Two Applications of Inverse Problems Techniques[2]

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Outline

Introduction

Before solving parameter estimation problems, it is necessary:

- To describe the math. model of the problem.
- To state some regularity properties of the parameter to solution map.
- Is it linear, nonlinear, differentiable, satisfies the tangential cone condition,...?
- To identify the type of noise (white noise, impulsive noise, ...)
- To find some prior information.
- To consider the problem dimensionality.
- These help us to identify the most appropriate regularization technique to be used.

Outline

Pricing Derivatives

- Asset (Petrobras, Vale S.A., Itau, Sabesp,...) price dynamics means history.
- Pricing: expectation is more important than history.
- Expectation here means beliefs that practitioners have.
- Expectation is hidden in derivative prices.
- Derivatives are designed to reduce exposure to some source of risk.
- The most simple and most traded derivatives are vanilla options.

Vanilla Options

- **European Call**: gives the right, but not the obligation, of buying a share of an asset for a fixed strike price at its maturity.
- **Euopean Put** similar to the call, but gives the right of selling.
- American Option (call and put) can be exercised any time before its maturity.
- Sometimes, American options are more expensive than the European ones.
- The prices of such contract take into account asset dynamics.

- Typically, the asset price dynamics is given by a semi-martingale:

$$S_t = \text{something}_t + \text{Martingale}_t.$$

- What is a martingale?

$$\mathbb{E}[|M_t|] < \infty \quad \text{e} \quad \mathbb{E}[M_t \,|\, \{M_l,\ l \leq s\}] = M_s.$$

The Black-Scholes Model (1973) 1/2

- Let $(\Omega, \mathcal{U}, \widetilde{\mathbb{P}})$ be a prob. space with filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$.
- An asset price at time $t \geq 0$ is given by

$$dS_t = S_t(rdt + \sigma dW_t),$$

where W_t is a (risk neutral) Brownian motion and S_0 is given.
- An European call option price is then given by:

$$C(t, S_t, T, K) = \mathrm{e}^{-r(T-t)}\widetilde{\mathbb{E}}[\max\{0, S_T - K\} \,|\, \mathcal{F}_t].$$

- Feynman-Kac, when T and K are fixed, $C(t,S)$ satisfies the Black-Scholes PDE:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r\,S\frac{\partial C}{\partial S} - r\,V = 0,\ 0 < t < T,\ S > 0,$$

with terminal condition

$$C(T,S) = \max\{0, S - K\}.$$

The Black-Scholes Model (1973) 2/2

- Its solution is given by:

$$C(t,S) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy,$$

$$d_1(t,S) = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

e

$$d_2(t,S) = d_1 - \sigma\sqrt{T-t}.$$

Dupire's Local Volatility Model (1994) 1/2

- Let $(\Omega, \mathcal{V}, \mathcal{F}, \widetilde{\mathbb{P}})$ be a filtered prob. space.
- the asset price S_t satisfies:

$$\begin{cases} dS_t & = & (r-q)\, S_t\, dt + \sigma(t, S_t) S_t d\widetilde{W}_t, \ t \geq 0 \\ S_0 & \text{is} & \text{given.} \end{cases}$$

- Again, European call option price is given by:

$$C(t, S_t, T, K) = \widetilde{\mathbb{E}}[\mathrm{e}^{-r(T-t)} \max\{0, S_T - K\} \,|\, \mathcal{F}_t].$$

Dupire's Local Volatility Model (1994) 2/2

Fixing $t = 0$ and $S_t = S_0$, it follows that:

$$C(0, S_0, T, K) = e^{-rT} \int_0^{\infty} \max\{0, S - K\} \varphi(S, T)\, dS$$

and applying Fokker-Planck equation to φ and integrating by parts we find:

$$\begin{cases} \dfrac{\partial C}{\partial T} = \dfrac{1}{2}\sigma^2(T, K; S_0) K^2 \dfrac{\partial^2 C}{\partial K^2} - (r - q) K \dfrac{\partial C}{\partial K} - qC, \; T > 0, \; K \geq 0 \\ \lim\limits_{K \to 0} C(T, K) = S_0, \; T > 0, \\ \lim\limits_{K \to +\infty} C(T, K) = 0, \; T > 0, \\ C(T = 0, K) = \max\{0, S_0 - K\}, \; K > 0. \end{cases} \tag{1}$$

Adaptation to Commodity Markets

Again, Let $(\Omega, \mathcal{V}, \mathcal{F}, \widetilde{\mathbb{P}})$ be a risk neutral filtered prob. space.
$y_{t,T} = \log(F_{t,T}/F_{0,T})$ is the log-future.
Assuming that $y_{t,T}$ does not depends on T, i.e. $y_{t,T} = y_t$, and y_t satisfies:

$$dy_t = -a(S_0; t, y_t)dt + \sqrt{2a(S_0; t, y_t)}dW_t.$$

Since, $F_{t,T} = F_{0,T}\mathrm{e}^{y_t}$, it follows that

$$\frac{dF_{t,T}}{F_{t,T}} = \sqrt{2a(S_0; t, \log(F_{t,T}/F_{0,T}))}dW_t$$

and a call option on $F_{t,T}$ with maturity. T' and strike K is given by

$$\begin{aligned} C(t, F_{t,T}, T, K) &= \widetilde{\mathbb{E}}[\mathrm{e}^{-r(T-t)}\max\{0, F_{t,T} - K\} \,|\, \mathcal{F}_t] \\ &= \widetilde{\mathbb{E}}[\mathrm{e}^{-r(T-t)}\max\{0, F_{0,T}\mathrm{e}^{y_t} - K\} \,|\, \mathcal{F}_t]. \end{aligned}$$

