A Tutorial on Options and Derivatives

Vinicius Albani

Federal University of Santa Catarina (UFSC)

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Outline

Introduction

- What is this about?
- What is the relationship with Game Theory?
- What are the mathematical tools?

Pricing Derivatives and Options

- Asset prices (Petrobras, Vale S.A., Itau,...) are naturally random.
- How to protect ourselves from large losses?
- Buying/Selling Derivatives and Options.
- Derivatives and Options are designed to reduce exposure to some source of risk.
- How to price options and derivatives appropriately?

Arbitrage

"A trading strategy that begins with no money, has zero probability of losing money, and has a positive probability of making money." Shreve (2004)

Arbitrage: an example[1]

Consider a roulette wheel that pays 2 : 1 when the outcome is red and nothing if the outcome is black.
The probabilities of the outcomes are:

$$\text{Red: } 70\% \quad \text{and} \quad \text{Black: } 30\%.$$

Playing many times, for each $\$1,00$ invested, we expect to receive:

$$2 \times 0,7 + 0 \times 0,3 = \$1,40.$$

A gambler sells for $\$60,00$ a ticket that pays $\$100,00$ if red and $0 if black.
Is it cheap or expansive?

[1] Souza and Zubelli (2016)

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A gambler sells for $\$60,00$ a ticket that pays $\$100,00$ if red and $\$0$ if black.
Is it cheap or expansive? Using the same ideas as above, the price must be $\$70,00$.

[1] Souza and Zubelli (2016)

Arbitrage: an example[2]

Possible situations:

- The gambler keeps the money.
- The gambler bets $\$60,00$ on roulette.
- The gambler bets $\$50,00$ on roulette and keeps $\$10,00$.

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What is the fair price? $\$50,00$**. It corresponds to the probabilities** $50\% - 50\%$**,why?**

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Risk Neutral Probability Measure

- The probability measure that allows to find the correct price of derivatives.
- No arbitrage opportunities or efficient market hypothesis.
- Completeness of Markets
- Uniqueness of prices.
- Hedging: A trading strategy that reduces exposure to the risk of losses.

Vanilla Options

- **European Call**: gives the right, but not the obligation, of buying a share of an asset for a fixed strike price at its maturity.
- **Euopean Put** similar to the call, but gives the right of selling.
- American Option (call and put) can be exercised any time before its maturity.
- Sometimes, American options are more expensive than the European ones.
- The prices of such contract take into account asset dynamics.
- Other options: Asian, Lookback, Basket, Spread... Real Options...

- Typically, the asset price dynamics is given by a semi-martingale:

$$S_t = \text{something}_t + \text{Martingale}_t.$$

- What is a martingale?

$$\mathbb{E}[|M_t|] < \infty \quad \text{e} \quad \mathbb{E}[M_t \,|\, \{M_l,\ l \leq s\}] = M_s.$$

Some Hints

Consider a time series of asset prices:

$$\{s_{t_0}, s_{t_1}, s_{t_2}, \dots, s_{t_n}\}, \quad \text{with} \quad \Delta t = t_i - t_{i-1},\ i = 1, 2, \dots, n.$$

Consider now the log-returns:

$$y_i = \log(s_{t_i}/s_{t_{i-1}}).$$

Assume that they are:

1. independent,
2. identically distributed
3. Gaussian

So,

$$\mathbb{E}[y] = \mu\Delta t = \frac{1}{n}\sum_{i=1}^{n} y_i \quad \text{e} \quad \mathbb{V}ar[y] = \sigma^2\Delta t = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - (\mu\Delta t)^2$$

Asset Prices and Brownian Motion

Since asset price evolves as:

$$\ln(S_t) = \ln(S_0) + \mu t + \text{“randomness”}$$

Such randomness can be modeled as

$$\sigma W_t$$

$W(t)$ is a Brownian motion, i.e.,

1. $W_0 = 0$ a.s.
2. $W_t - W_s$ is $N(0, t-s)$, $t > s$
3. $W_t - W_s$ is independent of W_s

Basic tools: SDEs and Itô's Lemma.

