MTM510037 — C*-Dynamical Systems and Crossed Products

Exercise Sheet 1 — Tensor Products of C*-Algebras

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1 Universality of the tensor product

Exercise 1.1 (Universality via multipliers)**.** *Let A and B be C*-algebras and let $\|\cdot\|_\mu$ be a C*-norm on $A \odot B$.*

*(1) Show that there exist nondegenerate *-homomorphisms*

$$\iota_A^\mu : A \to M(A \otimes_\mu B), \qquad \iota_B^\mu : B \to M(A \otimes_\mu B)$$

such that

$$\iota_A^\mu(a)\iota_B^\mu(b) = \iota_B^\mu(b)\iota_A^\mu(a) = a \odot b.$$

(2) (Universal property) Assume that $\mu = \max$. Let

$$\pi : A \to M(D), \qquad \rho : B \to M(D)$$

*be nondegenerate *-homomorphisms with commuting ranges. Show that there exists a unique nondegenerate *-homomorphism*

$$\pi \times_{\max} \rho : A \otimes_{\max} B \to M(D)$$

such that

$$(\pi \times_{\max} \rho) \circ \iota_A^{\max} = \pi, \qquad (\pi \times_{\max} \rho) \circ \iota_B^{\max} = \rho.$$

*(3) Show that this property characterizes $A \otimes_{\max} B$ up to canonical *-isomorphism.*

2 Functoriality

Exercise 2.1. *Let $\mu = \min$ or $\mu = \max$. Let*

$$\Phi : A \to A', \qquad \Psi : B \to B'$$

*be nondegenerate *-homomorphisms.*

*(1) Show that there exists a unique *-homomorphism*

$$\Phi \otimes_\mu \Psi : A \otimes_\mu B \to A' \otimes_\mu B'$$

compatible with the canonical inclusions.

(2) State and prove the analogous result when

$$\Phi : A \to M(A'), \qquad \Psi : B \to M(B').$$

Show that if Φ and Ψ are nondegenerate, then $\Phi \otimes_\mu \Psi$ is also nondegenerate.

(3) Conclude that the tensor product defines a bifunctor in the category of C-algebras with *-homomorphisms and also in the category with nondegenerate *-homomorphisms into multiplier algebras.*

3 Symmetries of the tensor product

Exercise 3.1 (Commutativity). *Show that there exists a canonical *-isomorphism*

$$A \otimes_\mu B \cong B \otimes_\mu A$$

induced by

$$a \odot b \mapsto b \odot a,$$

for $\mu = \min$ *or* $\mu = \max$.

Exercise 3.2 (Associativity). *Show that there exists a canonical *-isomorphism*

$$(A \otimes_\mu B) \otimes_\mu C \cong A \otimes_\mu (B \otimes_\mu C)$$

induced by the natural algebraic identification.

4 The commutative case

Exercise 4.1. *Let X and Y be locally compact Hausdorff spaces.*

*Show that there exists a natural *-isomorphism*

$$C_0(X) \otimes C_0(Y) \cong C_0(X \times Y),$$

where $\otimes$ denotes the unique C^*-tensor product (possible since commutative C^*-algebras are nuclear).

5 Tensor functionals and separation of points

Exercise 5.1. *Let A and B be C^*-algebras.*

(1) Let $\varphi \in A'$ and $\psi \in B'$ be bounded linear functionals. Show that the functional

$$\varphi \odot \psi : A \odot B \longrightarrow \mathbb{C}, \qquad (\varphi \odot \psi)\left(\sum_i a_i \odot b_i\right) = \sum_i \varphi(a_i)\psi(b_i),$$

is continuous for any *C^*-norm on $A \odot B$.*

*(*Hint: *first reduce to the positive case using linear combinations of positive functionals and use the known continuity of tensor products of states.)*

(2) Let $x \in A \otimes_{\min} B$ satisfy

$$(\varphi \otimes \psi)(x) = 0 \qquad \textit{for all } (\varphi, \psi) \in A' \times B'.$$

Show that $x = 0$.

(3) Show that if $x \geq 0$ in $A \otimes_{\min} B$, then in part (2) it suffices to assume

$$(\varphi \otimes \psi)(x) = 0 \quad \textit{only for states } \varphi \in S(A),\ \psi \in S(B).$$

6 Injectivity of the minimal tensor product

Exercise 6.1 (The minimal tensor preserves embeddings). *Let A, B, A', B' be C^*-algebras and*

$$\Phi : A \to A', \qquad \Psi : B \to B'$$

*injective *-homomorphisms.*

*Show that the *-homomorphism*

$$\Phi \otimes_{\min} \Psi : A \otimes_{\min} B \longrightarrow A' \otimes_{\min} B'$$

is injective.

7 Units in tensor products

Exercise 7.1 (When the tensor product is unital)**.** *Let A and B be C^*-algebras and $\|\cdot\|_\mu$ a C^*-norm on $A \odot B$. Denote by*

$$A \otimes_\mu B$$

the corresponding completion.

(1) Assume that A and B are unital. Show that $A \otimes_\mu B$ is unital and that

$$1_{A\otimes_\mu B} = 1_A \odot 1_B.$$

(2) Assume now that $A \otimes_\mu B$ is unital. Show that the nondegenerate inclusions

$$\iota_A^\mu : A \to M(A \otimes_\mu B), \qquad \iota_B^\mu : B \to M(A \otimes_\mu B)$$

take values in $A \otimes_\mu B$.

(3) Show that, in this case, A and B are unital.

*(*Hint: *use an approximate unit (e_i) of A and show that $\iota_A^\mu(e_i)$ converges strictly to $1_{A\otimes_\mu B}$; deduce that the limit belongs to the image of A.)*

(4) Conclude that

$$A \otimes_\mu B \text{ is unital } \iff A \text{ and } B \text{ are unital},$$

and that in this case

$$1_{A\otimes_\mu B} = 1_A \otimes 1_B.$$

8 Minimality of the minimal norm

Exercise 8.1. *Let A and B be C^*-algebras and let $A \odot B$ denote the algebraic tensor product.*

2. Let γ be a C^*-seminorm on $A \odot B$ such that

$$\gamma(a \otimes b) \neq 0 \qquad \textit{for all } a \neq 0,\ b \neq 0.$$

Show that

$$\|x\|_{\min} \leq \gamma(x), \qquad x \in A \odot B.$$

3. Conclude that $\|\cdot\|_{\min}$ is the smallest C^*-seminorm on $A \odot B$ that does not annihilate nonzero elementary tensors.

As an application of the exercise above:

Exercise 8.2 (Simplicity of the minimal tensor product)**.** *Let A and B be simple C^*-algebras. Show that $A \otimes_{\min} B$ is simple.*

Exercise 8.3. *Suppose that $\pi\colon A \otimes_{\min} B \to C$ is a $*$-homomorphism which is injective when restricted to $A \odot B$. Is π injective?*