EXERCISES 2 (INVERSE SEMIGROUPS, GROUPOIDS AND THEIR C*-ALGEBRAS)

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(1) Let $G = X \times X$ be the pair groupoid on a topological space X. We know that G is always a topological groupoid (with respect to the product topology). Prove that G is étale iff X is discrete.

(2) (Pullback groupoids) Let G be a topological groupoid and let $p\colon X \to G^0$ be a continuous surjective open map from a topological space X onto G^0. Define $p^*(G)^0 := X$, and

$$p^*(G)^1 := X \times_{p,r} G^1 \times_{s,p} X := \{(x, g, y) : p(x) = r(g), p(y) = s(g)\}$$

Show that $p^*(G)$ is a topological groupoid with respect to the product topology inherited from $X \times G^1 \times X$ and the following operations:

$$s(x, g, y) := y, \quad r(x, g, y) := x,$$

$$(x, g, y) \cdot (y, h, z) := (x, gh, z) \quad \text{for } s(g) = r(h),$$

$$(x, g, y)^{-1} := (y, g^{-1}, x).$$

Prove that the source and range maps on $p^*(G)$ are open if the source and range maps on G are open. Prove that $p^*(G)$ is étale if G is étale and p is a local homeomorphism.

Specialise to the case where $G = Y$ is a topological space (viewed as a topological groupoid with only units) and show that in this case $p^*(Y)$ can be identified with the groupoid of the equivalence relation $\sim$ on X induced by p, that is, $x \sim y$ iff $p(x) = p(y)$. In particular this is an étale groupoid if p is a local homeomorphism.

Further specialisation to the case in which $X = \sqcup_{i\in I} U_i$ is the disjoint union of an open cover $(U_i)_{i\in I}$ and the obvious map $p\colon X \to Y$ yields a groupoid $p^*(Y)$ which is (topologically) isomorphic to the Čech groupoid of the cover, which is therefore always an étale groupoid.

(3) Let Γ be a group and let $\Gamma_0 = \Gamma \sqcup \{0\}$ be the inverse semigroup Γ with a formal zero added. Let G be the groupoid of germs of the action θ of Γ_0 on $X := [0, 1]$ with $\theta_\gamma = \mathrm{Id}_X$ for all $\gamma \in \Gamma$ and $\theta_0 = \mathrm{Id}_{[0,1)}$. Check that G can be identified with the set $[0, 1] \cup \Gamma$ in which $1 \in [0, 1]$ is also the unit of Γ and the groupoid operations are the obvious ones: trivial on $[0, 1]$ (which is the unit space) and the group operations on Γ (which is the isotropy group at $x = 1$). The topology is the usual on $[0, 1]$ and if a sequence $(t_n) \subseteq [0, 1)$ converges to 1, then it also converges to every $\gamma \in \Gamma$.

Show that $C_c(G)$ is the set of all functions $f\colon G \to \mathbb{C}$ such that f is continuous on $[0, 1)$, $f(\gamma) \neq 0$ only for finitely many $\gamma \in \Gamma$ and

$$\lim_{x\to 1^-} f(x) = \sum_{\gamma\in\Gamma} f(\gamma).$$

Next show that the *-algebra of G can be canonically identified with

$$C_c(G) \cong \left\{ (\varphi, \psi) \in C[0,1] \oplus \mathbb{C}[\Gamma] : \varphi(1) = \sum_{\gamma \in G} \psi(\gamma) \right\}.$$

Notice that $\psi \mapsto \sum_{\gamma\in\Gamma} \psi(\gamma)$ is the integrated form of the trivial representation $\Gamma \to \mathbb{C}$, so it extends to a representation (a character) $\epsilon\colon C^*(\Gamma) \to \mathbb{C}$. Show that

$$C^*(G) \cong \{(\varphi, \psi) \in C[0,1] \oplus C^*(\Gamma) : \varphi(1) = \epsilon(\psi)\}.$$

Moreover, show that if Γ is amenable (that is, if $C^*(\Gamma) \cong C^*_r(\Gamma)$), then $C^*(G) \cong C^*_r(G)$, and otherwise

$$C^*_r(G) \cong C[0,1] \oplus C^*_r(\Gamma).$$

Hint: you may use that Γ is non-amenable iff $\ker(\lambda^\Gamma) \not\subseteq \ker(\epsilon)$ in order to show that $C[0,1] \oplus C^*_r(\Gamma)$ is a quotient of $C^*(G)$. Here $\lambda^\Gamma\colon C^*(\Gamma) \to C^*_r(\Gamma)$ denotes the left regular representation of Γ.

(4) Consider the snake with two heads groupoid $G = [0,1] \sqcup \{\gamma\}$ (the groupoid of the previous example with $\Gamma = \{1, \gamma\} \cong \mathbb{Z}_2$), but now consider on G the compact Hausdorff topology in which γ is an isolated point and $[0,1]$ carries the usual topology. Prove that G is a semi-étale topological groupoid. Is it étale? As in the étale case, endow the space $C(G) = C_c(G)$ of all continuous functions $G \to \mathbb{C}$ with the convolution product and involution:

$$(f * g)(x) = \sum_{yz=x} f(y)g(z), \quad f^*(x) := \overline{f(x^{-1})}.$$

Prove that $C(G)$ is a *-algebra with these operations and describe it proving that it is isomorphic to the C^*-algebra $C[0,1] \oplus \mathbb{C}$. Hint: The isomorphism should not be obvious! Observe that the algebra is unital (with $1 = \chi_{[0,1]}$, the characteristic functions of $[0,1]$) and we have two canonical complementary projections: $p = \delta_\gamma = \chi_{\{\gamma\}}$ and $q := 1 - p$.

