On the Simulation and Calibration of Jump-Diffusion Models in Finance[2]

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[2]Joint work with J.P. Zubelli

Outline

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1. The Direct Problem
2. The Inverse Problem
3. Numerical Examples and Applications
4. Final Considerations

The Jump Diffusion Model

Let $(\Omega, \mathcal{V}, \mathcal{F}, \widetilde{\mathbb{P}})$ be a filtered prob. space.

The asset price S_t satisfies:

$$S_t = S_0 + \int_0^t r S_{t'-} dt' + \int_0^t \sigma(t', S_{t'-}) S_{t'} dW_{t'} + \int_0^t \int_{\mathbb{R}} S_{t'-}(e^y - 1)\tilde{N}(dt' dy), \quad 0 \leq t \leq T,$$

where W is a Brownian motion,
$\tilde{N}$ is the compensated Poisson prob. measure on $[0, T] \times \mathbb{R}$,
N is the poisson measure and the compensator is $\nu(dy)dt$.
Cont and Tankov (2003).
This model allows asset price to jump. This happens quite often in practice!
Recall 2007/2008...

Vanilla Options

- **European Call**: gives the right, but not the obligation, of buying a share of an asset for a fixed strike price at its maturity.
- **Euopean Put** similar to the call, but gives the right of selling.
- American Option (call and put) can be exercised any time before its maturity.
- Sometimes, American options are more expensive than the European ones.
- The prices of such contract take into account asset dynamics.

European Call Prices

An European call option price is given by:

$$C(t, S_t, T, K) = \mathrm{e}^{-r(T-t)}\widetilde{\mathbb{E}}[\max\{0, S_T - K\} \,|\, \mathcal{F}_t].$$

If $\sigma > 0$ and the compensator ν satisfies $\int_{|y|>1} \mathrm{e}^{2y}\nu(dy)$, if, $\sigma < K$ then, by setting $t = 0$ and denoting τ the time to maturity and K the strike price, by Bentata and Cont (2015) the price of an European call option is the unique weak solution of

$$C_\tau(\tau, K) - \frac{1}{2}K^2\sigma(\tau, K)^2 C_{KK}(\tau, K) + rKC_K(\tau, K) = \int_{\mathbb{R}} \nu(dz)\mathrm{e}^z \left(C(\tau, K\mathrm{e}^{-z}) - C(\tau, K) - (\mathrm{e}^{-z} - 1)KC_K(\tau, K)\right),$$

with $\tau \geq 0$, $K > 0$, and the initial condition

$$C(0, K) = \max\{0, S_0 - K\}, \; K > 0. \tag{1}$$

Change of Variables

Make the change of variable $y = \log(K/S_0)$ and define

$$a(\tau, y) = \frac{1}{2}\sigma(\tau, S_0 e^y)^2 \text{ and } u(\tau, y) = C(\tau, S_0 e^y)/S_0.$$

So, defining $D = [0, T] \times \mathbb{R}$, the PIDE problem becomes

$$u_\tau(\tau, y) - a(\tau, y)\left(u_{yy}(\tau, y) - u_y(\tau, y)\right) + r u_y(\tau, y) = \int_{\mathbb{R}} \nu(dz) e^z \left(u(\tau, y - z) - u(\tau, y) - (e^{-z} - 1) u_y(\tau, y)\right),$$

with $\tau \geq 0$, $y \in \mathbb{R}$, and the initial condition

$$u(0, y) = \max\{0, 1 - e^y\}, \; y \in \mathbb{R}.$$

Double Exponential Tail

Instead of using ν in the PIDE, consider, as in Kindermann and Mayer (2011), the double-exponential tail of ν:

$$\varphi(y) = \varphi(\nu; y) = \begin{cases} \int_{-\infty}^{y} (e^y - e^x)\nu(dx), & y < 0 \\ \int_{y}^{\infty} (e^x - e^y)\nu(dx), & y > 0, \end{cases}$$

and the convolution operator

$$I_\varphi f(y) := \varphi * f(y) = \int_{\mathbb{R}} \varphi(y - x) f(x) dx.$$

Applying Lemma 2.6 in Bentata and Cont (2015) to the integral part of the PIDE, it follows that

$$\int_{\mathbb{R}} \nu(dz) e^z \left(u(\tau, y - z) - u(\tau, y) - (e^{-z} - 1) u_y(\tau, y)\right)$$
$$= \int_{\mathbb{R}} \varphi(y - z)(u_{yy}(\tau, z) - u_y(\tau, z)) dy.$$

