Calibrating Volatility Surfaces for Commodity Derivatives

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IMPA
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Support

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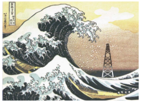

IMPA PRH/ANP-32

Tikhonov Reg.

- Black-Scholes/Dupire framework: non-const. vol.

$$F : \{\text{vol. surf. } \sigma\} \longmapsto \{\text{Call Prices surf. } C\}$$

- Vol. Calibration: given C find σ such that

$$F(\sigma) = C$$

  in a **robust way**.

- ”Standard” Tikhonov reg.: Find an element of

$$\text{argmin}_{\sigma \in D(F)} \{\|F(\sigma) - C\|^2 + \alpha f(\sigma)\}$$

Our Proposal

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Considering many different measurements in the same Tikhonov Reg. procedure:

$$\mathrm{argmin}_{\sigma \in D(F)} \{ \int_{S_{\min}}^{S_{\max}} \| F(s, \sigma(s)) - C(s) \|^2 ds + \alpha \mathcal{F}(\overrightarrow{\sigma}) \},$$

where $\overrightarrow{\sigma} = \{ s \mapsto \sigma(s) \}$.

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In other words, it is an "Online" version of Tikhonov Reg.

The Model

$(\Omega, \mathcal{V}, \mathbb{F}, \mathbb{Q})$ is a filtered probability space, with $\mathbb{F} = \{\mathbb{F}_t\}_{t\in\mathbb{R}}$ a filtration and $\mathbb{Q}$ a ”risk-neutral” measure.

$$\begin{aligned} & dF_{t,T} = \sqrt{\nu(t)} F_{t,T} d\widetilde{W}(t), \text{ for } t \in [0, T] \\ & F_{0,T} > 0 \text{ non-random and known.} \end{aligned} \tag{1}$$

$\{\widetilde{W}(t)\}_{t\in\mathbb{R}}$ - $\mathbb{Q}$-Brownian motion,
$\nu(t)$ - squared vol.

Local vol can be def. as[Dup94, Gat06]:

$$\sigma(F_{0,T}, t, f) = \sqrt{\widetilde{\mathbb{E}}[\nu(t) | F_{t,T} = f]}$$

European Vanilla Options on Futures

$C(\widetilde{T}, K, t, F_{t,T})$ on $F_{t,T}$, with maturity $\widetilde{T} \leq T$ and strike K.
Fix $t = 0$ and $F(t = 0, T) = F_{0,T}$.
$C(F_{0,T}, \widetilde{T}, K)$ satisfies:

$$\begin{cases} \dfrac{\partial C}{\partial \widetilde{T}} = \frac{1}{2}\sigma^2(F_{0,T}, \widetilde{T}, K) K^2 \dfrac{\partial^2 C}{\partial K^2} & 0 \leq \widetilde{T} \leq T,\ K \geq 0 \\ C(\widetilde{T} = 0, K) = (F_{0,T} - K)^+, & \text{for } K > 0. \end{cases} \tag{2}$$

Market Variables

Change of variables:

$$\tau = \widetilde{T} \text{ and } y = \log(KF_{0,T}).$$

Define

$$u(\tau, y) = C(\tau, F_{0,T}e^y)/F_{0,T} \text{ and } a(F_{0,T}, \tau, y) = \sigma^2(F_{0,T}, \tau, F_{0,T}e^y)/2.$$

Thus, $u(\tau, y)$ satisfies:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = a(F_{0,T}, \tau, y)\left(\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial y}\right) & 0 \le \tau \le T,\ y \in I \subset \mathbb{R} \\ u(0, y) = (1 - e^y)^+, & y \in \mathbb{R}, \end{cases} \tag{3}$$

A Change in Notation

Instead of depending on $F_{0,T}$, we assume that a depends on the initial spot commodity price $S(0)$:

$$a = a(S(0), \tau, y).$$

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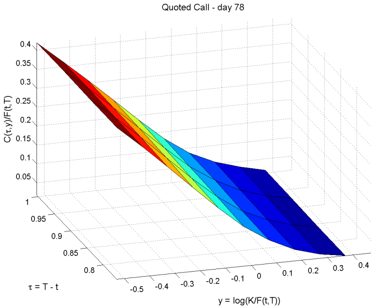
Then, call prices for futures with different maturities satisfy the same equation.