Change of Variables

Setting $t = 0$ and $F_{t,T} = F_{0,T}$, define $\tau = T'$ and

$$v(\tau, y) = C(\tau, F_{0,T}e^{y})/F_{0,T},$$

so, v satisfies the PDE:

$$\left\{\begin{array}{rcl} \dfrac{\partial v}{\partial \tau} & = & a(S_0;\tau,y)\left(\dfrac{\partial^2 v}{\partial y^2} - \dfrac{\partial v}{\partial y}\right) - rv,\ \tau > 0,\ y \in \mathbb{R} \\ \lim\limits_{y\to-\infty} v(\tau,y) & = & 1,\ \tau > 0, \\ \lim\limits_{y\to+\infty} v(\tau,y) & = & 0,\ \tau > 0, \\ v(0,y) & = & \max\{0, 1 - e^{y}\},\ \text{se } y \in \mathbb{R}. \end{array}\right. \tag{2}$$

Notation

- $D = (0, \infty) \times \mathbb{R}$.
- $a_1, a_2 \in \mathbb{R}$ s.t. $0 < a_1 \leq a_2 < +\infty$.
- a_0 is s.t. $a_1 \leq a_0 \leq a_2$ e $\nabla a_0 \in (L^2(D))^2$.
- Define the set

$$Q := \{a \in a_0 + H^{1+\varepsilon}(D) : a_1 \leq a \leq a_2\}, \tag{3}$$

with $\varepsilon > 0$.

Proposition

If $a \in Q$, then the Cauchy problem is well-posed.

The Direct operator

Define

$$
\begin{aligned}
F : Q \subset H^{1+\varepsilon}(D) &\longrightarrow L^2(D) \\
a &\longmapsto V(a) - V(a_0).
\end{aligned}
$$

By Crepey (2003); Egger and Engl (2005); De Cezaro et al. (2012):

(i) F is continuous and compact.

(ii) F is weakly continuous and weakly closed.

(iii) F is Fréchet differentiable with Lipschitz continuous derivative.

(iv) F satisfies the tangential cone condition.

The "Online" Model

To associate indexed families of local volatility surfaces to families of surfaces of option prices, adapting results from Haltmeier et al..

- Denote the index by $s \in [0, \overline{s}]$.

- The family of local volatility surfaces by:

$$\mathcal{A} : s \in [0, \overline{s}] \longmapsto a(s; \tau, y) \in Q.$$

  Define also the set:

$$\mathfrak{Q} = \{\mathcal{A} \in \mathcal{A}_0 + H^l(0, T, H^{1+\varepsilon}(D)) \ : \ a(s) \in Q, \ s \in [0, \overline{s}]\}.$$

- The family of option prices:

$$\mathcal{V}(\mathcal{A}) : s \longmapsto v(a(s)), \ s \in [0, \overline{s}].$$

- Then, define the direct operator:

$$\mathcal{F} : \mathcal{A} \in \mathfrak{Q} \subset H^l(0, T, H^{1+\varepsilon}(D)) \longmapsto \mathcal{V}(\mathcal{A}) - \mathcal{V}(\mathcal{A}_0) \in L^2(0, S, L^2(D)).$$

Properties of the Direct Operator (Online Model)

In Albani and Zubelli (2014), it is shown that if $l > 1/2$ in $H^l(0,T,H^{1+\varepsilon}(D))$, $\mathcal{A}$ is continuous w.r.t. s, then, $\mathcal{F}$ satisfies:

(i) It is continuous and compact.

(ii) It is weakly continuous and weakly closed.

(iii) It is Frechét differentiable with Lipschitz derivative.

(iv) It satisfies the tangential cone condition and it is injective.

(v) The kernel of $\mathcal{F}'(\mathcal{A}^\dagger)^*$ is trivial.

- Let $\tilde{v}$ be a surface of European call option prices.
- Assume that it is given by Dupire's equation.
- So, the corresponding local volatility surface $a^{\dagger}$, solution of

$$\tilde{v} = v(a^{\dagger}). \tag{4}$$

Unfortunately, only scarce and noisy data v^{δ} is available:

$$\|\tilde{v} - v^{\delta}\| \leq \delta,$$

with $\delta > 0$ (noise level).

Tikhonov-type Regularization

- The inverse problem is ill-posed.
- Tikhonov-type regularization leads us to find an element in

$$\operatorname{argmin}\left\{\|\mathcal{V}(\mathcal{A})-\mathcal{V}^{\delta}\|^2_{L^2(0,S,L^2(D))}+\alpha f_{\mathcal{A}_0}(\mathcal{A}) \ : \ \mathcal{A}\in\mathfrak{Q}\right\}, \tag{5}$$

where $\mathfrak{Q}$ is the set of indexed families of local vol. surf.:

$$\mathcal{A} : s\in[0,\overline{s}]\longmapsto a(s)\in Q,$$

and

$$Q := \{a\in a_0+H^{1+\varepsilon}(D) : a_1\leq a\leq a_2\}.$$

Variational theory gives us existence and stability of minimizers, as well as convergence and convergence-rate results.