The Domain of The Parameter to Solution Map

Assume that the restrictions of φ to $(-\infty,0)$ and $(0,+\infty)$ are in $W^{2,1}(-\infty,0)$ and $W^{2,1}(0,+\infty)$, respectively.
Consider the Banach space

$$X = H^{1+\varepsilon}(D) \times W^{2,1}(-\infty,0) \times W^{2,1}(0,+\infty),$$

let $0 < \underline{a} \leq \overline{a} < \infty$ be fixed constants and
$a_0 : D \to (\underline{a},\overline{a})$ be a fixed continuous function s.t. its weak derivatives are in $L^2(D)$.
Define:

$$\mathcal{D}(F) = \{(\tilde{a},\varphi_-,\varphi_+) \in X \ : \text{ let } a = \tilde{a} + a_0, \text{ be s.t. } \underline{a} \leq a \leq \overline{a}, \\ \text{let } \varphi \text{ be s.t., } \varphi = \varphi_- \text{ in } (-\infty,0) \text{ and } \varphi = \varphi_+ \text{ in } (0,+\infty)\}$$

For simplicity, write $(a,\varphi) \in \mathcal{D}(F)$, meaning that a and φ are given as in the definition of $\mathcal{D}(F)$.

Well-Posedness

Proposition

Let (a,φ) be in $\mathcal{D}(F)$, in addition, assume that $\|\varphi\|_{L^1(D)} < C^{-1}$, where the constant C depends on $\underline{a}$, $\overline{a}$ and r. Then, there exists a unique solution of the PIDE problem in $W^{1,2}_{2,loc}(D)$.

The Parameter-to-Solution Map

Definition

The direct operator $F : \mathcal{D}(F) \to W_2^{1,2}(D)$*, that associates* $(\tilde{a}, \varphi_-, \varphi_+)$ *to* $u(a,\varphi) - u(a_0, 0)$*, where* $u(a,\varphi)$ *is the solution of the PIDE problem, with* (a,φ) *in* $\mathcal{D}(F)$*.*

$F(\tilde{a}, \varphi_-, \varphi_+)$ is the solution of a PIDE problem with homogeneous boundary condition and source term $f = -I_\varphi(u(a_0,0)_{yy} - u(a_0,0)_y)$.

Properties of the Parameter-to-Solution Map

Proposition

1. *F is continuous.*
2. *F is weakly continuous and compact.*
3. *F is Frechét differentiable and satisfies*

$$\|F(a+h_1,\varphi+h_2) - F(a,\varphi) - F'(a,\varphi)h\|_{W^{1,2}_2(D)} \leq \frac{C}{1-K}\|h\|_X\|F(a+h_1,\varphi+h_2) - F(a,\varphi)\|_{W^{1,2}_2(D)},$$

for any $(a,\varphi) \in \mathcal{D}(F)$ *and any* $h = (h_1,h_2) \in X$*, s.t.* $(a+h_1,\varphi+h_2) \in \mathcal{D}(F)$.

Outline

The Inverse Problem and The Data

- Let $\tilde{v} = \tilde{v}(\tau, y)$ be a surface of European call option prices.
- Assume that it is given by PIDE problem.
- So, the corresponding pair $(a^{\dagger}, \varphi^{\dagger})$, solves the *inverse problem*

$$\tilde{u} = u(a^{\dagger}, \varphi^{\dagger}).$$

Unfortunately, only scarce and noisy dataset v^{δ} is available. v^{δ} and $\tilde{v}$ are related by

$$\|\tilde{u} - u^{\delta}\| \leq \delta,$$

with $\delta > 0$ (noise level).