Resulting Surfaces



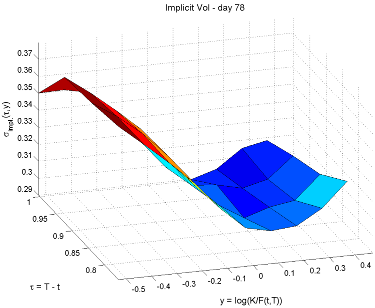
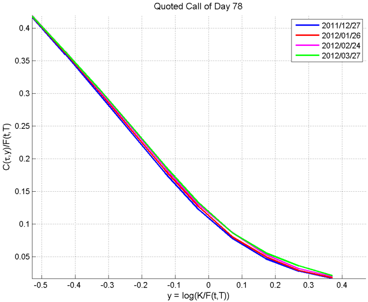


Figure: Normalized European call option prices on Futures of Brent oil (WTI) traded on 2011/03/16, where for each option's maturity $\widetilde{T}$, we consider the normalized prices $u(\widetilde{T}, K)$.

Another View



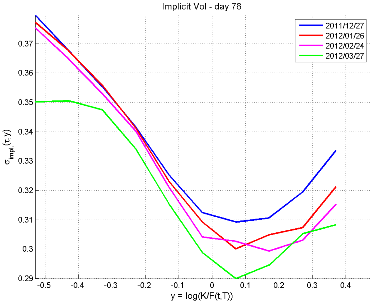


Figure: Normalized European call option prices on Futures of Brent oil (WTI) traded on 2011/03/16, where for each option's maturity T', the call prices $C(t,T,T')$ are divided by the current futures price $F(t,T)$ and the related implied vol.

The Forward Operator

We define the sets

$$Q = \{a \in a_0 + H^{1+\varepsilon}(D) \mid 0 < a_1 \leq a \leq a_2 < +\infty\} \tag{4}$$

$$\mathfrak{Q} := \{\mathcal{A} \in H^l(0,T,H^{1+\varepsilon}(D)) : a(s) \in Q,\ \forall s \in [0,S]\}, \tag{5}$$

and the operators

$$\begin{array}{rcl} F : [0,S] \times Q & \longrightarrow & W_2^{1,2}(D) \\ (s,a) & \longmapsto & u(s,a) - u(s,a_0), \end{array}$$

$$\begin{array}{rcl} \mathcal{U} : \mathfrak{Q} & \longrightarrow & L^2(0,S,W_2^{1,2}(D)), \\ \mathcal{A} & \longmapsto & \mathcal{U}(\mathcal{A}) : s \in [0,S] \mapsto F(s,a(s)) \in W_2^{1,2}(D). \end{array}$$

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We have proved regularity properties for them.

The Inverse Problem

IDEALIZED

Given the noiseless data $\widetilde{\mathcal{U}}$, we have to find some $\widetilde{\mathcal{A}} \in \mathfrak{Q}$ such that

$$\widetilde{\mathcal{U}} = \mathcal{U}(\widetilde{\mathcal{A}}). \tag{6}$$

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REALISTIC

Instead of considering $\widetilde{\mathcal{U}}$, we shall consider the noisy data $\mathcal{U}^\delta$. Thus, we have to find $\widetilde{\mathcal{A}} \in \mathfrak{Q}$ such that

$$\mathcal{U}^\delta = \widetilde{\mathcal{U}} + \mathcal{E} = \mathcal{U}(\widetilde{\mathcal{A}}) + \mathcal{E}. \tag{7}$$

$\mathcal{E}$ introduces all the uncertainties sources concerning the actual problem. The constant $\delta > 0$ is the noise level, i.e.,

$$\|\mathcal{U}^\delta - \widetilde{\mathcal{U}}\| = \|\mathcal{E}\| \leq \delta$$

SINCE, $\mathcal{U}(\cdot)$ IS COMPACT, PROBLEMS (6) AND (7) ARE ILL-POSED.

