Optimal Convergence Rates Results for Linear Inverse Problems in Hilbert Spaces

Vinicius Albani

Joint work with:
P. Elbau, M. de Hoop and O. Scherzer.

Computational Science Center
University of Vienna

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Outline

Preliminaries

- Let X and Y be real Hilbert spaces.
- Let $L : X \longrightarrow Y$ be a bounded linear operator.
- Given $y \in \mathcal{R}(L)$, we want to find $x \in X$ s.t.

$$Lx = y.$$

- However, we only observe $\tilde{y} \in Y$ s.t.

$$\|y - \tilde{y}\| \leq \delta.$$

- Since L can be non-injective, we look for

$$x^{\dagger} \in \text{argmin}\{\|x\| \ : \ Lx = y\}.$$

Tikhonov Regularization

Since the inverse of L is not necessarily continuous, we apply Tikhonov regularization and search for

$$x_\alpha(\tilde{y}) \in \operatorname{argmin}\{\|Lx - \tilde{y}\|_Y^2 + \alpha\|x\|_X^2 \ : \ x \in X\},$$

which is given by

$$x_\alpha(\tilde{y}) = (L^*L + \alpha I)^{-1} L^*\tilde{y}.$$

When $\delta \to 0$ and $\tilde{y}(\delta) \to y$, if $\alpha = \alpha(\delta)$ is appropriately chosen, then

$$x_{\alpha(\delta)}(\tilde{y}(\delta)) \to x^\dagger,$$

where the convergence is in norm and $x^\dagger$ is the minimum-norm sol.

General Regularization

More generaly, we define the regularized solutions $x_\alpha(\tilde{y})$ by

$$x_\alpha(\tilde{y}) = r_\alpha(L^*L)L^*\tilde{y}$$

where the continuous function $r_\alpha : [0,\infty) \to [0,\infty)$ satisfies:

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where the continuous function $r_\alpha : [0,\infty) \to [0,\infty)$ satisfies:

- there exists a constant $\rho \in (0,1)$ s.t.

  $$r_\alpha(\lambda) \leq \min\left\{\frac{1}{\lambda}, \frac{\rho}{\sqrt{\alpha\lambda}}\right\} \text{ for every } \lambda > 0, \alpha > 0,$$

  the error function $\tilde{r}_\alpha : [0,\infty) \to [0,\infty)$, defined by

  $$\tilde{r}_\alpha(\lambda) = (1 - \lambda r_\alpha(\lambda))^2,$$

  is decreasing,
- the map $\alpha \mapsto \tilde{r}_\alpha(\lambda)$ is continuous and increasing for each $\lambda \geq 0$, and
- there exists a constant $\tilde{\rho} \in (0,1)$ s.t.

$$(1-\rho)^2 \leq \tilde{r}_\alpha(\alpha) < \tilde{\rho} \quad \text{for all} \quad \alpha > 0.$$

Spectral Characterization

We search for conditions that imply a convergence rate

$$\|x_{\alpha(\delta)}(\tilde{y}) - x^\dagger\| = \mathcal{O}(\varphi(\delta))$$

for suitable functions α and φ.

In order to do that we make use of the spectral measure $E : \mathcal{B}([0,\infty)) \to \mathcal{L}(X,X)$

- $E_A : X \to X$ is a projection,
- $E_\emptyset = 0$ and $E_{[0,\infty)} = I$,
- $E_{A\cap B} = E_A E_B$,
- $E_{\cup_{n\in\mathbb{N}} A_n} = \sum_{n\in\mathbb{N}} E_{A_n}$ for pairwise disjoint sets A_n,
- for every bounded and continuous real function g, and all $x, \tilde{x} \in X$,

$$\langle \tilde{x}, g(L^*L)x\rangle = \int_0^\infty g(\lambda) d\langle \tilde{x}, E_\lambda x\rangle,$$

where $A \mapsto \langle \tilde{x}, E_A x\rangle$ is a measure.