Tikhonov-type Regularization: The Functional

Since the inverse problem can be ill-posed, Tikhonov-type regularization is applied:
Find an element of $\mathcal{D}(F)$ *that minimizes*

$$\mathcal{F}(x) = \phi(x) + \alpha f_{x_0}(x),$$

where

$$\phi(x) = \|F(x) - y^\delta\|_Y^p$$

is the data misfit or merit function, $\alpha > 0$ *is the regularization parameter and* f_{x_0} *is called regularization functional.*
The minimizers of the Tikhonov functional in $\mathcal{D} := \mathcal{D}(F) \cap \mathcal{D}(f_{x_0})$ are called Tikhonov minimizers or reconstructions, and are denoted by x_α^δ.

Tikhonov-type Regularization: Assumptions

Assumption (Scherzer et al. (2008))

1. *T_X and T_Y are topologies associated to X and Y, respectively, weaker than the norm topologies.*
2. *The exponent in the misfit satisfies $p \geq 1$.*
3. *The norm of Y is sequentially lower semi-continuous w.r.t. T_Y.*
4. *f_{x_0} is convex and sequentially lower semi-continuous w.r.t. T_X.*
5. *The objective set satisfies $\mathcal{D} \neq \emptyset$*
6. *For every $\alpha > 0$ and $M > 0$ the level set*

   $$M_\alpha(M) := \{x \in \mathcal{D} \; : \; \mathcal{F}(x) \leq M\}$$

   is sequentially pre-compact w.r.t. T_X.
7. *For every $\alpha, M > 0$, $M_\alpha(M)$ is sequentially closed w.r.t. T_X and the restriction of F to $M_\alpha(M)$ is sequentially continuous w.r.t. T_X and T_Y.*

Tikhonov-type Regularization: Well-Posedness

Proposition (See Scherzer et al. (2008))

1. *The existence of stable Tikhonov minimizers is guaranteed.*
2. *In addition, some sequences of Tikhonov minimizers converge to some f_{x_0}-minimizing solution whenever $\delta \to 0$ and $\alpha = \alpha(\delta)$ satisfies the limits:*

$$\lim_{\delta \to 0} \alpha(\delta) = 0 \quad \textit{and} \quad \lim_{\delta \to 0} \frac{\delta^p}{\alpha(\delta)} = 0.$$

Penalty Terms

$$g_{a_0}(a) = \|a - a_0\|^2_{H^{1+\varepsilon}(D)}$$

for the variable a and for the variable φ

$$h_{\varphi_0}(\varphi) = KL(\varphi_+|\varphi_{+,0}) + KL(\varphi'_+|\varphi'_{+,0}) + KL(\varphi''_+|\varphi''_{+,0}) + KL(\varphi_-|\varphi_{-,0}) + KL(\varphi'_-|\varphi'_{-,0}) + KL(\varphi''_-|\varphi''_{-,0})$$

where the KL stands for the Kullback-Leibler divergence

$$KL(\varphi_+|\varphi_{+,0}) = \int_0^{+\infty} \left[\varphi_+ \ln\left(\frac{\varphi_+}{\varphi_{+,0}}\right) + (\varphi_{+,0} - \varphi_+)\right] dx,$$

with $\varphi_0 > 0$ given.

g_{a_0} and h_{φ_0} are convex, weakly continuous and coercive. In addition, the level sets of the Kullback-Leibler divergence

$$\{\varphi \in L^1(\mathbb{R}) \ : \ KL(\varphi|\varphi_0) \leq C\}$$

are weakly pre-compact in $L^1(\mathbb{R})$. See Lemma 3.4 in Resmerita and Anderssen (2007).

The Splitting Strategy: Notation

Define the projections $P_1 : (a,\varphi) \mapsto a$ and $P_2 : (a,\varphi) \mapsto \varphi$.
For each φ, define:

1. the operator F_φ as $F_\varphi(a) = F(a,\varphi)$,
2. the Tikhonov-type functional $\mathcal{F}_\varphi(a) = \mathcal{F}(a,\varphi)$,
3. and the set $\mathcal{D}_\varphi = P_1(\mathcal{D}) \times \{\varphi\}$.