Tikhonov-type Regularization

Find an element of

$$\text{argmin}\left\{\|\mathcal{U}(\mathcal{A})-\mathcal{U}^{\delta}\|^2_{L^2(0,S,L^2(D))}+\alpha f_{\mathcal{A}_0}(\mathcal{A})\right\} \text{ subject to } \mathcal{A}\in\mathfrak{Q}. \tag{8}$$

Tikhonov functional:

$$\mathcal{F}^{\mathcal{U}^{\delta}}_{\mathcal{A}_0,\alpha}(\mathcal{A})=\|\mathcal{U}^{\delta}-\mathcal{U}(\mathcal{A})\|^2_{L^2(0,S,L^2(D))}+\alpha f_{\mathcal{A}_0}(\mathcal{A}). \tag{9}$$

Morozov's Principle [AR10, Mor66]

Definition

For $1 < \tau_1 \leq \tau_2$ *we choose* $\alpha = \alpha(\delta, \mathcal{U}^\delta) > 0$ *such that*

$$\tau_1 \delta \leq \| \mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}^\delta \| \leq \tau_2 \delta \tag{10}$$

holds for some $\mathcal{A}_\alpha^\delta$ *a minimizer of*

$$\mathcal{F}_{\mathcal{A}_0,\alpha}^{\mathcal{U}^\delta}(\mathcal{A}) = \| \mathcal{U}^\delta - \mathcal{U}(\mathcal{A}) \|_{L^2(0,S,L^2(D))}^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}).$$

We can use this definition to find the reg. par. appropriately in the vol. calib. prob.

Additional Assumptions

In order to apply some convex regularization tools we assume that

- $f_{\mathcal{A}_0}$ is convex.
- $f_{\mathcal{A}_0}$ is weakly lower semi-continuous.
- $f_{\mathcal{A}_0}$ is coercive.

By the continuity and compactness of the operator $\mathcal{U}$, we can state some results concerning existence, stability and convergence of minimizers for (9) [SGG$^+$08].

Some Results

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Theorem (Existence)

For every data $\mathcal{U}^\delta \in L^2(0,S,L^2(D))$*, there exists* $\mathcal{A}^\delta_\alpha \in \mathfrak{Q}$ *minimizing*

$$\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha}(\mathcal{A}) = \|\mathcal{U}^\delta - \mathcal{U}(\mathcal{A})\|^2_{L^2(0,S,L^2(D))} + \alpha f_{\mathcal{A}_0}(\mathcal{A}).$$

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Definition (Stability)

The minimizers of (9) are stable if for every sequence $\{\mathcal{U}_k\}_{k\in\mathbb{N}} \subset L^2(0,S,W_2^{1,2}(D))$ converging strongly to $\mathcal{U}$, the sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}} \subset \mathfrak{Q}$ of minimizers of $\mathcal{F}_{\mathcal{A}_0,\alpha}^{\mathcal{U}^k}(\cdot)$ has a subsequence converging weakly to $\widetilde{\mathcal{A}}$, a minimizer of $\mathcal{F}_{\mathcal{A}_0,\alpha}^{\mathcal{U}}(\cdot)$.

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Theorem (Stability)

The minimizers of (9) are stable. Furthermore, if there exists a solution to (6), then there is at least one $f_{\mathcal{A}_0}$*-minimizing solution for such problem.*

Convergence

Theorem (Convergence)

We assume that there exist $\mathcal{A}^\dagger \in \mathfrak{Q}$ *solving the noiseless Inv. Prob. (6) and the map* $\alpha : (0,\infty) \to (0,\infty)$*, satisfies*

$$\lim_{\delta\to 0} \alpha(\delta) = 0 \quad \textit{and} \quad \lim_{\delta\to 0} \frac{\delta^2}{\alpha(\delta)} = 0. \tag{11}$$

Thus, when $\delta \to 0$*, and* $\mathcal{U}^\delta \to \widetilde{\mathcal{U}}$ *it follows that the minimizers* $\mathcal{A}_\alpha^\delta$ *converges weakly to* $\mathcal{A}^\dagger$*.*

Convergence Rates

Convergence Rates

Theorem (Convergence Rates)

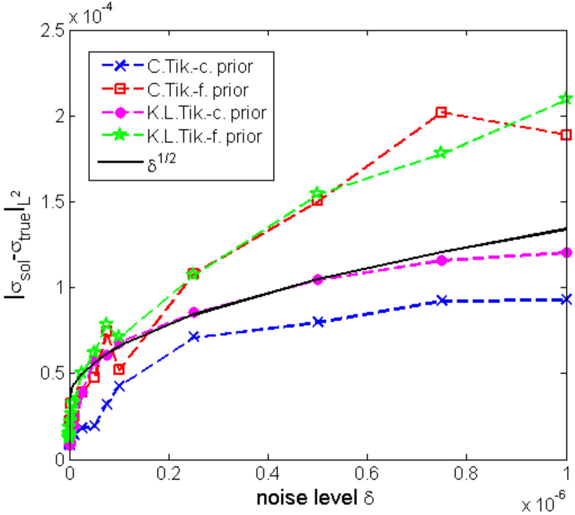
Let the map $\alpha : (0,\infty) \to (0,\infty)$ *be such that* $\alpha(\delta) \approx \delta$*.Then*