Let us consider the following:

- Replace $\mathcal{V}$ by a numerical approximation $\mathcal{V}_m$ in Y_m.

- Replace $\mathfrak{Q}$ by the finite dimensional set $\mathfrak{Q}_n = \mathfrak{Q} \cap X_n$;

- $Y_m \subset Y_{m+1} \subset \ldots \subset L^2(0, S, L^2(D))$ and $X_n \subset X_{n+1} \subset \ldots \subset H^l(0, T, H^{1+\varepsilon}(D))$, satisfy

$$\overline{\cup_{m\in\mathbb{N}} Y_m} = L^2(0, S, L^2(D)) \quad \text{and} \quad \overline{\cup_{n\in\mathbb{N}} X_n} = H^l(0, T, H^{1+\varepsilon}(D)).$$

Now we have the minimization problem:

$$\operatorname{argmin}\left\{\|\mathcal{V}_m(\mathcal{A}) - \mathcal{V}^\delta\|_{L^2(0,S,L^2(D))} + \alpha f_{\mathcal{A}_0}(\mathcal{A}) \ : \ \mathcal{A} \in \mathfrak{Q}_n\right\}. \tag{6}$$

Results

In the minimization problem

$$\operatorname{argmin}\left\{\|\mathcal{V}_m(\mathcal{A})-\mathcal{V}^\delta\|_{L^2(0,S,L^2(D))}+\alpha f_{\mathcal{A}_0}(\mathcal{A})\ :\ \mathcal{A}\in\mathfrak{Q}_n\right\},$$

choose appropriately α and n through the discrepancy principle:

$$\|V_m(a^{\delta,\alpha}_{m,n})-V^\delta\|\leq\lambda\delta.$$

Denote the vector of futures by $\mathbb{F}$, we must find $(\mathcal{A}_{m,n}^{\delta,\alpha};\mathbb{F})$ in

$$\text{argmin}\left\{\|P(\mathbb{F})\mathcal{V}_n(\mathcal{A})-\mathcal{V}^\delta\|^2+\psi_{\mathcal{A}_0}(\mathcal{A};\mathbb{F})\right\}, \quad (7)$$

where

$$\psi_{\mathcal{A}_0}(\mathcal{A};\mathbb{F}) = \alpha_1\sum_{l=0}^{L}\|a(s_l)-a_0(s_l)\|^2+\alpha_2\sum_{l=0}^{L}\|\partial_{y,m}a(s_l)\|^2+ \alpha_3\sum_{l=0}^{L}\|\partial_{\tau,m}a(s_l)\|^2+\alpha_4\sum_{l=0}^{L}\|q(\mathbb{F}(s_l),s_l)-q(\hat{\mathbb{F}}(s_l),s_l)\|^2+ \alpha_5\|\mathbb{F}-\hat{\mathbb{F}}\|^2+\frac{\alpha_6}{\Delta s^2}\sum_{l=1}^{L}\|a(s_l)-a(s_{l-1})\|^2.$$

$q(\mathbb{F}(s_l),s_l)$ represents boundary and initial conditions, and $\hat{\mathbb{F}}$ are the observed futures.

Since a and $\mathbb{F}$ are independent variables, so, we split the minimization as:

1. Fix $\mathbb{F}$ and minimize w.r.t. a.
2. Fix a and minimize w.r.t. $\mathbb{F}$.

Repeat until some tolerance is satisfied.

Numerical Solution

- Dupire's PDE is solved by a Crank-Nicolson scheme.
- The minimization of the Tikhonov-type functional are solved by the gradient descent method.
- The steps are chosen by Wolfe's rules.
- The iterations cease whenever the tolerance is satisfied:

$$\frac{\|V(A^k) - V^\delta\|}{\|V^\delta\|} < tol,$$

typically $tol = 0.01$.

Asset Price Correction

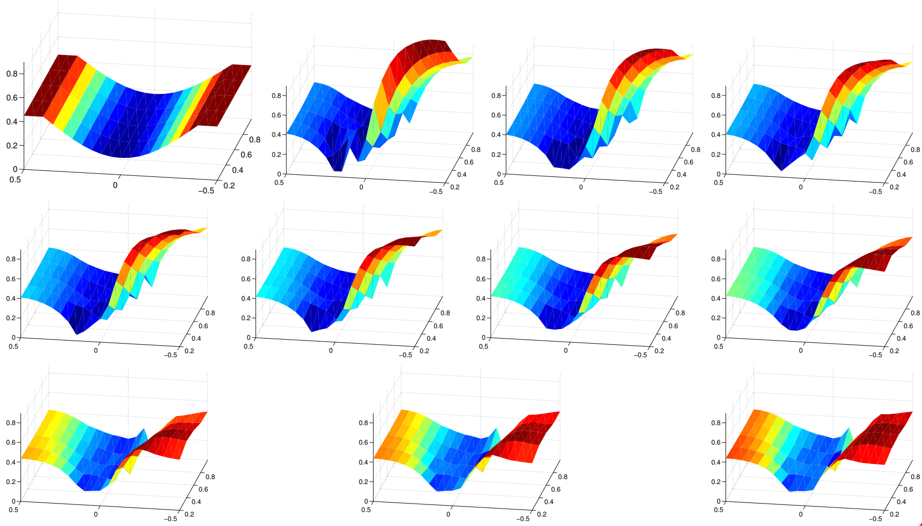
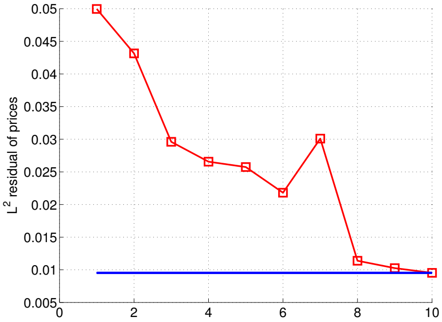