Similarly, define F_a, $\mathcal{F}_a$ and $\mathcal{D}_a$.

Proposition

Whenever $\mathcal{D}$ is replaced by $P_1(\mathcal{D})$ or $P_2(\mathcal{D})$, F by F_a or F_φ and $\mathcal{F}$ by $\mathcal{F}_a$ or $\mathcal{F}_\varphi$, existence of stable Tikhonov minimizers is guaranteed, for each $a \in P_1(\mathcal{D})$ and $\varphi \in P_2(\mathcal{D})$.

The Splitting Strategy: The Algorithm

For any $a \in P_1(\mathcal{D})$ (or $\varphi \in P_2(\mathcal{D})$), set $a^0 = a$ ($\varphi^0 = \varphi$) and consider the iterations with $n \in \mathbb{N}$:

$$\varphi^n \in \operatorname{argmin}\left\{\mathcal{F}_{a^{n-1}}(\varphi) \ : \ \varphi \in P_2(\mathcal{D})\right\}$$
$$a^n \in \operatorname{argmin}\left\{\mathcal{F}_{\varphi^n}(a) \ : \ a \in P_1(\mathcal{D})\right\}.$$

Repeat the iterations until some termination criteria.
If the algorithm starts with φ instead of a, the order of the two iterations must be reversed.

The Splitting Strategy: Stationary Points

Definition

A stationary point of the functional $\mathcal{F}$ is some point $\hat{x} = (\hat{a}, \hat{\varphi}) \in \mathcal{D}$, such that

$$\hat{a} \in \textit{argmin}\{\mathcal{F}_{\hat{\varphi}}(a) \;:\; a \in P_1(\mathcal{D})\} \;\; \textit{and} \;\; \hat{\varphi} \in \textit{argmin}\{\mathcal{F}_{\hat{a}}(\varphi) \;:\; \varphi \in P_2(\mathcal{D})\}.$$

Proposition

For every initializing pair $(w, z) \in \mathcal{D}$, any convergent subsequence produced by the splitting algorithm converges to some "stable" stationary point of $\mathcal{F}$.

Stable w.r.t. perturbations in the data.

The Splitting Strategy: Regularization Technique

Proposition

If the initializing pair and $x^\dagger$ is inside some ball $B(x^*;r)$ and $\lambda > (1+\eta)/(1-\eta)$ is fixed, then there exist constants $\alpha_1, \alpha_2 > 0$ such that for a finite n, the iterates of the splitting algorithm satisfy

$$\|F(w^n, z^n) - y^\delta\|_Y \geq \lambda\delta > \|F(w^{n+1}, z^{n+1}) - y^\delta\|_Y.$$

Proposition

Every sequence of solutions obtained by the splitting algorithm, satisfying the discrepancy in the previous proposition, when $\delta \searrow 0$, has a subsequence converging w.r.t. T_X to some solution of the inverse problem.

The parameter-to-solution map satisfies all the conditions to apply the splitting strategy.

Outline

Numerical Solution

- The PIDE is solved by a Crank-Nicolson-like scheme, with the integral in the explicit part.
- The minimization of both Tikhonov-type functionals are solved by the gradient descent method.
- The iterations cease whenever the tolerance is satisfied:

$$\frac{\|u(a^k,\varphi^k) - u^\delta\|}{\|u^\delta\|} < tol,$$

typically $tol = 0.01$.