$$D_{\xi^\dagger}(\mathcal{A}^\delta, \mathcal{A}^\dagger) = \mathcal{O}(\delta) \quad \textit{and} \quad \|\mathcal{U}(\mathcal{A}^\delta) - \mathcal{U}^\delta\| = \mathcal{O}(\delta).$$

Then, we conclude that:

- This theorem quantifies how reliable solutions are.
- For example, if $f_{\mathcal{A}_0}(\mathcal{A}) = \|\mathcal{A} - \mathcal{A}_0\|$, then $\mathcal{A}^\delta_\alpha \to \mathcal{A}^\dagger$.

Convergence Rates (cont.)



The Classical One

$$f_{\mathcal{A}_0}(\mathcal{A}) = \|\mathcal{A} - \mathcal{A}_0\|^2_{H^l(0,T,H^{1+\varepsilon}(D))}$$

Kullback-Leibler

$$f_{\mathcal{A}_0}(\mathcal{A}) = \int_0^S \int_D \log(a(s,\tau,y)/a_0(s,\tau,y)) - (a(s,\tau,y) - a_0(s,\tau,y))d\tau dyds$$

In order to be mathematically precise, we assume that D is bounded when considering the second functional.

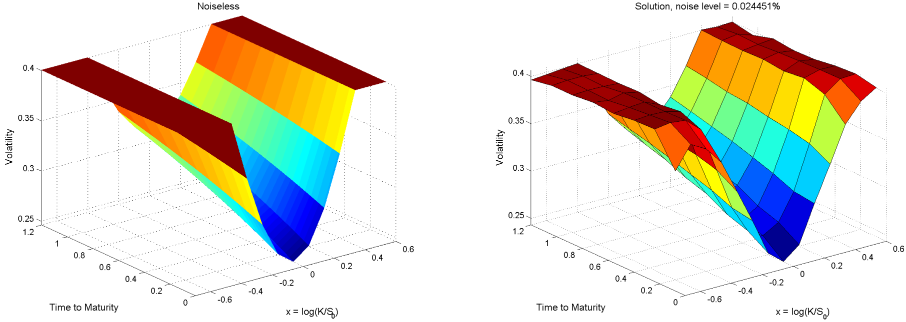
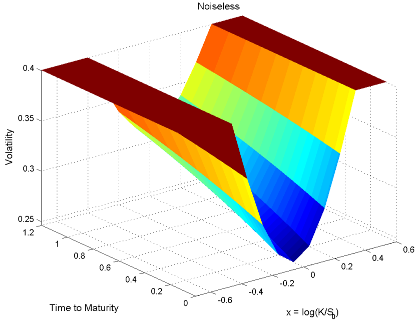
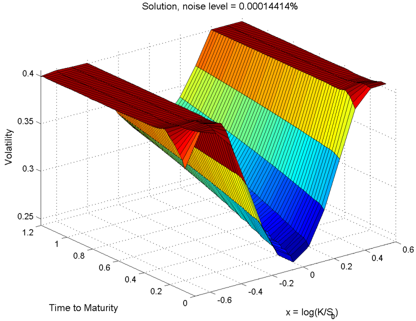


Figure: The first image shows the true volatility surface, the second is the reconstructed one and the third is the relative error.

Synthetic Data (cont.)





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Online × Standard Tikhonov

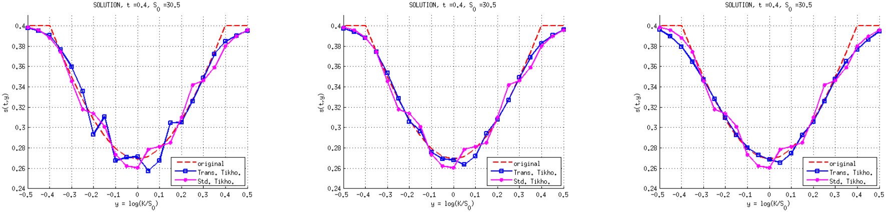


Figure: As the data amount increases, the reconstructions become better.