Figure: Local vol. after correction of the underlying asset prices.

Asset Price Correction



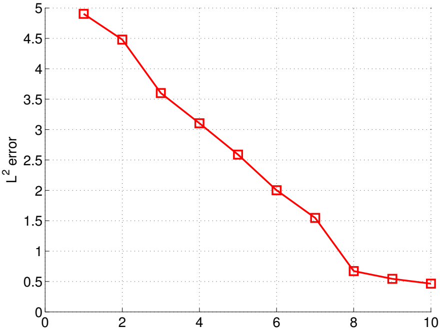


Figure: Esq.: Normalized Residual. Dir.: Normalized Error.

Asset Price Correction

	F_{0,τ_1}	F_{0,τ_2}	F_{0,τ_3}	F_{0,τ_4}	F_{0,τ_5}
$\mathbb{F}_{true}$	1.0809	1.0951	1.0309	0.9412	0.9000
$\mathbb{F}^0$	1.0269	1.0404	0.9794	0.8942	0.8550
$\mathbb{F}^{10}$	1.0801	1.0922	1.0262	0.9369	0.8936

Table: Futures Prices: True, Initial an after 10 iterations.

Henry Hub Natural Gas Data

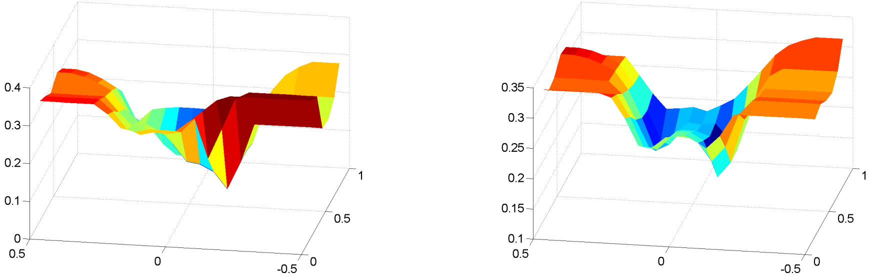


Figure: Local vol. reconstructions with original (left) and corrected (right) prices.

Henry Hub Natural Gas Data

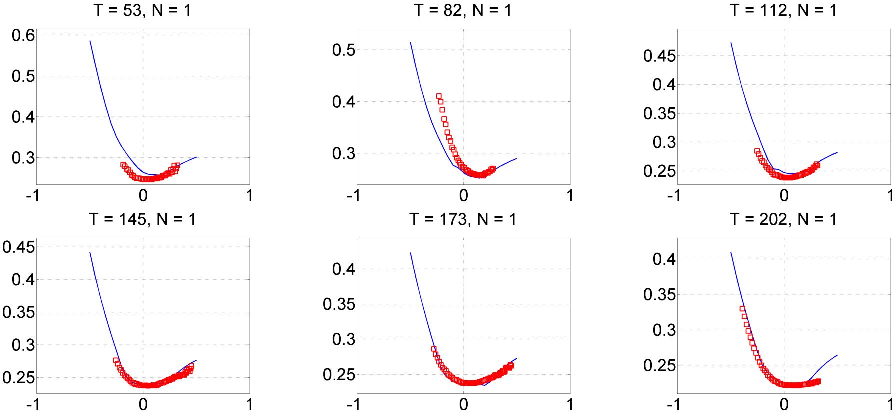


Figure: Implied volatility: Market (squares) and reconstructions (continuous line).

Henry Hub Natural Gas Data

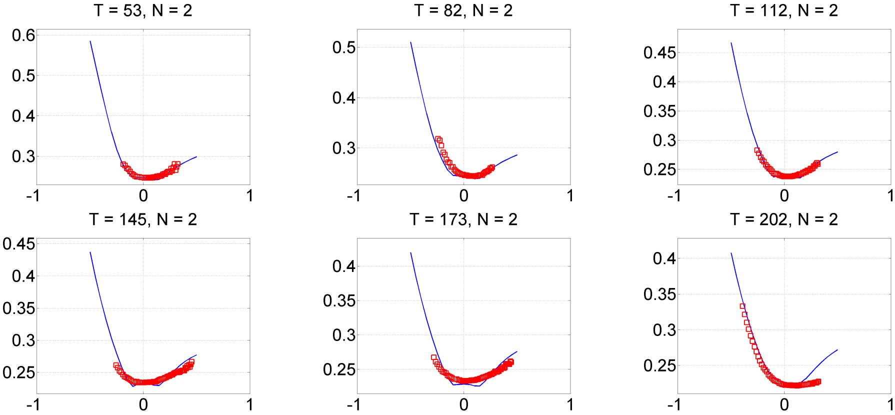


Figure: Implied volatility: Market (squares) and reconstructions (continuous line).