Synthetic Data: Local Volatility Calibration

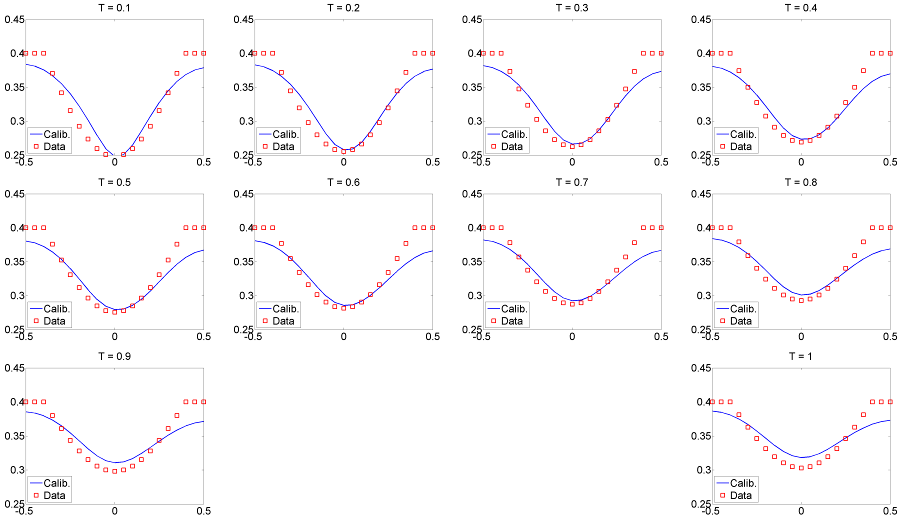

Figure: Original and Calibrated Local volatility surfaces.

Synthetic Data: Local Volatility Calibration

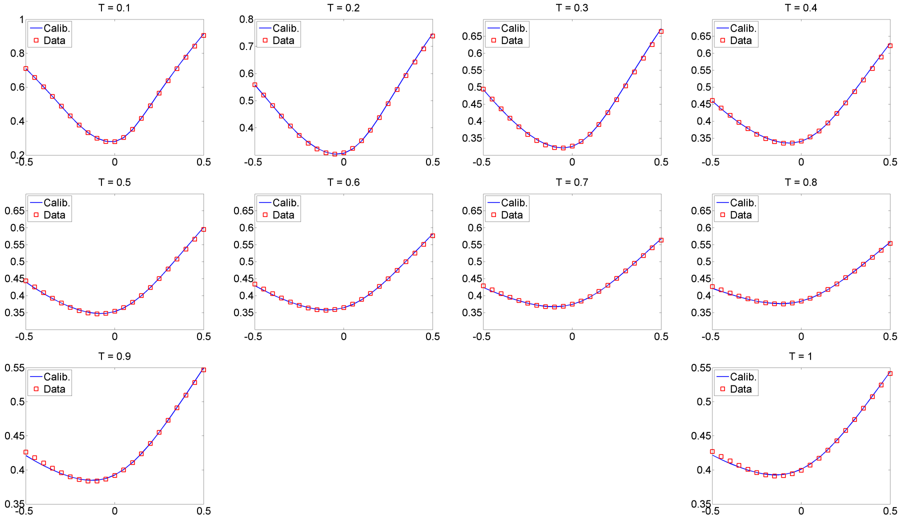

Figure: Adherence to data: Implied volatilities

Synthetic Data: Double Exponential Tail and Jump-Size Dist.

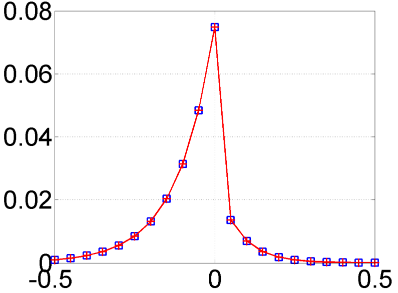

Figure: Left: true (line with crosses) and reconstructed (line with squares) double-exponential tail functions. Right: true (line with crosses) and reconstructed (line with squares) jump-size distributions.

Synthetic Data: Splitting Strategy and Local Vol. Calibration

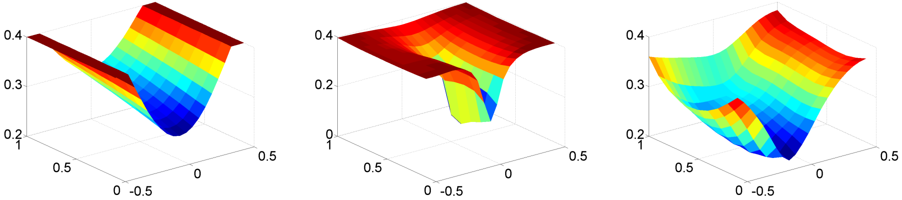

Figure: Reconstruction of the local volatility surface: original (left), after one step (center) and after two steps (right).