Convergence Rates for Exact Data

Proposition

Let $\varphi : (0,\infty) \to (0,\infty)$ *be an increasing function with the property*

$$\varphi(\lambda)\tilde{r}_\alpha(\lambda)^\mu \leq A\varphi(\alpha), \quad \textit{for all } \lambda > 0, \alpha > 0$$

with $\mu \in (0,1)$ *and* $A > 0$ *fixed constants.*
Then,

$$\|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(\varphi(\alpha)) \quad \Leftrightarrow \quad \|E_{[0,\lambda]}x^\dagger\|^2 = \mathcal{O}(\varphi(\lambda)).$$

Proof of Proposition 1

For the first implication:
By the definition of $\tilde{r}_\alpha$, we can write, if $0 < \alpha \le \|L\|^2$,

$$\|x_\alpha(y) - x^\dagger\|^2 = \|\tilde{r}_\alpha(L^*L)^{\frac{1}{2}}x^\dagger\|^2 = \int_0^{\|L\|^2} \tilde{r}_\alpha(\lambda)d\|E_\lambda x^\dagger\|^2$$
$$\ge \int_0^\alpha \tilde{r}_\alpha(\lambda)d\|E_\lambda x^\dagger\|^2 \ge \tilde{r}_\alpha(\alpha)\|E_{[0,\alpha]}x^\dagger\|^2$$
$$\ge (1-\rho)^2\|E_{[0,\alpha]}x^\dagger\|^2 \quad (1)$$

Since we have assumed that

$$\|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(\varphi(\alpha)),$$

the result follows.

Proof of Proposition 1

For the converse, we integrate by parts to find:

$$\|x_\alpha(y) - x^\dagger\|^2 = \tilde{r}_\alpha(\|L\|^2)\|x^\dagger\| + \int_0^\alpha \|E_{[0,\lambda]}x^\dagger\|^2 d(-\tilde{r}_\alpha)(\lambda) + \int_\alpha^{\|L\|^2} \|E_{[0,\lambda]}x^\dagger\|^2 d(-\tilde{r}_\alpha)(\lambda). \quad (2)$$

The first term can be estimated with the condition $\varphi(\lambda)\tilde{r}_\alpha(\lambda)^\mu \leq A\varphi(\alpha)$:

$$\tilde{r}_\alpha(\|L\|^2)\|x^\dagger\| = \mathcal{O}(\varphi(\alpha)^{\frac{1}{\mu}}).$$

For the second term we use the hypothesis $\|E_{[0,\lambda]}x^\dagger\|^2 = \mathcal{O}(\varphi(\lambda))$:

$$\int_0^\alpha \|E_{[0,\lambda]}x^\dagger\|^2 d(-\tilde{r}_\alpha)(\lambda) \leq \|E_{[0,\alpha]}x^\dagger\|^2(1 - \tilde{r}_\alpha(\alpha)) \leq (1-(1-\rho)^2)\|E_{[0,\alpha]}x^\dagger\|^2 = \mathcal{O}(\varphi(\alpha)) \quad (3)$$

Proof of Proposition 1

For the last term, we use again the hypothesis $\|E_{[0,\lambda]}x^\dagger\|^2 = \mathcal{O}(\varphi(\lambda))$ and the condition $\varphi(\lambda)\tilde{r}_\alpha(\lambda)^\mu \leq A\varphi(\alpha)$:

$$\begin{aligned}\int_\alpha^{\|L\|^2} \|E_{[0,\lambda]}x^\dagger\|^2 d(-\tilde{r}_\alpha)(\lambda) &\leq C\int_\alpha^{\|L\|^2} \varphi(\lambda) d(-\tilde{r}_\alpha)(\lambda)\\ &\leq C\cdot A\varphi(\alpha)\int_\alpha^{\|L\|^2} \tilde{r}_\alpha(\lambda)^{-\mu} d(-\tilde{r}_\alpha)(\lambda)\\ &= \frac{C\cdot A}{1-\mu}\varphi(\alpha)(\tilde{r}_\alpha(\alpha)^{1-\mu} - \tilde{r}_\alpha(\|L\|^2)^{1-\mu}) = \mathcal{O}(\varphi(\alpha)) \qquad (4)\end{aligned}$$

This ends the proof.

Applications

In the case of Tikhonov regularization we have:

$$r_\alpha(\lambda) = \frac{1}{\alpha+\lambda} \qquad \text{and} \qquad \tilde{r}_\alpha(\lambda) = (1-\lambda r_\alpha(\lambda))^2 = \frac{\alpha^2}{(\alpha+\lambda)^2}.$$

In particular $\tilde{r}_\alpha(\alpha) = \frac{1}{4}$.
To recover the result in Neubauer (1997), let us consider $\varphi(\alpha) = \alpha^{2\nu}$, then

$$\tilde{r}_\alpha(\lambda)^\mu \varphi(\lambda) = \frac{\alpha^{2\mu}\lambda^{2\nu}}{(\alpha+\lambda)^{2\mu}} = \frac{\alpha^{2\mu-2\nu}}{(\alpha+\lambda)^{2\mu-2\nu}} \frac{\lambda^{2\nu}}{(\alpha+\lambda)^{2\nu}} \alpha^{2\nu} \le \alpha^{2\nu},$$

for every $\mu \ge \nu$ with $\nu \in (0,1)$, and arbitrary $\alpha, \lambda > 0$.
So, the proposition gives us the equivalence

$$\|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(\alpha^{2\nu}) \quad \Leftrightarrow \quad \|E_{[0,\lambda]}x^\dagger\|^2 = \mathcal{O}(\lambda^{2\nu}).$$

