EXERCISES 2 (HILBERT MODULES AND FELL BUNDLES)

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(1) Let X be a locally compact Hausdorff space. Prove that a positive function $h \in C_0(X)$ is strictly positive if and only if $h(x) > 0$ for all $x \in X$. Prove that $C_0(X)$ is σ-unital if and only if X is σ-compact, that is, $X = \cup_{n=1}^{\infty} K_n$ for some sequence of compacts $K_n \subseteq X$.

(2) A *quasi-basis* (or *frame*) for a Hilbert B-module $\mathcal{E}$ is a family of vectors $(\xi_i)_{i\in I}$ in $\mathcal{E}$ such that

$$\sum_{i\in I} \xi_i \langle \xi_i \,|\, \eta \rangle = \eta \quad \text{for all } \eta \in \mathcal{E}.$$

If B is unital, an *orthogonal basis* for $\mathcal{E}$ is a quasi-basis as above with the additional property that $\langle \xi_i \,|\, \xi_j \rangle_B = 0$ if $i \neq j$. If, in addition, $\|\xi_i\| = 1$ for all i, we say that the basis is *orthonormal.* Prove that the Hilbert B-module $\ell^2(I, B) := \ell^2(I) \otimes_{\mathbb{C}} B$ has an orthonormal basis of cardinality $|I|$ if B is unital (and I is some index set). In particular, the standard B-module $\mathcal{H}_B = B^\infty$ has a countable orthonormal basis. Using Kasparov's stabilisation theorem, prove that every countably generated Hilbert B-module has a countable quasi-basis.

Give an example of a Hilbert B-module over a unital C^*-algebra that has no orthogonal basis. Hint: consider the ideal $C_0(0, 1]$ of continuous functions $[0, 1] \to \mathbb{C}$ vanishing at 0 as a Hilbert module over $B = C[0, 1]$.

(3) Let $\mathcal{E}$ be Hilbert B-module for some C^*-algebra B. Prove that the following assertions are equivalent:
- $\mathcal{L}_B(\mathcal{E}) = \mathcal{K}_B(\mathcal{E})$;
- the identity operator $\mathrm{Id}_{\mathcal{E}}$ is compact, that is, $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_B(\mathcal{E})$;
- $\mathcal{K}_B(\mathcal{E})$ is unital;
- there is $n \in \mathbb{N}$ and $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{E}$ such that for all $\zeta \in \mathcal{E}$,

$$\zeta = \sum_{i=1}^{n} \xi_i \langle \eta_i \,|\, \zeta \rangle_B$$

- every B-linear operator $T\colon \mathcal{E} \to \mathcal{E}$ is compact, that is, if $T(\xi + \eta \cdot b) = T(\xi) + T(\eta) \cdot b$ for all $\xi, \eta \in \mathcal{E}$, $b \in B$, then $T \in \mathcal{K}_B(\mathcal{E})$;
- every B-linear operator $T\colon \mathcal{E} \to \mathcal{F}$ is compact, that is, $T \in \mathcal{K}_B(\mathcal{E}, \mathcal{F})$ for every Hilbert B-module $\mathcal{F}$;
- every adjointable operator $S\colon \mathcal{F} \to \mathcal{E}$ is compact, that is, $\mathcal{L}_B(\mathcal{F}, \mathcal{E}) = \mathcal{K}_B(\mathcal{F}, \mathcal{E})$ for every Hilbert B-module $\mathcal{F}$.

Now assume that B is unital (and hence σ-unital). Then we know (from Eliezer's Lecture) that above assertions are equivalent to the fact that $\mathcal{E}$ is isomorphic to an (orthogonal) direct summand of B^n for some $n \in \mathbb{N}$.

Using this prove that there are $\xi_1, \ldots, \xi_n \in \mathcal{E}$ with

$$\sum_{i=1}^{n} |\xi_i\rangle\langle\xi_i| = \mathrm{Id}_{\mathcal{E}}.$$

In other words, $\mathcal{E}$ admits a finite quasi-basis. Conclude that the existence of a finite quasi-basis is also equivalent to the above equivalent assertions in case B is unital.

(4) Let $A := \mathcal{O}_n$ be the Cuntz algebra, that is, the universal C^*-algebra generated by n isometries $S_1, \ldots, S_n$ with $S_i^* S_j = \delta_{i,j} 1$ and $S_1 S_1^* + \ldots + S_n S_n^* = 1$. Prove that $A^n \cong A$ as Hilbert A-modules. Conversely, if B is a unital C^*-algebra such that $B^n \cong B$ as Hilbert B-modules, then there are isometries $S_1, \ldots, S_n \in B$ with $S_i^* S_j = \delta_{i,j} 1$ and $S_1 S_1^* + \ldots + S_n S_n^* = 1$. In other words, $A = \mathcal{O}_n$ is the universal unital C^*-algebra with $A^n \cong A$.

(5) Let B be a C^*-algebra. Prove that $\mathcal{H}_B \cong B$ as Hilbert B-modules if and only if B is stable, that is, $\mathcal{K} \otimes B \cong B$.

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