- The Itô's integral can be defined as the "limit" of Riemann's sums:

$$\int_0^T X_t dW_t = \lim_{n\to\infty} \sum_{j=1}^{n} X_{t_i} \left(W_{t_i} - W_{t_{i-1}} \right)$$

- An SDE is then a stochastic process defined as:

$$X_T = X_0 + \int_0^T a_t dt + \int_0^T b_t dW_t$$

or

$$dX_t = a_t dt + b_t dW_t.$$

- The Itôs Lemma says that, if $f(t,x)$ is $C^{1,2}$, then,

$$\begin{aligned} df(t,X_t) &= \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)d\langle X_t, X_t\rangle \\ &= \left[\frac{\partial f}{\partial t}(t,X_t) + \frac{\partial f}{\partial x}(t,X_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)b_t^2\right] dt + \frac{\partial f}{\partial x}(t,X_t)b_t dW_t \end{aligned}$$

The Black-Scholes Model (1973) 1/2[3]

- Let $(\Omega, \mathcal{U}, \widetilde{\mathbb{P}})$ be a prob. space with filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$.
- An asset price at time $t \geq 0$ is given by the SDE:

$$dS_t = S_t(\mu dt + \sigma dW_t),$$

where W_t is a Brownian motion and S_0 is given.

How to price an European option in this setting?

[3] Black and Scholes (1973)

The Black-Scholes Trading Strategy[4]

Main assumptions:

- No transaction costs.
- The asset does not pay dividends.
- No arbitrage.
- Asset price is divisible.
- Short selling is permitted.
- Trading can take place continuously in time.
- Risk-free interest rate and volatility are known and constant.

[4] Korn and Korn (2001)

The Black-Scholes Trading Strategy

Consider the riskless portfolio:

$$Y_t = \phi_t P_t + \pi_t S_t - C(t, S_t),$$

where $P_t = P_0 \exp(r \cdot t)$ is a money market account with risk-free interest rate $r > 0$
ϕ and π represent the amount of investment on each asset.

Since the portfolio is riskless

$$dY_t = rY_t dt.$$

On the other hand,

$$dY_t = \phi_t dP_t + \pi_t dS_t + dC(t, S_t).$$

The Black-Scholes Trading Strategy

$$dY_t = \phi_t dP_t + \pi_t dS_t + dC(t, S_t) = \phi_t r P_t dt + \pi_t(\mu S_t dt + \sigma S_t dW_t)$$
$$- \left[\frac{\partial C}{\partial t}(t, S_t)dt + \frac{\partial C}{\partial x}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(t, S_t)\sigma^2 S_t^2 dt\right]$$
$$= \phi_t r P_t dt + \pi_t(\mu S_t dt + \sigma S_t dW_t)$$
$$- \left[\frac{\partial C}{\partial t}(t, S_t)dt + \frac{\partial C}{\partial x}(t, S_t)(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(t, S_t)\sigma^2 S_t^2 dt\right].$$

Making $\pi_t = \frac{\partial C}{\partial x}(t, S_t)$, and recalling that $Y_t = \phi_t P_t + \pi_t S_t - C(t, S_t)$, it follows that

$$0 = rC(t, S_t) - rS_t \frac{\partial C}{\partial x}(t, S_t) - \frac{\partial C}{\partial t}(t, S_t) - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial x^2}(t, S_t).$$

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$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r\,S\frac{\partial C}{\partial S} - r\,C = 0,\ 0 < t < T,\ S > 0,$$

with terminal condition

$$C(T,S) = \max\{0, S-K\}.$$

Its solution is given by:

$$C(t,S) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where,

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy,$$

$$d_1(t,S) = \frac{\log(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2(t,S) = d_1 - \sigma\sqrt{T-t}.$$

Risk-Neutral Pricing

The no-arbitrage assumption implies the existence of the risk-neutral prob. meas. $\mathbb{Q}$.

- Under $\mathbb{Q}$, S_t satisfies

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t,$$

  where $\widetilde{W}_t$ is a $\mathbb{Q}$-Brownian motion.

- An European call option price is then given by:

$$C(t, S_t, T, K) = \mathrm{e}^{-r(T-t)}\widetilde{\mathbb{E}}[\max\{0, S_T - K\} \,|\, \mathcal{F}_t].$$

- More generally, European options with a payoff $f(S_T)$ satisfy

$$V(t, S_t) = \mathrm{e}^{-r(T-t)}\widetilde{\mathbb{E}}[f(S_T) \,|\, \mathcal{F}_t].$$

Feynman-Kac Theorem

Under suitable assumptions, an option with maturity T and payoff $f(S_T)$

$$V(t, S_t) = \mathrm{e}^{-r(T-t)}\widetilde{\mathbb{E}}[f(S_T)\,|\,\mathcal{F}_t].$$

is the solution of the PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\,S\frac{\partial V}{\partial S} - r\,v = 0,\ 0 < t < T,\ S > 0,$$

with terminal condition

$$V(T, S) = f(S).$$

- Nice model but too simple.
- Constant Parameters.
- Generalize the model.
- Consider non-constant parameters.
- Possible drawbacks.
- Model calibration.

