synthetic Data: Local Volatility.

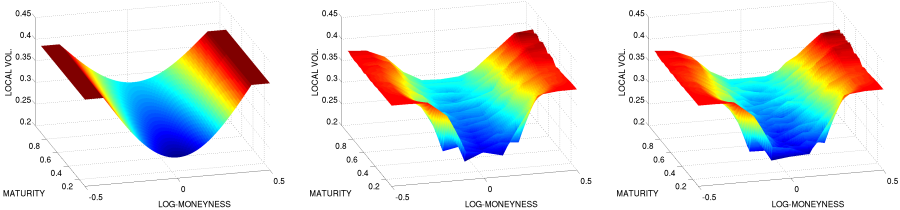
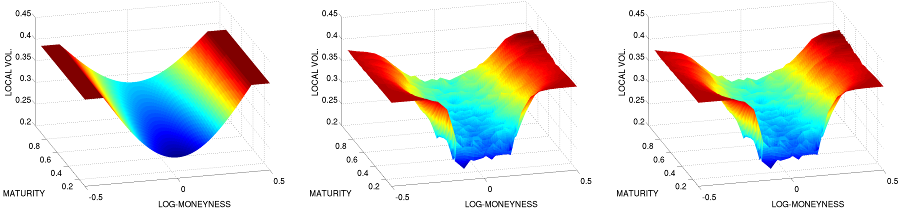


Figure: Left: Original. Center and right.: Reconstructions with noisy data.

Synthetic Data: Local Volatility.



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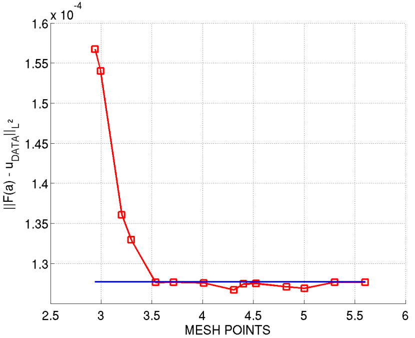
Synthetic Data: Residual and Error Evolutions.



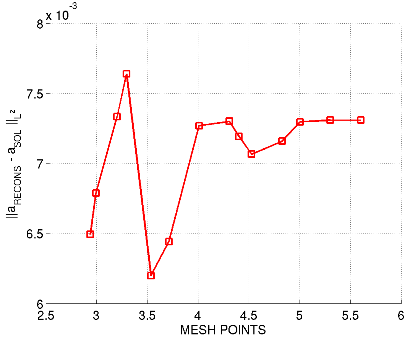


Figure: Left: Residual $\times$ discretization level. Right: Error $\times$ discretization level.

Online × Standard Calibration

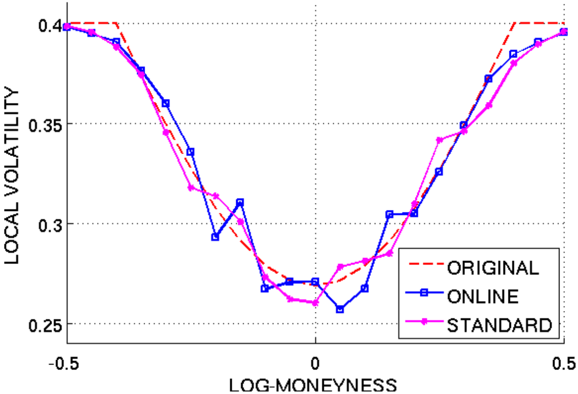
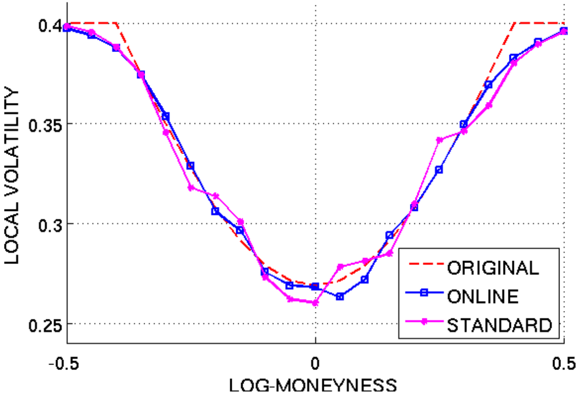


Figure: More data, better results!