Applications

Let us consider logarithmic convergence rates in Tikhonov regularization, i.e., set:

$$r_\alpha(\lambda) = \frac{1}{\alpha+\lambda}, \quad \tilde{r}_\alpha(\lambda) = \frac{\alpha^2}{(\alpha+\lambda)^2} \quad \text{and} \quad \varphi(\alpha) = \frac{1}{|\log\alpha|^\nu}.$$

Let also $0 < \nu < \mu < 1$ and $0 < \alpha \leq e^{-\frac{\nu}{\mu}}$. So, the map $\lambda \mapsto \varphi(\lambda)\tilde{r}_\alpha(\lambda)^\mu$ is decreasing on $[\alpha, e^{-\frac{\nu}{\mu}})$:

$$(\varphi\tilde{r}_\alpha^\mu)'(\lambda) = \frac{\alpha^{2\mu}}{(\alpha+\lambda)^{2\mu+1}|\log\lambda|^{\nu+1}}\left(\nu\frac{\alpha+\lambda}{\lambda} - 2\mu|\log\lambda|\right)$$
$$\leq -\frac{2(\mu|\log\lambda| - \nu)\alpha^{2\mu}}{(\alpha+\lambda)^{2\mu+1}|\log\lambda|^{\nu+1}} \leq 0 \qquad (5)$$

So, $\varphi(\lambda)\tilde{r}_\alpha(\lambda)^\mu \leq \varphi(\alpha)$, and $A = 1$.
Therefore, we have the equivalence:

$$\|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(|\log\alpha|^{-\nu}) \quad \Leftrightarrow \quad \|E_{[0,\lambda]}x^\dagger\|^2 = \mathcal{O}(|\log\lambda|^{-\nu}).$$

Convergence Rates for Noisy Data

Proposition

Let $\varphi : [0,\infty) \to [0,\infty)$ *be a strictly increasing function s.t.* $\varphi(0) = 0$ *and*

$$\varphi(\gamma\alpha) \leq g(\gamma)\varphi(\alpha) \quad \textit{for all} \quad \alpha > 0, \gamma > 0$$

for some increasing function $g : (0,\infty) \to (0,\infty)$.
Moreover, we assume that there exist constants $C > 0$ *and* $\tilde{C} > 0$ *s.t.*

$$\frac{\tilde{r}_\alpha(\lambda)}{\tilde{r}_\beta(\lambda)} \leq C\frac{\varphi(\alpha)}{\varphi(\beta)} \quad \textit{for all} \quad 0 < \alpha \leq \beta \leq \lambda,$$
$$\frac{\tilde{r}_\alpha(\lambda)}{\tilde{r}_\beta(\lambda)} \geq \tilde{C}\frac{\varphi(\alpha)}{\varphi(\beta)} \quad \textit{for all} \quad 0 < \lambda \leq \alpha \leq \beta.$$

Let us define: $\tilde{\varphi} = \sqrt{\alpha\varphi(\alpha)}$ *and* $\psi(\delta) = \dfrac{\delta^2}{\tilde{\varphi}^{-1}(\delta)}$.

Then, the equivalence holds:

$$\|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(\varphi(\alpha)) \quad \Leftrightarrow \quad \sup_{\tilde{y}\in\overline{B}_\delta(y)} \inf_{\alpha>0} \|x_\alpha(\tilde{y}) - x^\dagger\|^2 = \mathcal{O}(\psi(\delta))$$

Proof of Proposition 2

Given φ, the function ψ is defined s.t.:

$$\frac{\delta^2}{\alpha} = \varphi(\alpha) \Leftrightarrow \frac{\delta^2}{\alpha} = \psi(\delta) = \frac{\delta^2}{\tilde{\varphi}^{-1}(\delta)} \Leftrightarrow \tilde{\varphi}(\alpha) = \sqrt{\alpha\varphi(\alpha)} = \delta.$$

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Main step: if we choose for each $\delta > 0$, the regularization parameter α_δ through

$$\alpha_\delta \| x_{\alpha_\delta}(y) - x^\dagger \|^2 = \delta^2, \tag{6}$$

then, there exist constants $c, C > 0$ s.t.

$$c\frac{\delta^2}{\alpha_\delta} \leq \sup_{\tilde{y} \in \overline{B}_\delta(y)} \inf_{\alpha > 0} \| x_\alpha(\tilde{y}) - x^\dagger \|^2 \leq C\frac{\delta^2}{\alpha_\delta}.$$

(For the lower bound we must require that $\alpha_\delta \in \sigma(LL^*)$.)

