# EXERCISES 1 (INVERSE SEMIGROUPS, GROUPOIDS AND THEIR C*-ALGEBRAS)

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(1) Let $S$ be an inverse semigroup and $e \in E(S)$. Show that $S(e) := \{s \in S : s^*s = e = ss^*\}$ is a group (with the operation of $S$).

(2) Let $G$ be a semigroup with unit. Assume that for every $g \in G$, there is a *unique* $h \in G$ with $ghg = g$. Prove that $G$ is a group.

(3) Let $S$ be the universal unital semigroup generated by two elements $s, t$ with the relation $ts = 1$. Prove that $S$ is isomorphic to the Toeplitz semigroup, that is, the $*$-subsemigroup of $\mathbb{B}(\ell^2\mathbb{N})$ generated by the unilateral shift (the operator sending $\delta_n \mapsto \delta_{n+1}$).

(4) Let $E$ be a semilattice. Recall that a character of $E$ is a non-zero homomorphism $\chi\colon E \to \{0,1\} \subseteq \mathbb{C}$. Given a character $\chi$, prove that $\mathcal{F}_\chi := \chi^{-1}(1)$ is a *filter*, meaning a non-empty subset $\mathcal{F} \subseteq E$ such that $e \geq f \in \mathcal{F}$ implies $e \in \mathcal{F}$ and $e, f \in \mathcal{F}$ implies $ef \in \mathcal{F}$. Prove that the assignment $\chi \mapsto \mathcal{F}_\chi$ is a bijection between characters and filters.

Now define $I_\chi := \chi^{-1}(0)$. Prove that $I$ is a (possibly empty) proper prime ideal of $E$ (where proper means different from $E$ and prime ideal means $ef \in I$ iff $e \in I$ or $f \in I$) and that the assignment $\chi \mapsto I_\chi$ is a bijection between characters and proper ideals. In particular, the assignment $\mathcal{F} \mapsto \mathcal{F}^c$ is a bijection between filters and proper prime ideals.

In the case of a semilattice with zero $0 \in E$, observe that characters preserving zero ($\chi(0) = 0$) correspond to filters not containing 0 or to proper prime ideals containing 0.

(5) Given a semilattice $E$, we shall write $\widehat{E}$ for the space of all characters $\chi\colon E \to \{0,1\} \subseteq \mathbb{C}$. When endowed with the topology of pointwise convergence, $\widehat{E}$ will be called the spectrum of $E$. Prove that $\widehat{E}$ is a locally compact Hausdorff space and that its topology is the same as the induced topology from the product space $\{0,1\}^E$. Given $e \in E$, define $U_e := \{\chi \in \widehat{E} : \chi(e) = 1\}$. Prove that the sets $U_e$, $e \in E$ together with their complements $U_e^c$ form a subbasis for the topology of $\widehat{E}$.

(6) Consider $E = \{e_0, e_1, e_2, \ldots\}$ the semilattice with $e_0 > e_1 > e_2 > \ldots$. Recall that $E$ is (isomorphic to) the semilattice of idempotents of the Toeplitz semigroup. Compute the spectrum $\widehat{E}$ of $E$ showing that it is homeomorphic to the one-point (i.e. Alexandroff) compactification $\mathbb{N}_\infty = \mathbb{N} \sqcup \{\infty\}$ of the natural numbers. In particular, $C^*(E) \cong C(\mathbb{N}_\infty)$.

(7) Now consider $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the natural order as a semilattice (that is, $0 < 1 < 2 < 3 < \ldots$). This is a semilattice with zero. Compute its spectrum $\widehat{\mathbb{N}}$ and its reduced spectrum $\widehat{\mathbb{N}}_0$, showing that both are homeomorphic to $\mathbb{N}$ as a discrete space. In particular, $C^*(\mathbb{N}) \cong C_0(\mathbb{N}) \cong C_0^*(\mathbb{N})$.

(8) Given two $*$-algebras $A, B$, assuming that both admit a maximal $C^*$-seminorm, prove that the same holds for its direct sum $A \oplus B$ and that

$C^*(A \oplus B) \cong C^*(A) \oplus C^*(B)$, where $C^*(A)$ denotes the universal enveloping $C^*$-algebra of $A$. Conclude, in particular, that $C^*(S) \cong C^*_0(S) \oplus \mathbb{C}$. Prove also that $C^*_r(S) \cong C^*_{r,0}(S) \oplus \mathbb{C}$.

(9) Let $I$ be an arbitrary set and define $S$ to be $S := I \times I \sqcup \{0\}$ endowed with the operation $(i, j) \cdot (k, l) := (i, l)$ if $j = k$ and 0 otherwise and where 0 works as a zero element: $0x = x0 = 0$. Prove that $S$ is an inverse semigroup with $(i, j)^* := (j, i)$ and that $C^*_0(S)$ is isomorphic to the $C^*$-algebra $\mathrm{K}(\ell^2 I)$ of compact operators on the Hilbert space $\ell^2 I$ (of square-summable functions $I \to \mathbb{C}$). Using that this $C^*$-algebra is simple, conclude that $C^*_{r,0}(S) \cong C^*_0(S) \cong \mathrm{K}(\ell^2 I)$.

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