Online × Standard Calibration



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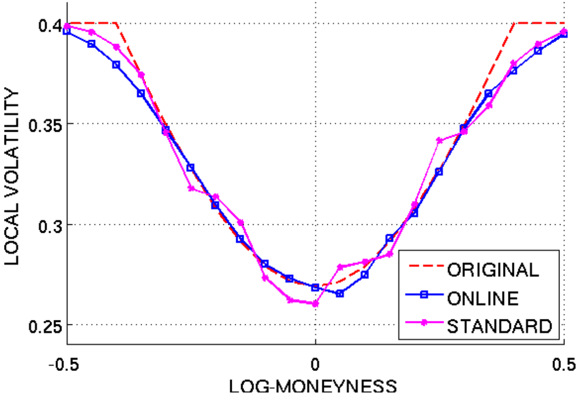

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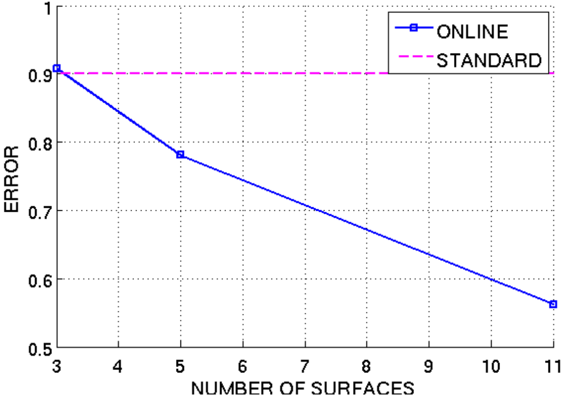
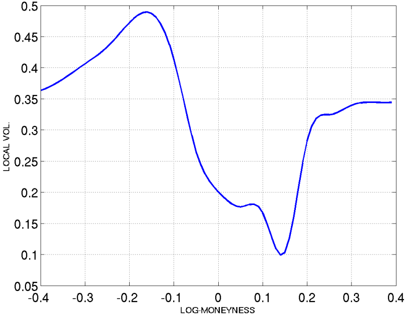


Figure: L^2 distance between original and reconstructed local vol.

WTI Local and Implied Volatilities



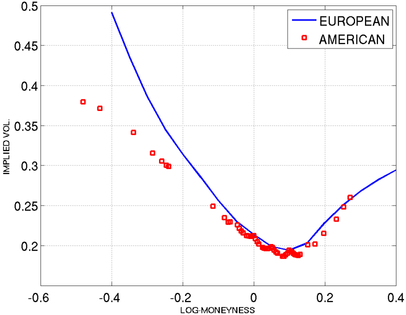
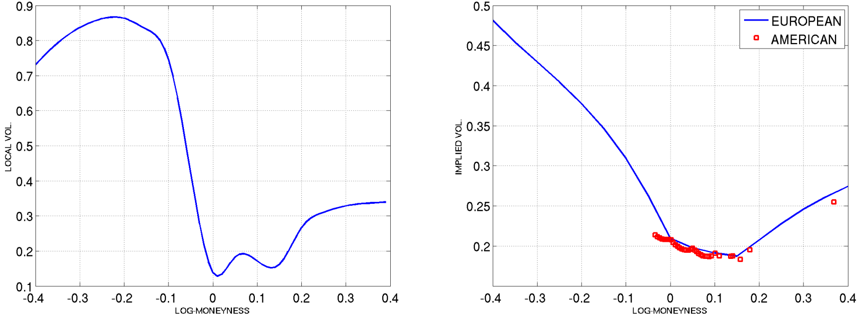


Figure: Left: Local Volatility. Right: Implied Vol. of model (cont.) and market (squares).

WTI Local and Implied Volatilities



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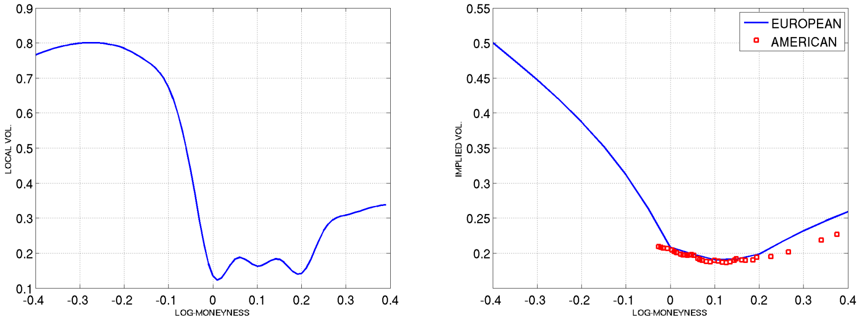