(5) Now consider again the snake with two heads with the non-Hausdorff (but compact locally Hausdorff) topology but forget its groupoid structure: view it only as a space $Y := [0,1] \sqcup \{\gamma\}$. Consider the cover of Y consisting of the two open subsets $U_1 := [0,1]$ and $U_2 := [0,1) \sqcup \{\gamma\} \cong [0,1]$ and form the corresponding Čech groupoid $H = p^*(Y)$ (as in the (2)), which is an étale compact Hausdorff groupoid. It is the groupoid of the equivalence relation on $[0,1] \sqcup [0,1]$ that identifies all points in the two copies of $[0,1]$ except at 1. Prove that the *-algebra of H is

$$C_c(H) \cong \{f \in C([0,1], \mathbb{M}_2(\mathbb{C})) : f(1) \text{ diagonal}\}.$$

In particular $C^*(H) \cong C^*_r(H) \cong C_c(H)$ is isomorphic to the C^*-algebra above. Describe the regular representations $\lambda_x\colon C_c(H) \to \mathbb{B}(\ell^2(H_x))$ of H and the conditional expectation $C^*_r(H) \to C[0,1]$ via the isomorphism given above.

(6) Let S be a E-unitary inverse semigroup and let $\Gamma := G_{\max}(S)$ be the maximal group homomorphic image of S. Given an action $\theta\colon S \to \mathrm{pHomeo}(X)$ of S on a topological space, prove that it induces a partial action $\tilde{\theta}\colon \Gamma \to \mathrm{pHomeo}(X)$ with domains $\tilde{D}_\gamma := \cup_{\sigma(s)=\gamma} D_s$, where $\sigma\colon S \to \Gamma$ is the quotient homomorphism, and such that $\tilde{\theta}_\gamma(x) := \theta_s(x)$ if $x \in D_{s^*s} = D_{s^*}$ with $\sigma(s) = \gamma$.

Then show that there is a canonical isomorphism of topological groupoids $S \ltimes X \cong \Gamma \ltimes X$. Here $\Gamma \ltimes X$ denotes the transformation groupoid for the partial action $\tilde{\theta}$. It has X as unit space and the space of arrows is $(\Gamma \ltimes X)^1 := \{(\gamma, x) : x \in \tilde{D}_{\gamma^{-1}}\} \subseteq \Gamma \times X$ with the induced product topology. The groupoid operations of $\Gamma \rtimes X$ are as "usual".

In particular $S \ltimes X$ is (locally compact and) Hausdorff whenever X is (locally compact and) Hausdorff and S is E-unitary.

(7) Given a groupoid G and $U \subseteq G^0$, we define $G_U := s^{-1}(U) \cap r^{-1}(U)$. Check that G_U is a subgroupoid of G with $U = G^0_U$; we call it the restriction of G to U. Show that G_U is always a "hereditary" subgroupoid in the sense that $G_U \cdot G \cdot G_U \subseteq G_U$ and that it is an "ideal" in the sense that $G_U \cdot G, G \cdot G_U \subseteq G_U$ if U is G-invariant, meaning that $s^{-1}(U) = r^{-1}(U)$.

All this works in a purely algebraic level, no topology is necessary or required. Now, if G is a topological groupoid and if $U \subseteq G^0$ is open (resp. closed), then G_U is an open (resp. closed) subgroupoid in G. In particular, G_U is locally compact if G is locally compact and U is open or closed. What do you get for G_U if $U = \{x\}$ with $x \in G^0$?

Show that if G is étale and U is open or closed, then G_U is also étale.

(8) Let G be an étale groupoid with G^0 locally compact and Hausdorff. Let $U \subseteq G^0$ be an open subset. Show that the canonical map $C_c(G_U) \to C_c(G)$ given by $f \mapsto \tilde{f}$ (extension by zero) is a well-defined injective *-homomorphism that identifies $C_c(G_U)$ with a hereditary *-subalgebra of $C_c(G)$, meaning that $C_c(G_U) \cdot C_c(G) \cdot C_c(G_U) \subseteq C_c(G_U)$. Moreover, if U is G-invariant, show that $C_c(G_U)$ is identified with an ideal in $C_c(G)$ and that the quotient *-algebra $C_c(G)/C_c(G_U)$ is canonically isomorphic to $C_c(G_F)$, where $F := G^0 \backslash U$ (which is closed and G-invariant). Hence we get a short exact sequence of *-algebras:

$$0 \to C_c(G_U) \to C_c(G) \to C_c(G_F) \to 0. \tag{0.1}$$

In particular, $C_c(G)$ cannot be simple (no proper *-ideals) unless G is *minimal* in the sense that the only G-invariant open (or closed) subsets of G^0 are $\emptyset$ and G^0.

(9) Show that (0.1) extends to a short exact sequence of C^*-algebras:

$$0 \to C^*(G_U) \to C^*(G) \to C^*(G_F) \to 0.$$

(10) Show that the sequence (0.1) also extends to a sequence

$$0 \to C^*_r(G_U) \to C^*_r(G) \to C^*_r(G_F) \to 0$$

which is exact on the left and on the right, meaning that the map $C^*_r(G_U) \to C^*_r(G)$ is always injective, the map $C^*_r(G) \to C^*_r(G_F)$ is always surjective, but it is not, in general, exact in the middle: the ideal $C^*_r(G_U)$ in $C^*_r(G)$ is always contained in $\ker(C^*_r(G) \to C^*_r(G_F))$, but this is eventually bigger! As an example for this, consider the groupoid $G = [0,1] \cup \Gamma$ from (3) (with the non-Hausdorff étale topology) for a non-amenable group Γ and $U := [0,1)$.

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