EXERCISES 1 (HILBERT MODULES AND FELL BUNDLES)

ALCIDES BUSS

(1) Let $\mathcal{E}, \mathcal{F}$ Hilbert B-modules and $\mathcal{E}_0 \subseteq \mathcal{E}$ a dense B-submodule. Suppose $S\colon \mathcal{E}_0 \to \mathcal{F}$ and $T\colon \mathcal{F} \to \mathcal{E}$ are linear maps with T bounded and satisfying $\langle S(\xi) \,|\, \eta\rangle_B = \langle \xi \,|\, T(\eta)\rangle$ for all $\xi \in \mathcal{E}_0$ and $\eta \in \mathcal{F}$. Prove that S extends to an adjointable operator $S\colon \mathcal{E} \to \mathcal{F}$ with $S^* = T$.

(2) Let B be a C^*-algebra and define

$$\ell^2(\mathbb{N}, B) := \left\{ f\colon \mathbb{N} \to B : \sum_{n=1}^{\infty} f(n)^* f(n) \text{ converges inconditionally in } B \right\}.$$

Prove that $\ell^2(\mathbb{N}, B)$ has a canonical structure of a Hilbert B-module and with this structure it is isomorphic to the tensor product $\ell^2(\mathbb{N}) \otimes_{\mathbb{C}} B$ (where we use the obvious map $\mathbb{C} \to \mathcal{L}(B) = \mathcal{M}(B)$ to form the (internal) tensor product). Moreover, prove that both Hilbert B-modules can be canonically identified with the (Hilbert B-module) direct sum $\oplus_{n=1}^{\infty} B_n$ with $B_n = B$ (as a Hilbert B-module) for all n. Generalize (i.e. formulate and prove) all the above statements to arbitrary index sets I in place of $\mathbb{N}$.

(3) The goal of this exercise is to prove that every Hilbert module can be represented as a TRO (Ternary Ring of Operators). Recall that a TRO is a closed subspace $E \subseteq \mathbb{B}(H, K)$ for K, H Hilbert spaces, such that $EE^*E \subseteq E$. One may assume that $K = H$ by replacing both K and H by its direct sum $H \oplus K$ and representing $\mathbb{B}(H, K) \hookrightarrow \mathbb{B}(H \oplus K)$ in the canonical way. Check and write the details. Also check that for a TRO E, $A := \overline{\operatorname{span}}\, EE^*$ and $B = \overline{\operatorname{span}}\, E^*E$ are C^*-algebras and one has $\overline{\operatorname{span}}\, AE = \overline{\operatorname{span}}(EE^*E) = \overline{\operatorname{span}}(EB) = E$. Conclude that E may be viewed as a Hilbert $A - B$-bimodule in a canonical way.

Now let $\mathcal{E}$ be a Hilbert module over a C^*-algebra B. Represent B into $\mathbb{B}(H)$ via some representation $\pi\colon B \to \mathbb{B}(H)$. Consider the Hilbert space $K := \mathcal{E} \otimes_\pi B$ (internal tensor product of Hilbert modules) and define

$$\tilde{\pi}\colon \mathcal{E} \to \mathbb{B}(H, K), \quad \tilde{\pi}(\xi)(\zeta) := \xi \otimes \zeta.$$

Also define $\rho\colon \mathcal{L}_B(\mathcal{E}) \to \mathbb{B}(K)$ by $\rho(T)(\eta \otimes \zeta) := T(\eta) \otimes \zeta$. Prove that these both maps are well-defined continuous linear maps (you also have to prove that $\rho(T)$ can be extended to $K = \mathcal{E} \otimes_\pi H$) and that ρ is a unital *-homomorphism (that is, ρ is a representation of $\mathcal{L}(\mathcal{E})$). Moreover, check the following properties:

- $\tilde{\pi}(\xi)\pi(b) = \tilde{\pi}(\xi \cdot b)$;
- $\rho(T)\tilde{\pi}(\xi) = \tilde{\pi}(T(\xi))$;
- $\tilde{\pi}(\xi)^*\tilde{\pi}(\eta) = \pi(\langle \xi \,|\, \eta\rangle_B)$;
- $\tilde{\pi}(\xi)\tilde{\pi}(\eta)^* = \rho(\langle\!\langle \xi | \eta \rangle\!\rangle)$, where $\langle\!\langle \xi | \eta \rangle\!\rangle := \theta_{\xi,\eta}$,

for all $\xi, \eta \in \mathcal{E}$, $b \in B$, $T \in \mathcal{L}(\mathcal{E})$. Conclude that if π is faithful (e.g. the Gelfand-Naimark representation), then $\tilde{\pi}$ and ρ are also faithful (and indeed, isometric) so that $\mathcal{E}$ can identified with the closed subspace $E := \tilde{\pi}(\mathcal{E}) \subseteq \mathbb{B}(H, K)$, which is a TRO. Also check that the C^*-algebra $A = \overline{\operatorname{span}}\, EE^* \subseteq \mathbb{B}(K)$ identifies with $\mathbb{K}(\mathcal{E})$ via ρ and that the C^*-algebra $\overline{\operatorname{span}}\, E^*E \subseteq \mathbb{B}(H)$ identifies with the ideal $I := \overline{\operatorname{span}}\, \langle \mathcal{E} \,|\, \mathcal{E}\rangle \subseteq B$ via π.