Henry Hub Natural Gas Data

Vencimento	10/29/13	11/27/13	12/27/13	01/29/14	02/26/14	03/27/14
Original	3.62	3.78	3.87	3.87	3.83	3.77
Ajustado	3.62	3.82	3.87	3.87	3.84	3.77

Table: Original and corrected future prices.

Calibração Online com Dados Sintéticos

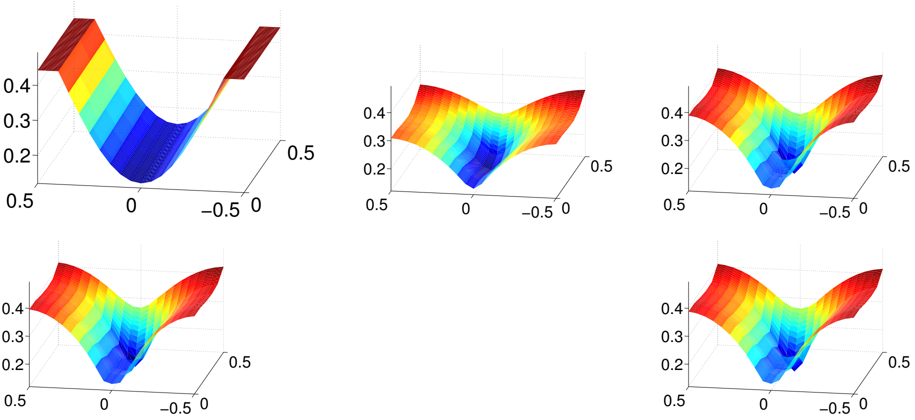
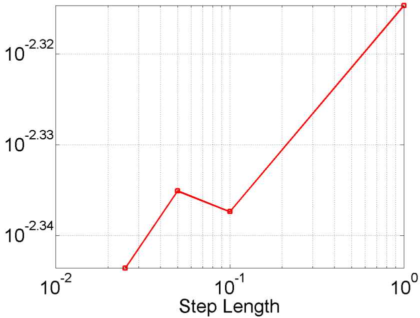


Figure: Vol. local original e reconstruções, a medida que aumentam os dados.

Online Calibration with Synthetic Data



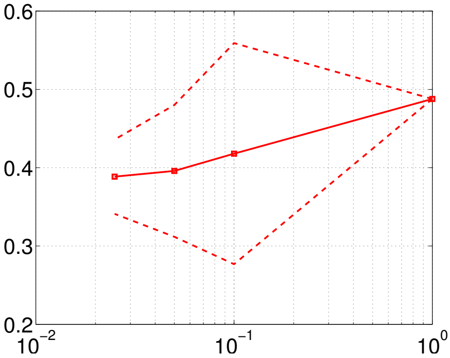


Figure: Left: Normalized Residual vs. Δs (squares). Right: Mean (squares) and std. deviation (dashed line) of normalized error.

Online Calibration with Henry Hub Data

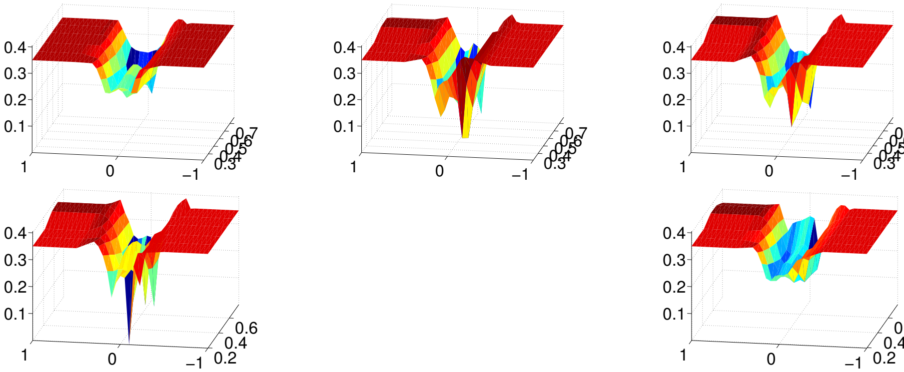


Figure: Local vol.: 04-Set-2013, 05-Set-2013, 09-Set-2013 and 10-Set-2013.

Online Calibration with Henry Hub Data

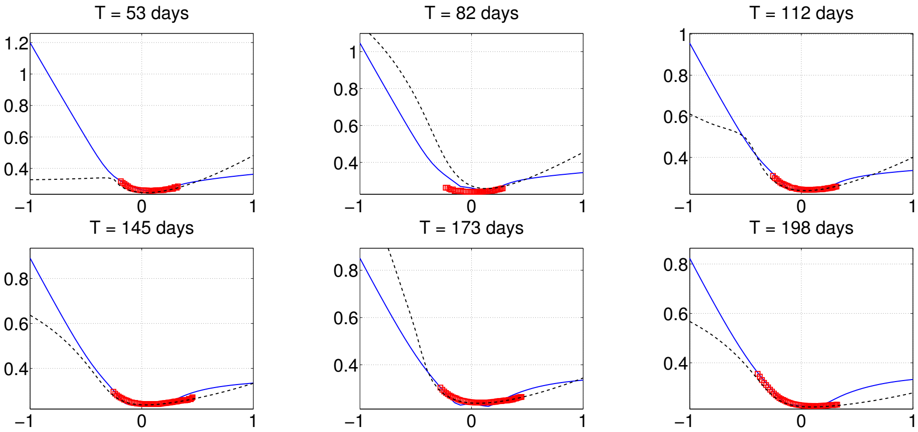


Figure: Implied volatility: Market (squares), SVI (dashed), and reconstructions (continuous line).