Proof of Proposition 2

Note that,

$$\varphi(\gamma\alpha) \leq g(\gamma)\varphi(\alpha) \Rightarrow \psi(\tilde{\gamma}\delta) \leq \frac{\tilde{\gamma}^2}{\tilde{g}^{-1}(\tilde{\gamma})}\psi(\delta),$$

with $\tilde{g}(\gamma) = \sqrt{\gamma g(\gamma)}$.

By assuming $\|x_\alpha(y) - x^\dagger\|^2 = O(\varphi(\alpha))$ for all $\alpha > 0$, we have

$$\frac{\delta^2}{\alpha_\delta} \leq \tilde{c}\varphi(\alpha_\delta) \Rightarrow \tilde{\varphi}^{-1}\left(\frac{\delta}{\sqrt{\tilde{c}}}\right) \leq \alpha_\delta \Rightarrow \frac{\delta^2}{\alpha_\delta} \leq \tilde{c}\psi\left(\frac{\delta}{\sqrt{\tilde{c}}}\right) = O(\psi(\delta)).$$

Proof of Proposition 2

Conversely, assume further that δ is s.t. $\alpha_\delta \in \sigma(LL^*)$. Then we have

$$c\psi(\delta) \geq \sup_{\tilde{y}\in\overline{B}_\delta(y)} \inf_{\alpha>0} \|x_\alpha(\tilde{y}) - x^\dagger\|^2 \geq C_0 \frac{\delta^2}{\alpha_\delta}$$

which implies that, by $\varphi(\gamma\alpha) \leq g(\gamma)\varphi(\alpha)$ and the definition of ψ and :

$$\tilde{\varphi}^{-1}(\delta) \leq \frac{c}{C_0}\alpha_\delta \Rightarrow \frac{\delta^2}{\alpha_\delta} \leq \frac{c}{C_0}\varphi\left(\frac{c}{C_0}\alpha_\delta\right) = \mathcal{O}(\varphi(\alpha_\delta))$$

For $\alpha \notin \sigma(LL^*)$, let us consider

$$\alpha_- = \sup\{\tilde{\alpha} \in \sigma(LL^*) \cup \{0\} \,:\, \tilde{\alpha} < \alpha\}, \quad \alpha_+ = \sup\{\tilde{\alpha} \in \sigma(LL^*) \,:\, \tilde{\alpha} > \alpha\}.$$

Note that

$$\|x_\alpha(y) - x^\dagger\|^2 = \int_0^{\alpha_-} \tilde{r}_\alpha(\lambda)d\|E_\lambda x^\dagger\| + \int_{\alpha_+}^{\|L\|^2} \tilde{r}_\alpha(\lambda)d\|E_\lambda x^\dagger\|.$$

Proof of Proposition 2

This and the hypotheses on φ and $\tilde{r}_\alpha$ imply that:

$$\|x_\alpha(y) - x^\dagger\|^2 \leq$$

$$\|x_{\alpha_-}(y) - x^\dagger\|^2 \sup_{\lambda \in [0, \alpha_-]} \frac{\tilde{r}_\alpha(\lambda)}{\tilde{r}_{\alpha_-}(\lambda)} + \|x_{\alpha_+}(y) - x^\dagger\|^2 \sup_{\lambda \in [\alpha_+, \|L\|^2]} \frac{\tilde{r}_\alpha(\lambda)}{\tilde{r}_{\alpha_+}(\lambda)}$$

$$\leq \frac{\hat{c}}{\tilde{C}} \varphi(\alpha_-) \frac{\varphi(\alpha)}{\varphi(\alpha_-)} + C\hat{c}\varphi(\alpha_+) \frac{\varphi(\alpha)}{\varphi(\alpha_+)} = \mathcal{O}(\varphi(\alpha)). \quad (7)$$

Proof of Proposition 2 (Main Step)

Upper bound:

$$\|x_\alpha(\tilde{y}) - x^\dagger\|^2 \leq \left(\|x_\alpha(\tilde{y}) - x_\alpha(y)\| + \|x_\alpha(y) - x^\dagger\|\right)^2$$