Dupire's Local Volatility Model (1994) 1/2[5]

- Let $(\Omega, \mathcal{V}, \mathcal{F}, \widetilde{\mathbb{P}})$ be a filtered prob. space.
- The asset price S_t satisfies:

$$\begin{cases} dS_t &= (r-q)\, S_t\, dt + \sigma(t, S_t) S_t d\widetilde{W}_t, \ t \geq 0 \\ S_0 & \text{is given.} \end{cases}$$

- Again, European call option price is given by:

$$C(t, S_t, T, K) = \widetilde{\mathbb{E}}[\mathrm{e}^{-r(T-t)} \max\{0, S_T - K\} \,|\, \mathcal{F}_t].$$

[5] Dupire (1994)

Dupire's Local Volatility Model (1994) 2/2

Fixing $t = 0$ and $S_t = S_0$, it follows that:

$$C(0, S_0, T, K) = \mathrm{e}^{-rT} \int_0^\infty \max\{0, S - K\} \varphi(S, T)\, dS$$

and applying Fokker-Planck equation to φ and integrating by parts we find:

$$\begin{cases} \dfrac{\partial C}{\partial T} = \dfrac{1}{2}\sigma^2(T, K; S_0) K^2 \dfrac{\partial^2 C}{\partial K^2} - (r - q) K \dfrac{\partial C}{\partial K} - qC, \; T > 0, \; K \geq 0 \\ \lim\limits_{K \to 0} C(T, K) = S_0, \; T > 0, \\ \lim\limits_{K \to +\infty} C(T, K) = 0, \; T > 0, \\ C(T = 0, K) = \max\{0, S_0 - K\}, \; K > 0. \end{cases} \quad (1)$$

Heston Model (1993)[6]

- Let $(\Omega, \mathcal{V}, \mathcal{F}, \mathbb{P})$ be a filtered prob. space.

- the asset price S_t satisfies:

$$\begin{cases} dS_t & = & \mu S_t\, dt + \sqrt{v_t} S_t d\widetilde{W}_t, \ t \geq 0 \\ dv_t & = & -\lambda(v_t - \overline{v})dt + \eta\sqrt{v_t} dZ_t, \\ S_0, v_0 & \text{are} & \text{given}, \\ d\langle W_t, Z_t \rangle & = & \rho\, dt \end{cases}$$

- Again, European call option price is given by:

$$C(t, S_t, T, K) = \widetilde{\mathbb{E}}[e^{-r(T-t)} \max\{0, S_T - K\} \,|\, \mathcal{F}_t].$$

[6] volguide

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Simulation and Evaluation of Options

- European options can be priced with PDE's in some cases.
- American options are related to free boundary problems.
- For more general models options may not have a PDE representation.
- It is necessary to simulate the SDE's
- Option prices can be evaluated through Monte Carlo integration.

- Finite difference and FEM methods to solve PDE problems
- Similar methods can solve free boundary problems.
- SDE's can be numerically solved by e.g. Euler-Maruyama or Milstein schemes[7].
- Monte Carlo integration.

[7] See Higham (2001)

Model Calibration

1. One of the main problems in Math. Finance and Applied Math. in general.
2. Well-defined direct problem.
3. Calibration problems is in general ill-posed
4. Some regularization technique is needed.

Model Calibration

In Math. Fin. there are two main situations:

- Calibration using historical prices.
  1. Related to risk analysis.
  2. Main techniques: MLE, Kalman Filter, etc.
- Calibration using implicit data (European Call and Put prices).
  1. Related to option pricing.
  2. Needs regularization.

Local Volatility Calibration

- Let $\tilde{C}$ be a surface of European call option prices.
- Assume that it is given by Dupire's equation.
- So, the corresponding local volatility surface σ^2_{TRUE}, solution of

$$\tilde{C} = C(\sigma^2_{TRUE}). \qquad (2)$$

Unfortunately, only scarce and noisy data C^δ is available:

$$\|\tilde{C} - C^\delta\| \leq \delta,$$

with $\delta > 0$ (noise level).