Online Calibration with WTI Data

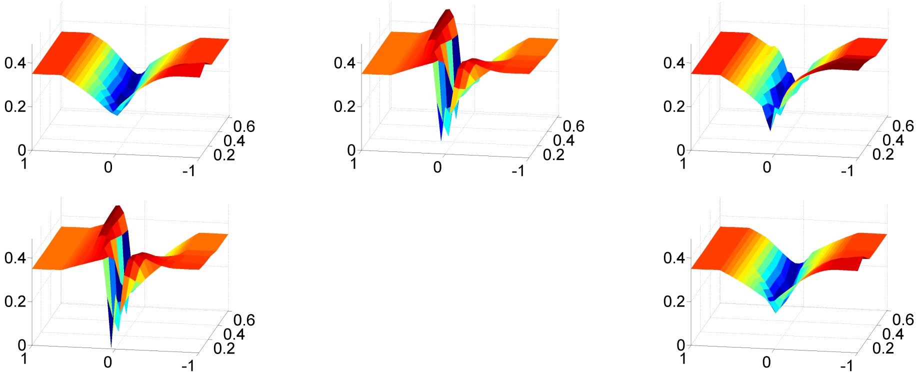


Figure: Local volatility: 04-Set-2013, 05-Set-2013, 09-Set-2013 and 10-Set-2013.

Online Calibration with WTI Data

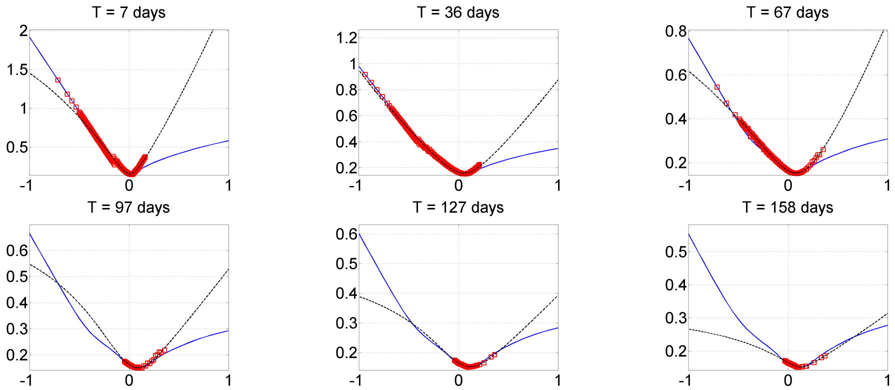


Figure: Implied Volatility: market (squares), SVI (dashed), and reconstructions (continuous line).

Pricing Exotic Option

Consider the Heston model:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^1, \quad 0 \le t \le T_{\max} \\ dV_t &= \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dW_t^2, \end{aligned} \tag{8}$$

Evaluate the price of *European Asian Options* with srike K, maturity $T_{\max}$ and payoff

$$A(T_{\max}) := \max \left\{ 0, \frac{1}{N} \sum_{j=0}^{N} S_{t_j} - K \right\},$$

where $t_j = j \cdot \Delta t$ and $\Delta t = T_{\max}/N$.

Pricing Exotic Option

	Local Volatility			Black & Scholes		
$\log(K/S_0)$	0	-0.1	0.1	0	-0.1	0.1
$\tau = 0.1$	0.0247	0.0387	0.0985	0.0067	0.0478	0.0519
$\tau = 0.5$	0.0189	0.0317	0.0495	0.0076	0.0576	0.1246
$\tau = 1.0$	0.0157	0.0103	0.0057	0.0757	0.1436	0.2370
$\tau = 1.5$	0.0400	0.0420	0.0426	0.1244	0.1791	0.2592

Table: Relative errors.

- We have introduced an adaptation of Dupire's model to commodity markets.
- We also applied calibration techniques based on Tikhonov-type regularization.
- Considered underlying asset as unknowns improving reconstructions.
- The online model also improves reconstructions.
- How to calibrate local volatility and jump-size distributions simultaneously?

Outline

For example, consider a population of *E. coli*.

- Typically rod-shaped unicellular organisms.
- Its volume falls between $0.6 - 0.7\mu m^3$.
- Extensively studied *in vitro* and *in vivo*.

Math. Model

Let $n(t,x)$ denote the population density of cells of "size" x at time t.
So, n satisfies

$$\partial_t n(t,x) + \partial_x[g(x)n(t,x)] = \int_0^\infty k(x,x')n(t,x')dx', \tag{9}$$

$g(x)$ = microscopic growth rate of individuals at size x,
$k(x,x')$ = proportion of cells of size x' that divide into cells of size x and $x'-x$.

Under this generality, the model is hard to calibrate and to make predictions.