Since $Lr_\alpha(L^*L) = r_\alpha(LL^*)L$ and $r_\alpha(\lambda) \leq \frac{\rho}{\sqrt{\alpha\lambda}}$, it follows that

$$\|x_\alpha(\tilde{y}) - x_\alpha(y)\|^2 = \langle \tilde{y} - y, r_\alpha^2(LL^*)LL^*(\tilde{y} - y)\rangle \leq \delta^2 \max_{\lambda>0} \lambda r_\alpha^2(\lambda) \leq \rho^2 \frac{\delta^2}{\alpha}.$$

So,

$$\sup_{\tilde{y}\in\overline{B}_\delta(y)} \inf_{\alpha>0} \|x_\alpha(\tilde{y}) - x^\dagger\|^2 \leq \inf_{\alpha>0} \left(\|x_\alpha(y) - x^\dagger\| + \frac{\delta}{\sqrt{\alpha}}\right)^2 = \mathcal{O}\left(\frac{\delta^2}{\alpha_\delta}\right)$$

Proof of Proposition 2 (Main Step)

Lower bound:

$$\|x_\alpha(\tilde{y}) - x^\dagger\|^2 = \|x_\alpha(y) - x^\dagger\|^2 + \langle \tilde{y} - y, r_\alpha(LL^*)^2 LL^*(\tilde{y} - y)\rangle + 2\langle r_\alpha(LL^*)(\tilde{y} - y), r_\alpha(LL^*)LL^* y - y\rangle. \quad (8)$$

Consider $\alpha_\delta \in \sigma(LL^*)$. So, the spectral measure F of LL^* satisfies $F_{[a_\delta, 2\alpha_\delta]} \neq 0$, with $a_\delta \in (0, \alpha_\delta)$ and $\tilde{r}_\alpha(a_\delta) < \tilde{\rho}$.
Suppose that,

$$z_\delta = F_{[a_\delta, 2\alpha_\delta]}(r_{\alpha_\delta}(LL^*)LL^* y - y) \neq 0 \text{ and define } \tilde{y} = y + \delta \frac{z_\delta}{\|z_\delta\|}.$$

So,

$$\|x_\alpha(\tilde{y}) - x^\dagger\|^2 = \|x_\alpha(y) - x^\dagger\|^2 + \frac{\delta^2}{\|z_\delta\|^2}\langle z_\delta, r_\alpha(LL^*)^2 LL^* z_\delta\rangle + 2\frac{\delta}{\|z_\delta\|}\langle r_\alpha(LL^*) z_\delta, z_\delta\rangle. \quad (9)$$

Proof of Proposition 2 (Main Step)

Using the inequality $\lambda r_\alpha^2(\lambda) \geq (1-\sqrt{\tilde{r}_\alpha(a_\delta)})^2/(2\alpha_\delta)$ in $[a_\delta, 2\alpha_\delta]$, we have:

$$\sup_{\tilde{y}\in\overline{B}_\delta(y)} \inf_{\alpha>0} \|x_\alpha(\tilde{y}) - x^\dagger\|^2 \geq \inf_{\alpha>0}\left(\|x_\alpha(y) - x^\dagger\|^2 + \delta^2 \frac{(1-\sqrt{\tilde{r}_\alpha(a_\delta)})^2}{2\alpha_\delta}\right)$$
$$\geq \min\left\{\|x_{\alpha_\delta}(y) - x^\dagger\|^2, \delta^2\frac{(1-\sqrt{\tilde{r}_\alpha(a_\delta)})^2}{2\alpha_\delta}\right\} \geq \frac{(1-\sqrt{\tilde{\rho}})^2}{2}\frac{\delta^2}{\alpha_\delta}$$

If z_δ is zero, choose an nonzero element in $\mathcal{R}(F_{[a_\delta, 2\alpha_\delta]})$.

