EXERCISES 2 (C*-DYNAMICAL SYSTEMS)

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(1) Let N be a normal closed subgroup of a locally compact group G. Then N and also the quotient group G/N are locally compact groups and hence carry Haar measures (you don't need to prove this!). Prove that it is possible to choose Haar measures μ, ν, ω on G, N and G/N such that the following equation holds for all $f \in C_c(G)$:

$$\int_G f(s)\,\mathrm{d}\mu(s) = \int_{G/N}\left(\int_N f(st)\,\mathrm{d}\nu(t)\right)\mathrm{d}\omega(sN).$$

(2) An extension of topological groups is an exact sequence of the form $N \hookrightarrow G \twoheadrightarrow K$ consisting of a continuous open surjective homomorphism $\pi\colon G \twoheadrightarrow K$ and an injective continuous homomorphism $\iota\colon N \hookrightarrow G$ which is open (and hence a topological isomorphism) onto its image $\mathrm{Im}(\iota) = \ker(\pi)$.

   (a) Show that any sequence as above is "isomorphic" to a sequence of the form $N \hookrightarrow G \twoheadrightarrow G/N$, where N is a normal subgroup of G (considered as a topological group with the subspace topology), $N \hookrightarrow G$ is the embedding and $G \twoheadrightarrow G/N$ is the quotient map. It is part of the exercise to explain the meaning of "isomorphic" here.

   (b) Show that an exact sequence of topological groups *splits* in the sense that there is a continuous homomorphism $\sigma\colon K \to G$ such that $\pi(\sigma(k)) = k$ if and only if there is a (continuous) action $\theta\colon K \to \mathrm{Aut}(N)$ of K on N by (continuous) group automorphisms such that G is isomorphic to the semidirect product $N \rtimes_\theta K$ and the original exact sequence is isomorphic to the canonical one $N \hookrightarrow N \rtimes_\theta K \twoheadrightarrow K$, where $N \hookrightarrow N \rtimes_\theta K$ sends $n \mapsto (n, e)$ and $N \rtimes_\theta K \twoheadrightarrow K$ sends $(n, k) \mapsto k$.

(3) Show that the construction of full crossed products $A \mapsto A \rtimes_\alpha G$ is functorial in the following sense: fix a locally compact G and suppose that (A, α) and (B, β) are C^*-algebras carrying G-actions α and β, respectively. Suppose that $\varphi\colon A \to B$ is a *-homomorphism which is G-equivariant, meaning that $\varphi(\alpha_t(a)) = \beta_t(\varphi(a))$ for all $a \in A$, $t \in G$. Show that φ induces a *-homomorphism $\varphi \rtimes G\colon A \rtimes_\alpha G \to B \rtimes_\beta G$. Moreover, if $\psi\colon B \to C$ is another G-equivariant *-homomorphism between C^*-algebras with G-actions, then $(\psi \circ \varphi) \rtimes G = (\psi \rtimes G) \circ (\varphi \rtimes G)$.

   Formulate and prove an analogous result for reduced crossed products.

(4) Suppose that a locally compact group G acts on C^*-algebras A and B via actions α and β. Prove that there is a "tensor product action" $\alpha \otimes \beta$ of G on the minimal tensor product $A \otimes B$ given on elementary tensors by $(\alpha \otimes \beta)_t(a \otimes b) = \alpha_t(a) \otimes \beta_t(b)$.

   If β is the trivial action, prove that there is a canonical isomorphism of C^*-algebras

$$(A \otimes B) \rtimes_{\alpha\otimes\beta,\mathrm{r}} G \cong (A \rtimes_{\alpha,\mathrm{r}} G) \otimes B.$$

   Remark.: There is an analogous result for maximal tensor products and maximal crossed products that we will see in the lectures.

(5) Let (A, G, α) be a C^*-dynamical system. Let $(e_i)_{i\in I}$ be an approximate unit A and let $(\varphi_V)_{V\in\mathcal{V}}$ be the "standard" approximate unit for the group algebra $\mathbb{C}[G] = C_c(G)$ (with respect to the inductive limit topology) consisting of functions $\varphi_V \in C_c^+(G)$ with $\mathrm{supp}(\varphi_V) \subseteq V$ and $\int_G \varphi_V(t)\,\mathrm{d}t = 1$. Here $\mathcal{V}$ denotes the directed set of all open neighborhoods of $e \in G$ with $V_1 \leq V_2$ iff $V_2 \subseteq V_1$. Endow $\mathcal{V} \times I$ with the product (directed) order: $(V_1, i_1) \leq (V_2, i_2)$ iff $V_1 \leq V_2$ and $i_1 \leq i_2$. Show that $(\varphi_V \otimes e_i)_{(V,i)\in\mathcal{V}\times I}$ is an approximate unit for $A \rtimes_{\alpha,\mathrm{alg}} G = C_c(G, A)$ with respect to the inductive limit topology, that is, prove that $(\varphi_V \otimes e_i) * f(t) \to f(t)$ uniformly with controlled supports. In particular $(\varphi_V \otimes e_i)_{(V,i)\in\mathcal{V}\times I}$ also serves as an approximate unit for $L^1(G, A)$, $A \rtimes_\alpha G$ or $A \rtimes_{\alpha,\mathrm{r}} G$.

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