Online × Standard Tikhonov (cont.)

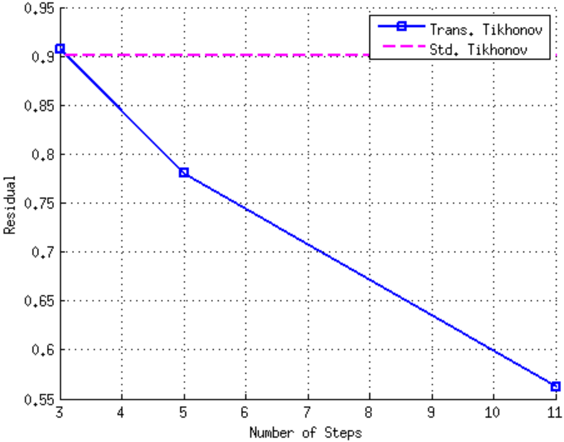


Figure: L^2 distance between original and reconstructions.

Heston Model

A popular model in practice [Gat06].

$$\begin{aligned}
&dS(t) = rS(t)dt + \sqrt{\nu(t)}S(t)dW_1(t) \text{ with } t \geq 0\\
&d\nu(t) = \kappa(\theta - \nu(t))dt + \eta\sqrt{\nu(t)}dW_2(t)\\
&S(0) = S_0 \text{ and } \nu(0) = \nu_0.
\end{aligned} \tag{12}$$

W_1 and W_2 are correlated $\widetilde{\mathbb{P}}$-Brownian motions, with correlation ρ.
Note that

$$(\nu_0, \theta, \kappa, \eta, \rho)$$

are usually estimated from market data.

Question:

Given the widespread use of Heston by practitioners,

what would prices given by Heston yield as local vol.?

Remember that

$$\sigma^2(S_0, T, K) = \mathbb{E}^{\widetilde{\mathbb{P}}}[\nu(T)|S(T) = K].$$

Use the parameters

$$(\nu_0, \theta, \kappa, \eta, \rho)$$

to simulate (12) and formula

$$C(S_0, T, K) = \mathbb{E}^{\widetilde{\mathbb{P}}}[(S_0 - K)^+]$$

by a Monte Carlo integration to interpolate real data.
Use this interpolated data as $\mathcal{U}^\delta$ in the analysis presented above in order to find $\widetilde{\mathcal{A}}$.

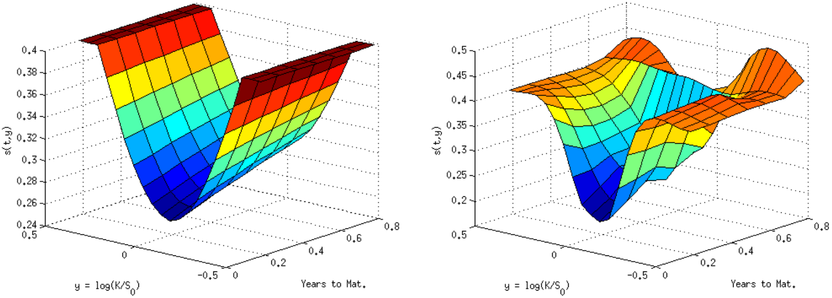
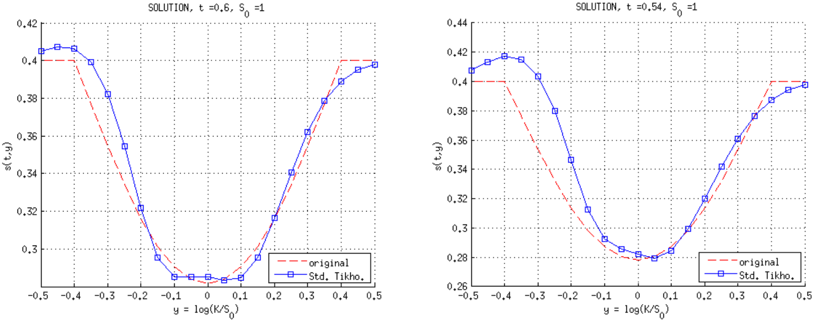


Figure: Original × Reconstruction: Heston data.