To recover the result in Neubauer (1997), let us consider Tikhonov regularization and

$$\varphi(\alpha) = \alpha^{2\nu}, \quad \text{and} \quad r_\alpha(\lambda) = \frac{1}{\alpha+\lambda}$$

In this particular case,

$$\varphi(\gamma\alpha) = g(\gamma)\varphi(\alpha) \quad \text{with} \quad g(\gamma) = \gamma^{2\nu} \leq C(1+\gamma^2),$$

$$\tilde{\varphi}(\alpha) = \alpha^{\frac{1+2\nu}{2}} \quad \text{and} \quad \psi(\delta) = \frac{\delta^2}{\delta^{\frac{2}{1+2\nu}}} = \delta^{\frac{4\nu}{1+2\nu}}.$$

So, we have the equivalences:

$$\sup_{\tilde{y}\in\overline{B}_\delta(y)} \inf_{\alpha>0} \|x_\alpha(\tilde{y}) - x^\dagger\|^2 = \mathcal{O}(\delta^{\frac{4\nu}{1+2\nu}}) \Leftrightarrow \|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(\alpha^{2\nu})$$

$$\Leftrightarrow \|E_{[0,\lambda]}x^\dagger\|^2 = \mathcal{O}(\lambda^{2\nu}). \quad (10)$$

Applications

We now consider logarithmic source conditions and Tikhonov regularization:

$$r_\alpha(\lambda) = \frac{1}{\alpha + \lambda} \qquad \text{and} \qquad \varphi(\alpha) = \frac{1}{|\log \alpha|^\nu}.$$

The concavity of the logarithmic function gives us:

$$\varphi(\gamma\alpha) \leq g(\gamma)\varphi(\alpha) \quad \text{with} \quad g(\gamma) = \begin{cases} 1, & \gamma \leq 1, \\ \gamma, & \gamma > 1. \end{cases}$$

To find ψ, we must solve the implicit equation:

$$\psi(\delta) = \left| \log \frac{\delta^2}{\psi(\delta)} \right|^{-\nu}.$$

However, we cannot solve this explicitly and find when $\delta \to 0$:

$$c|\log \delta|^{-\nu} \leq \psi(\delta) \leq C|\log \delta|^{-\nu}.$$

Therefore,

$$\sup_{\tilde{y} \in \overline{B}_\delta(y)} \inf_{\alpha > 0} \|x_\alpha(\tilde{y}) - x^\dagger\|^2 = \mathcal{O}(|\log \delta|^{-\nu}) \Leftrightarrow \|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(|\log \delta|^{-\nu})$$

$$\Leftrightarrow \|E_{[0,\lambda]} x^\dagger\|^2 = \mathcal{O}(|\log \delta|^{-\nu}). \tag{11}$$

Approximative Source Conditions

Let us consider the distance between $x^\dagger$ and $\mathcal{R}(\varphi(L^*L))$, instead of standard source conditions.
So, let us define the distance function d_φ w.r.t. a continuous $\varphi : [0,\infty) \to [0,\infty)$:

$$d_\varphi(R) = \inf_{\xi \in \overline{B}_R(0)} \|x^\dagger - \varphi(L^*L)\xi\|.$$

The following proposition extends a result in Flemming et al. (2011):

Proposition

Let $\varphi : [0,\infty) \to [0,\infty)$ be increasing and continuous with $\varphi(0) = 0$ so that there exists a constant $A > 0$ with

$$\sqrt{\tilde{r}_\alpha(\lambda)}\varphi(\lambda) \leq A\varphi(\alpha) \quad \textit{for all} \quad \lambda > 0, \alpha > 0.$$

Then, for every $\nu \in (0,1)$ we have

$$\|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(\varphi(\alpha)^{2\nu}) \Leftrightarrow d_\varphi(R) = \mathcal{O}(R^{-\frac{\nu}{1-\nu}}).$$

Proof of Proposition 3:

Note that:

$$\|x_\alpha(y) - x^\dagger\| = \|\tilde{r}_\alpha(L^*L)^{\frac{1}{2}} x^\dagger\| \leq \|\tilde{r}_\alpha(L^*L)^{\frac{1}{2}}(x^\dagger - \varphi(L^*L)\xi)\| + \|\tilde{r}_\alpha(L^*L)^{\frac{1}{2}}\varphi(L^*L)\xi\|$$

We use $\|\tilde{r}_\alpha(L^*L)^{\frac{1}{2}}\| \leq 1$ and $\sqrt{\tilde{r}_\alpha(\lambda)}\varphi(\lambda) \leq A\varphi(\alpha)$ to estimate the first and second terms respectively and find:

$$\|\tilde{r}_\alpha(L^*L)^{\frac{1}{2}}\varphi(L^*L)\xi\|^2 = \int_0^{\|L\|^2} \tilde{r}_\alpha(\lambda)\varphi^2(\lambda)d\|E_\lambda\xi\|^2 \leq A^2\varphi(\alpha)^2\|\xi\|^2.$$