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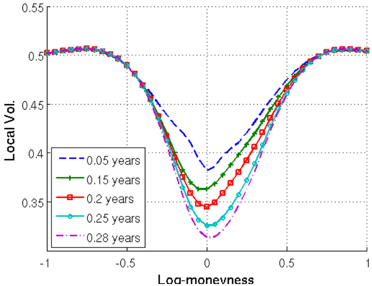
Local Vol.: Henry Hub Nat. Gas



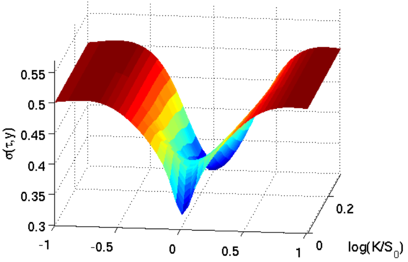
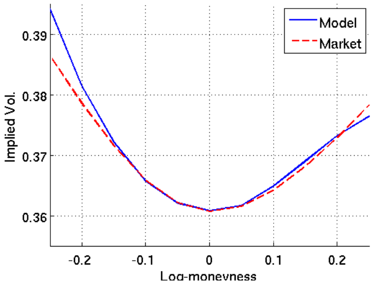


Figure: Left: local vol. reconstructed for some maturities. Right: reconstructed local vol. surface.

Implied Volatility Comparison



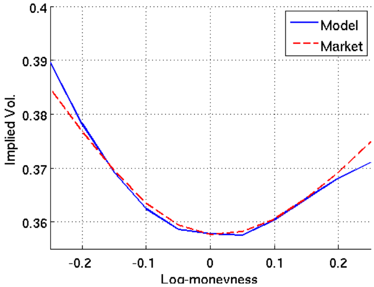
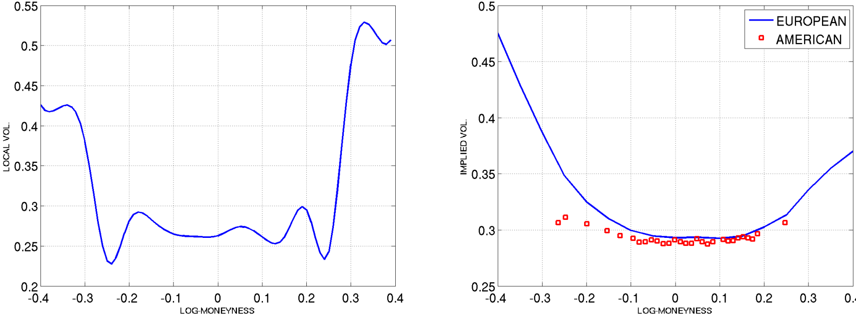


Figure: Implied vol. (Black) for market prices (dashed) and model prices (continuous) for two maturities.



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HH Local and Implied Volatilities

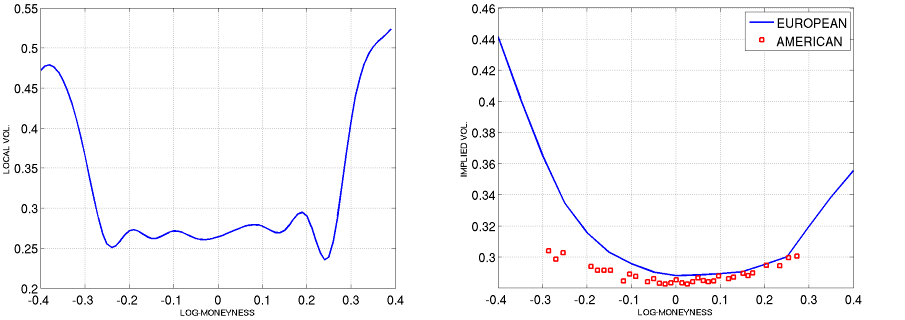

Figure: Left: Local Volatility. Right: Implied Vol. of model (cont.) and market (squares).

Conclusions and Future Directions

Conclusions:

- Applied Dupire's local vol. model to commodity markets.
- Solution of Local vol. calibration by convex regularization.
- *Online* setting: associate families of local volatility surfaces to call option prices.
- Morozov discrepancy principle.
- Numerical tests with market as well as synthetic data.
- The model has the required properties and the desirable one: robustness, reliability, simple calibration and smile adherence.

Future directions

- Application of particle and Kalman filtering techniques.
- Convex risk measures associated to local volatility.

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