Synthetic Data: Splitting Strategy and Double Exp. Tail Calib.

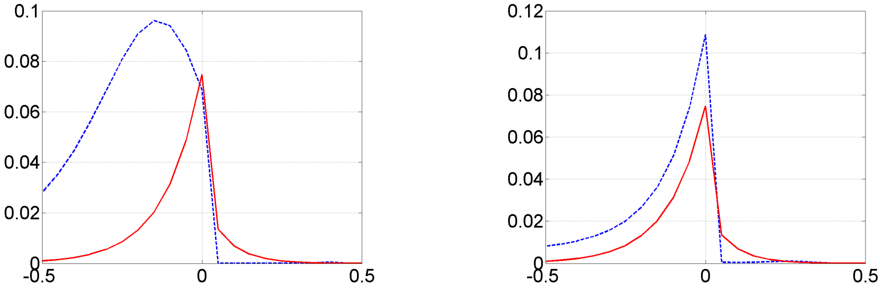

Figure: Reconstruction of the double exponential tail: after one step (left) and after two steps (right). Continuous line: true. Dashed line: reconstruction.

The Splitting Algorithm with DAX Options

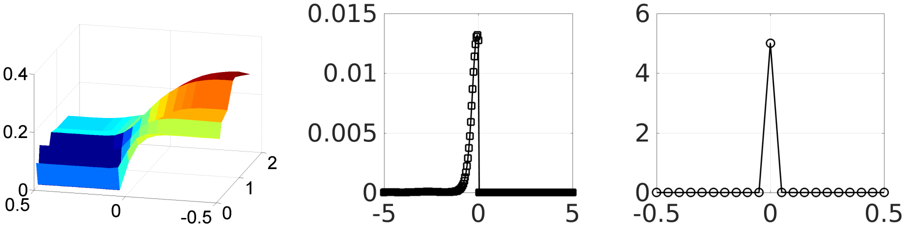

Figure: Reconstructions from Dax options of local volatility surface (left), double exponential tail (center) and jump-size density function (right).

The Splitting Algorithm with DAX Options

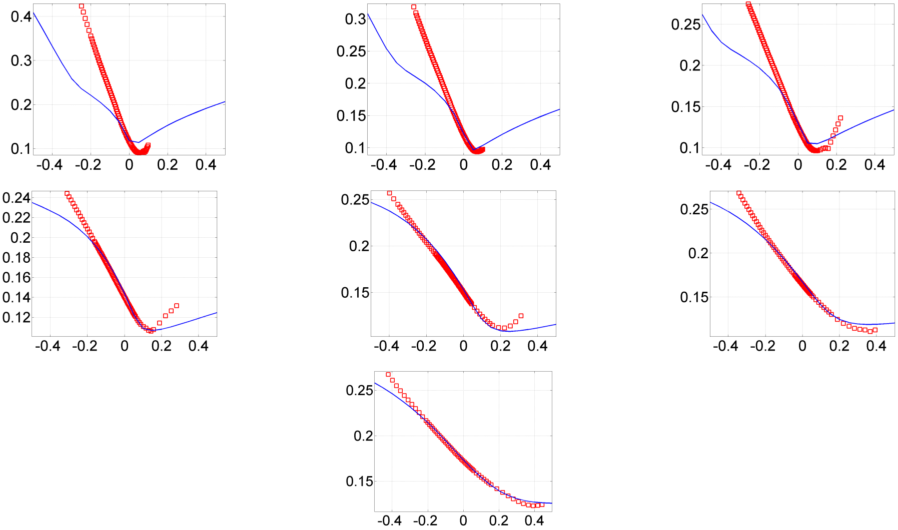

Figure: Market (squares) and model (continuous line) implied volatility of DAX options.

Outline

Final considerations

- We have considered the simultaneous calibration of local vol. and jump-size dist.
- We have stated the regularity properties of the parameter-to-solution map.
- Tikhonov-type regularization was used to solve the inverse problems separately.
- We have applied a splitting strategy to solve the simultaneous calib. prob.
- We provided numerical examples.
- We also provided examples with real data.

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