So, $\|x_\alpha(y) - x^\dagger\| \leq \|x^\dagger - \varphi(L^*L)\xi\| + A\varphi(\alpha)\|\xi\|$, which leads to:

$$\|x_\alpha(y) - x^\dagger\| \leq d_\varphi(R) + A\varphi(\alpha)R.$$

If $d_\varphi(R) = \mathcal{O}(R^{-\frac{\nu}{1-\nu}})$, we choose R s.t. both terms are balanced, i.e. $R = \varphi(\alpha)^{\nu-1}$, so,

$$\|x_\alpha(y) - x^\dagger\| = \mathcal{O}(\varphi^\nu(\alpha)).$$

Proof of Proposition 3:

Conversely, we have that $\|E_{[0,\lambda]}x^\dagger\| = \mathcal{O}(\varphi(\lambda)^\nu)$, and define:

the operator $T = \varphi(L^*L)|_{\mathcal{R}(E_{(\alpha,\infty)})}$ and the element $\xi_\alpha = T^{-1}E_{(\alpha,\infty)}x^\dagger$.

So,

$$\|x^\dagger - \varphi(L^*L)\xi_\alpha\|^2 = \|E_{[0,\alpha]}x^\dagger\|^2 \leq C\varphi(\alpha)^{2\nu}$$

and

$$\|\xi_\alpha\|^2 = \int_0^{\|L\|^2} \varphi(\lambda)^{-2} d\|E_\lambda x^\dagger\|^2 \leq c^2\varphi(\alpha)^{2\nu-2}.$$

By setting $R = c\varphi(\alpha)^{\nu-1}$, then

$$d_\varphi(c\varphi(\alpha)^{\nu-1}) \leq c'\varphi(\alpha)^\nu, \quad \text{i.e.} \quad d_\varphi(R) \leq \tilde{C}R^{-\frac{\nu}{1-\nu}}.$$

Relation to Variational Inequalities

The following proposition extends a result in Andreev et al. (2015):

Proposition

Let $\varphi : [0,\infty) \to [0,\infty)$ *be increasing and continuous and* $\nu \in (0,1)$. *Then,*

$$\langle x^\dagger, x\rangle \leq C\|\varphi(L^*L)x\|^\nu \|x\|^{1-\nu} \quad \Leftrightarrow \quad \|E_{[0,\lambda]}x^\dagger\|^2 = \mathcal{O}(\varphi(\lambda)^\nu)$$

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Proof:

By assuming :

$$\begin{aligned}\|E_{[0,\lambda]}x^{\dagger}\|^2 = \langle x^{\dagger}, E_{[0,\lambda]}x^{\dagger}\rangle \\ \leq C\|\varphi(L^*L)E_{[0,\lambda]}x^{\dagger}\|^{\nu}\|E_{[0,\lambda]}x^{\dagger}\|^{1-\nu} \\ \leq C\varphi(\lambda)^{\nu}\|E_{[0,\lambda]}x^{\dagger}\|. \qquad (12)\end{aligned}$$

Proof of Proposition 4

Conversely, let $\Lambda > 0$ be arbitrary. So,

$$|\langle E_{[0,\Lambda]}x^\dagger, x\rangle| \leq \|E_{[0,\Lambda]}x^\dagger\|\|x\| \leq C\varphi(\Lambda)^\nu\|x\|.$$

Let us consider $T = \varphi(L^*L)|_{\mathcal{R}(E_{[\Lambda,\infty)})}$, it follows that

$$\begin{aligned}|\langle E_{[\Lambda,\infty)}x^\dagger, x\rangle| &= |\langle T^{-1}E_{[\Lambda,\infty)}x^\dagger, TE_{[\Lambda,\infty)}x\rangle| \\ &\leq \|TE_{[\Lambda,\infty)}x\|\sqrt{\lim_{\varepsilon\downarrow 0}\int_{\Lambda-\varepsilon}^{\|L\|^2}\varphi(\lambda)^{-2}d\|E_\lambda x^\dagger\|^2} \qquad (13)\end{aligned}$$

After integration by parts, it follows that

$$|\langle E_{[\Lambda,\infty)}x^\dagger, x\rangle| \leq c\varphi(\Lambda)^{\nu-1}\|\varphi(L^*L)x\|.$$

Choosing $\Lambda = \inf\{\lambda > 0 \ : \ |\langle E_{[0,\lambda]}x^\dagger, x\rangle| \geq \frac{1}{2}|\langle x^\dagger, x\rangle|\}$, we get

$$\langle x^\dagger, x\rangle \leq 2|\langle E_{[0,\lambda]}x^\dagger, x\rangle|^{1-\nu}|\langle E_{[\lambda,\infty)}x^\dagger, x\rangle|^\nu \leq c'\|\varphi(L^*L)x\|^\nu\|x\|^{1-\nu}.$$