A Simplified Model

Consider the following simplified version:

$$\begin{cases} \partial_t n(t,x) + \partial_x n(t,x) + B(x)n(t,x) = 4B(2x)n(t,2x), \; x,t \geq 0, \\ \\ n(t, x=0) = 0, \; t > 0, \\ \\ n(0,x) = n^0(x) \geq 0, \; x \geq 0. \end{cases} \tag{10}$$

The choice of $g \equiv 1$ was made and that a natural alternative would be that of an affine function.

The Stable-Size Distribution

There exist a unique an eigenpair λ_0 and $N = N(x)$, s.t., after a time re-normalization, the limit

$$n(t,x)e^{-\lambda_0 t} \longrightarrow \rho N(x), \qquad \text{as} \qquad t \to \infty, \tag{11}$$

holds under weighted L^p topologies, and the pair (λ_0, N) is a solution for

$$\begin{cases} \partial_x N(x) + (\lambda_0 + B(x))N(x) = 4B(2x)N(2x), \; x \geq 0, \\ \\ N(x=0) = 0, \\ \\ N(x) > 0, \; \text{for } x > 0, \int_0^\infty N(x)dx = 1. \end{cases} \tag{12}$$

Such N is the so-called stable-size distribution.

UFSC

Let the birth rate B be a measurable function and satisfy

$$0 < B_m \leq B(x) \leq B_M < \infty. \tag{13}$$

Then, we can define the *direct problem* as, given a birth rate B satisfying such conditions, finding the eigenpair (λ_0, N) of Problem (12).

Theorem (Perthame and Zubelli (2007))

The map

$$B \mapsto (\lambda_0, N),$$

from $L^\infty(\mathbb{R}_+)$ into $[B_m, B_M] \times L^1 \cap L^\infty(\mathbb{R}_+)$ is:

1. *continuous under the weak-$*$ topology of $L^\infty(\mathbb{R}_+)$,*
2. *locally Lipschitz continuous under the strong topology of $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$,*
3. *of class C^1 in $L^2(\mathbb{R}_+)$.*

UFSC

The Inverse Problem

It is to recover the birth rate B from noisy data N and the rate λ_0.

If the measurement N were smooth, one could directly solve for B, the PDE

$$4B(y)N(y) = B(y/2)\,N(y/2) + \lambda_0 N(y/2) + 2\partial_y N(y/2)\,,\; y > 0. \tag{14}$$

This is well-posed as long as N satisfies, e.g. $\partial_y N(y/2)$ is in L^p, for some $p \geq 1$.
However, this is not the case for reasonable practical data.

Inverse Problem Regularization: Tikhonov

Find minimizers for the following Tikhonov-type functional:

$$\mathcal{F}(B) = \|N(B) - N^{\text{obs}}\|^2_{L^2(\mathbb{R}_+)} + \alpha f_{B_0}(B), \tag{15}$$

with $B \in L^2(\mathbb{R}_+)$, satisfying (13), and $\alpha = 0.05$.
The penalization functional used are:

Smoothing: $f_{B_0}(B) = 0.01\|B - B_0\|^2_{L^2(\mathbb{R}_+)} + \|\partial_x B\|^2_{L^2(\mathbb{R}_+)}$, and

Kullback-Leibler: $f_{B_0}(B) = \int_0^\infty B(x)\log(B_0(x)/B(x)) - (B_0(x) - B(x))dx$.

where $B_0(x)$ is assumed constant.

Bayesian Techniques

Suppose that

- N and B are random variables.
- the data is corrupted by a Gaussian noise, with distribution $N(0, Id)$.
- the noise is additive and independent of N.

So, the likelihood function is

$$\pi(N|B) \propto \exp\left[-\|N(B) - N^{\text{obs}}\|^2_{L^2(\mathbb{R}_+)}\right]$$

The prior distribution can be chosen as

$$\pi_{prior}(B) \propto \exp\left[-\alpha\left(\|B - B_0\|^2_{L^2(\mathbb{R}_+)} + \|\partial_x B\|_{L^1(\mathbb{R}_+)}\right)\right].$$

By Bayes Theorem:

$$\pi_{posterior}(B|N^{\text{obs}}) \propto \pi_{prior}(B) \times \pi(N^{\text{obs}}|B).$$

Estimators

1. Maximum a posteriori (MAP):

$$B_{MAP} \in \arg\max \pi_{posterior}(B|N)$$

2. Conditional Mean:

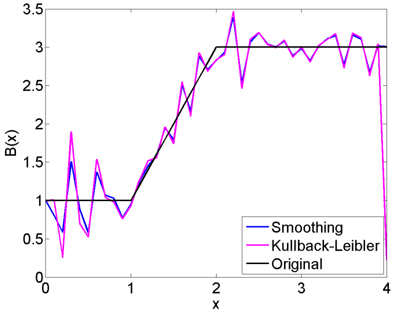
$$B_{CM} = \int B\pi_{posterior}(B|Nobs)dB,$$

   if the integral converges.

3. Other point estimators.

It is also possible to explore the posterior density by using a MCMC method.