- The inverse problem is ill-posed.
- Tikhonov-type regularization leads us to find an element in

$$\text{argmin}\left\{\|C(\sigma^2) - C^\delta\|^2 + \alpha f(\sigma^2) \ : \ \sigma^2 \in Q\right\},$$

where Q is the set of suitable local vol. surfaces:

$$Q := \{\sigma^2 \in \sigma_0^2 + H^{1+\varepsilon}(D) : a_1 \leq \sigma^2 \leq a_2\}.$$

- Variational theory gives us existence and stability of minimizers, as well as convergence and convergence-rate results.

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Numerical Solution[8]

- Dupire's PDE is solved by a Crank-Nicolson scheme.
- The minimization of the Tikhonov-type functional is solved by the gradient descent method.
- The steps are chosen by Wolfe's rules.
- The iterations cease whenever the tolerance is satisfied:

$$\frac{\|C(\sigma_k^2) - C^\delta\|}{\|C^\delta\|} < \text{tol},$$

typically tol$= 0.01$.

[8] Albani et al. (2018)

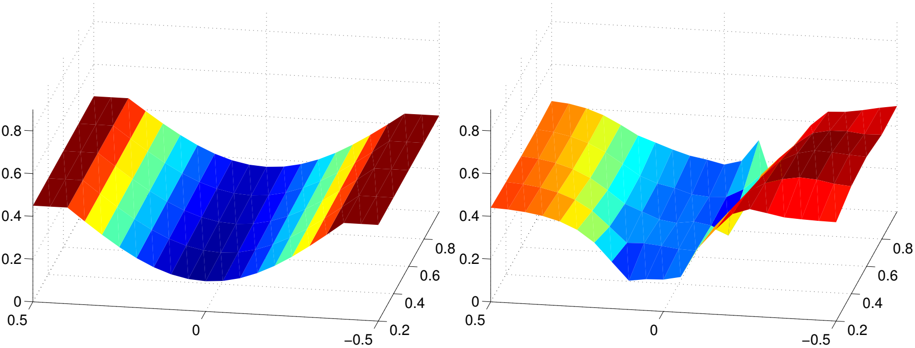

Figure: Local vol.: True (left) and reconstructed (right).

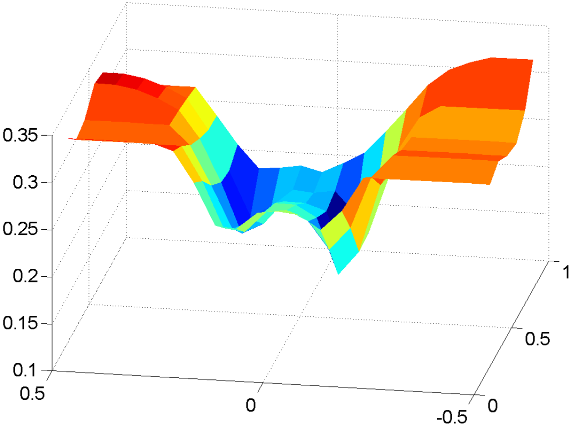

Figure: Local vol. reconstruction.

Consider the Heston model:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^1, \quad 0 \leq t \leq T_{\max} \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2, \end{aligned} \tag{3}$$

Evaluate the price of *European Asian Options* with srike K, maturity $T_{\max}$ and payoff

$$A(T_{\max}) := \max\left\{0, \frac{1}{N}\sum_{j=0}^{N} S_{t_j} - K\right\},$$

where $t_j = j \cdot \Delta t$ and $\Delta t = T_{\max}/N$.

Pricing Exotic Option

$\log(K/S_0)$	Local Volatility 0	Local Volatility -0.1	Local Volatility 0.1	Black & Scholes 0	Black & Scholes -0.1	Black & Scholes 0.1
$\tau = 0.1$	0.0247	0.0387	0.0985	0.0067	0.0478	0.0519
$\tau = 0.5$	0.0189	0.0317	0.0495	0.0076	0.0576	0.1246
$\tau = 1.0$	0.0157	0.0103	0.0057	0.0757	0.1436	0.2370
$\tau = 1.5$	0.0400	0.0420	0.0426	0.1244	0.1791	0.2592

Table: Relative errors.

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