Proposition

Let $\varphi : (0,\infty) \to (0,\infty)$ *be increasing and continuous, and* $\psi(\lambda) \geq c\varphi(\lambda)^{\mu}$ *for* $c > 0$ *and* $\mu < 1$*. Then,*

$$x^{\dagger} \in \mathcal{R}(\varphi(L^*L)) \Rightarrow \|E_{[0,\lambda]}x^{\dagger}\| = \mathcal{O}(\varphi(\lambda)) \Rightarrow x^{\dagger} \in \mathcal{R}(\psi(L^*L)).$$

Standard Source Condition

Proposition

Let $\varphi : (0,\infty) \to (0,\infty)$ *be increasing and continuous, and* $\psi(\lambda) \geq c\varphi(\lambda)^{\mu}$ *for* $c > 0$ *and* $\mu < 1$*. Then,*

$$x^{\dagger} \in \mathcal{R}(\varphi(L^*L)) \Rightarrow \|E_{[0,\lambda]}x^{\dagger}\| = \mathcal{O}(\varphi(\lambda)) \Rightarrow x^{\dagger} \in \mathcal{R}(\psi(L^*L)).$$

Proof: The first implication follows by $x^{\dagger} = \varphi(L^*L)w$, which implies that

$$\|E_{[0,\lambda]}x^{\dagger}\| = \|\varphi(L^*L)E_{[0,\lambda]}w\| \leq \varphi(\lambda)\|w\|.$$

The second implication can be seen from:

$$c^2 \int_0^{\|L\|^2} \psi(\lambda)^{-2} d\|E_{\lambda}x^{\dagger}\|^2 \leq \int_0^{\|L\|^2} \varphi(\lambda)^{-2\mu} d\|E_{\lambda}x^{\dagger}\|^2$$
$$= \frac{\|x^{\dagger}\|^2}{\varphi(\|L\|^2)^{2\mu}} - \lim_{\lambda\to 0} \frac{\|E_{[0,\lambda]}x^{\dagger}\|^2}{\varphi(\lambda)^{2\mu}} + 2\mu \int_0^{\|L\|^2} \frac{\|E_{[0,\lambda]}x^{\dagger}\|^2}{\varphi(\lambda)^{1+2\mu}} d\varphi(\lambda) < \infty. \quad (14)$$

Concluding Remarks

We have stated the following equivalences:

- $\|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(\varphi(\alpha)) \Leftrightarrow \|E_{[0,\lambda]}x^\dagger\|^2 = \mathcal{O}(\varphi(\lambda))$.
- $\|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(\varphi(\alpha)) \Leftrightarrow \sup_{\tilde{y}\in\overline{B}_\delta(y)} \inf_{\alpha>0} \|x_\alpha(\tilde{y}) - x^\dagger\|^2 = \mathcal{O}(\psi(\delta))$,

  where $\varphi(\gamma\alpha) \leq g(\gamma)\varphi(\alpha)$, $\tilde{\varphi} = \sqrt{\alpha\varphi(\alpha)}$ and $\psi(\delta) = \dfrac{\delta^2}{\tilde{\varphi}^{-1}(\delta)}$.
- $\|x_\alpha(y) - x^\dagger\|^2 = \mathcal{O}(\varphi^{2\nu}(\alpha)) \Leftrightarrow d_\varphi(R) = \mathcal{O}(R^{-\frac{\nu}{1-\nu}})$,

  where $d_\varphi(R) = \inf_{\xi\in\overline{B}_R(0)} \|x^\dagger - \varphi(L^*L)\xi\|$ and $\sqrt{\tilde{r}_\alpha(\lambda)}\varphi(\lambda) \leq A\varphi(\lambda)$.
- $\langle x^\dagger, x\rangle \leq C\|\varphi(L^*L)x\|^\nu\|x\|^{1-\nu} \quad \Leftrightarrow \quad \|E_{[0,\lambda]}x^\dagger\|^2 = \mathcal{O}(\varphi(\lambda)^\nu)$.
- $x^\dagger \in \mathcal{R}(\varphi(L^*L)) \Rightarrow \|E_{[0,\lambda]}x^\dagger\| = \mathcal{O}(\varphi(\lambda)) \Rightarrow x^\dagger \in \mathcal{R}(\psi(L^*L))$,

  with $\psi(\lambda) \geq c\varphi(\lambda)^\mu$

These results are part of Albani et al. (2016).

Thank you!

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