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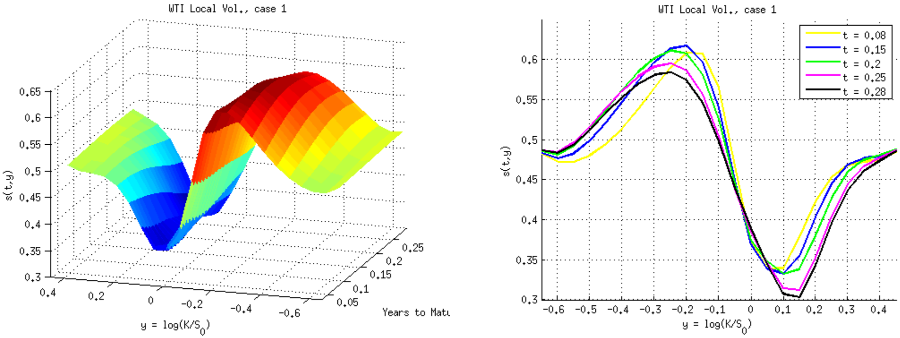


Figure: Reconstructions with the standard reg. functional for Online Tikhonov

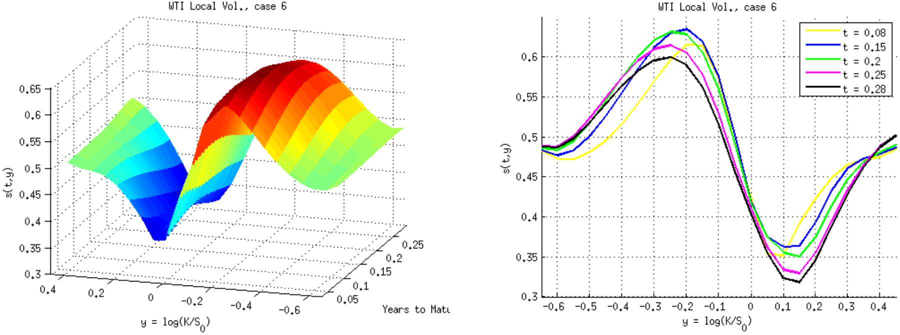
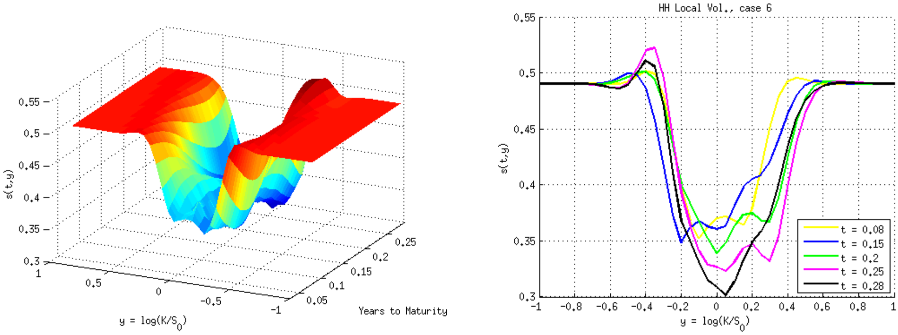


Figure: Reconstructions with the standard reg. functional for Online Tikhonov

Henry Hub



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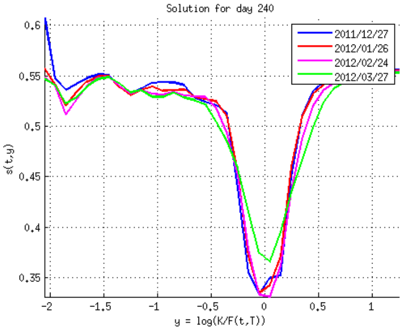

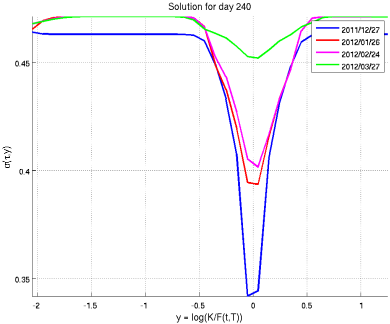

Figure: Day 239: Reconstructions with Classical and Kullback-Leibler functionals, respectively.

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- Reconstructions with the online version of Tikhonov became better as we increased the amount of data.
- We used Heston model for interpolating Vanilla option prices.

THANK YOU!

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