Numerical Results: Synthetic Data



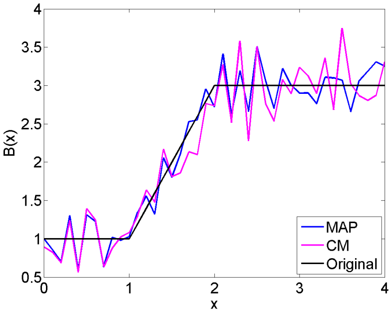
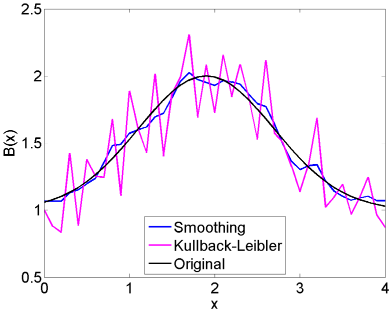


Figure: Reconstructions of a non-smooth *B* using Tikhonov regularization (left), and statistical techniques (right).

Numerical Results: Synthetic Data



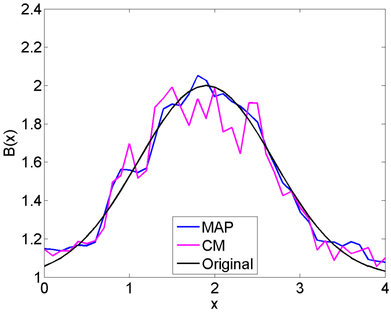


Figure: Reconstructions of a smooth B using Tikhonov regularization (left), and statistical techniques (right).

Numerical Results: Real Data – *E. coli*

Data from Doumic et al. (2010).

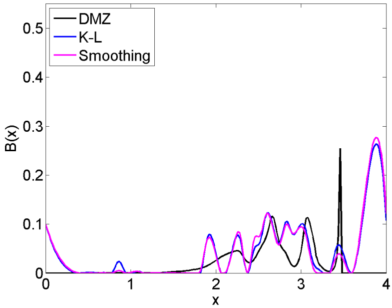


Figure: Reconstructions of *B* using Tikhonov-type (Smoothing and Kullbacl-Leibler) regularization (left), and statistical techniques (right).

Numerical Results: Real Data – *E. coli*

Data from Doumic et al. (2010).

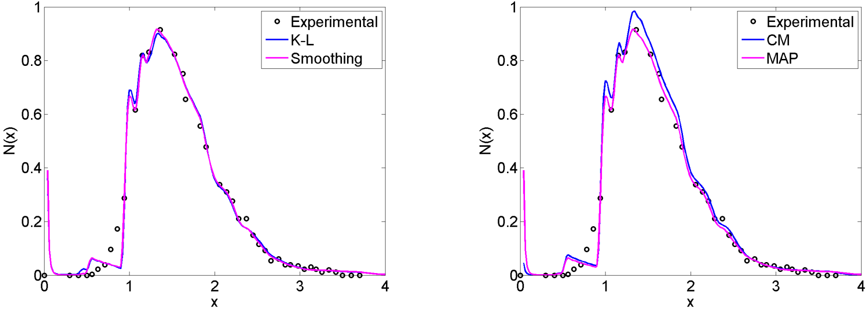


Figure: The density N corresponding to the reconstructions of B using Tikhonov-type (Smoothing and Kullbacl-Leibler) regularization (left), and statistical techniques (right).

1. Statistical Inverse Problems techniques are more versatile than Tikhonov reg.
2. However, they can be more computationally intensive.
3. We found similar results with Tikhonov reg. and point estimators.
4. So, they are at least as good as Tikhonov reg.
5. MAP and Tikhonov reg. are the same thing, at least intuitively.

Concluding Remarks

These inversion techniques can be used in many different applications, such as,

1. image processing (denoising, deblurring, ...)
2. medical imaging (CT, EIT, ...)
3. Geophysics
4. Math. Finance
5. Fluid dynamics
6. Biomath
7. and so on...

Albani, V. and Zubelli, J. P. Online Local Volatility Calibration by Convex Regularization. *Appl. Anal. Discrete Math.*, 8(2):243–268, 2014. doi: 10.2298/AADM140811012A.

Crepey, S. Calibration of the local volatility in a generalized Black-Scholes model using Tikhonov regularization. *SIAM Journal of Mathematical Analysis*, 34:1183–1206, 2003.

De Cezaro, A., Scherzer, O., and Zubelli, J. P. Convex regularization of local volatility models from option prices: Convergence analysis and rates. *Nonlinear Analysis*, 75(4):2398–2415, 2012. doi: 10.1016/j.na.2011.10.037,.

Doumic, M., Maia, P., and Zubelli, J. On the calibration of a size-structured population model from experimental data. *Acta Biotheoretica*, 58(4): 405–413, 2010. ISSN 0001-5342. doi: 10.1007/s10441-010-9114-9. URL http://dx.doi.org/10.1007/s10441-010-9114-9.

Egger, H. and Engl, H. Tikhonov Regularization Applied to the Inverse Problem of Option Pricing: Convergence analysis and Rates. *Inverse Problems*, 21: 1027–1045, 2005.

Haltmeier, M., Scherzer, O., and Leitão, A. Tikhonov and iterative regularization methods for embedded inverse problems. www.industrial-geometry.at/uploads/emb_preprint.pdf.

Perthame, B. and Zubelli, J. P. On the inverse problem for a size-structured population model. *Inverse Problems*, 23:1037, 2007. doi: doi:10.1088/0266-5611/23/3/012.