A Large Deviation Principle for Gibbs States on countable Markov shifts at zero temperature

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Jointly with
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Motivation

Common belief (among people working in countable Markov shifts)

"The thermodynamic formalism for topologically mixing countable Markov shifts with BIP property is similar to that of subshifts of finite type defined on finite alphabets."

Setting:

- $X = \Sigma_{\mathbf{A}}(S)$ is a Markov subshift of a one-dimensional lattice: $\Sigma_{\mathbf{A}}(S) \subseteq S^{\mathbb{N}}$ (or $S^{\mathbb{Z}}$).
  where $S = \{-1, +1\}, \{1, 2, ..., k\}$ or $\mathbb{N}$ [our case].
- The potential $f : X \to \mathbb{R}$ always more then continuous (Lipschitz, Hölder, summable variation, Walters).
- Shift map
  $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}; \sigma(x) = \sigma(x_0, x_1, x_2, ...) = (x_1, x_2, x_3, ...)$.

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Infinite matrix $A : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$

$$\Sigma_A(\mathbb{N}) := \{x \in \mathbb{N}^{\mathbb{N}} : A(x_i, x_{i+1}) = 1, \ \forall i \geq 0\}.$$

cylinders: Fix $a \in \mathbb{N}$.

$$[a] = \{x = (a, x_1, x_2, ...) \in \Sigma_A(\mathbb{N})\}$$

$$[a_0 a_1 ... a_{n-1}] = \{x = (a_0, a_1, ..., a_{n-1}, x_n, x_{n+1}, ...) \in \Sigma_A(\mathbb{N})\}$$

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The matrix $\mathbf{A}$ is *finitely primitive*, when there exist a finite subset $\mathbb{F} \subseteq \mathbb{N}$ and an integer $K_0 \geq 0$ such that, for any pair of symbols $i, j \in \mathbb{N}$ such that $[i] \neq \emptyset$ and $[j] \neq \emptyset$, one can find $\ell_1, \ell_2, \ldots, \ell_{K_0} \in \mathbb{F}$ satisfying

$$\mathbf{A}(i, \ell_1)\mathbf{A}(\ell_1, \ell_2) \cdots \mathbf{A}(\ell_{K_0}, j) = 1.$$

The matrix (or the shift) **A** has the *big image property (BIP)*, when there exist a finite subset $\mathbb{F} \subseteq \mathbb{N}$ such that, for any symbol with $[i] \neq \emptyset$, there exist ℓ_i and r_i in $\mathbb{F}$ such that $A(\ell_i, i) = A(i, r_i) = 1$.

In topologically mixing Markov shifts:

A has the BIP property $\Longleftrightarrow$ **A** is finitely primitive.

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In topologically mixing Markov shifts:

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Beyond the Finite Primitive case

Renewal shifts

Example: Let $A = (a_{ij})_{i,j\in\mathbb{N}}$ be the transition matrix such that there exists an increasing sequence of naturals $(d_i)_{i\in\mathbb{N}}$ for which

$$a_{00} = a_{i+1,i} = a_{1,d_i} = 1, \ \forall\, i \in \mathbb{N}$$

and the others coefficients are zero.

A topologically mixing Markov shift without the BIP property

Renewal Shift

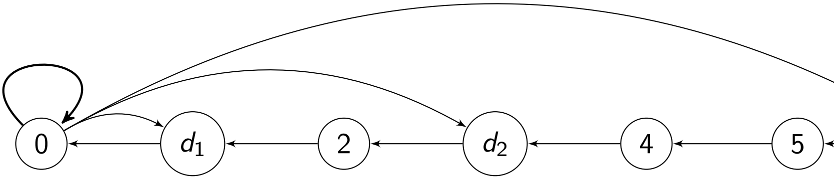

Figura : Example of Renewal shift.

$f : \Sigma_A(\mathbb{N}) \to \mathbb{R}$ has *summable variation* when

$$\mathrm{Var}(f) := \sum_{k=1}^{\infty} \mathrm{Var}_k(f) < \infty.$$

where

$$\mathrm{Var}_k(f) := \sup_{\substack{x,y\in\Sigma_A(\mathbb{N}) \\ d(x,y)\leq r^k}} [f(x) - f(y)] \;\; \forall\, k \geq 1 \;\; [r \in (0,1)]$$

f is locally Hölder continuous when there exist $H_f > 0$ such that $\mathrm{Var}_k(f) \leq H_f r^k$, $\forall k \geq 1$.

f can be unbounded!

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Thermodynamic Formalism

Let $f : \Sigma_A(\mathbb{N}) \to \mathbb{R}$ with summable variation.

For each $\beta > 0$ we define:
Pressure

$$P(\beta f) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n(x)=x} \exp\left(\sum_{i=0}^{n-1} \beta f(\sigma^i x)\right) 1_{[a]}(x),$$

The definition doesn't depend of the symbol a because we always assume that $\Sigma_A(\mathbb{N})$ is topologically mixing.

For each β the *equilibrium measures* μ_β (for a potential βf) are the probability measures satisfying the *variational principle*: $P(\beta f)$:

$$P(\beta f) = \sup_{m \in M_\sigma} \left\{ h(m) + \int \beta f dm; \int \beta f dm > -\infty \right\} = h(\mu_\beta) + \int \beta f d\mu_\beta$$

where $h(m)$ is the Kolmogorov-Sinai entropy of the measure m.

$M_\sigma =$ invariant (for σ) probability measures defined over the borel sets of $\Sigma_A(\mathbb{N})$

(Sarig, Mauldin-Urbanski)

Theorem (Ruelle-Perron-Frobenius)

Let $\Sigma_A(\mathbb{N})$ be a finitely primitive shift, f a potential with summable variation and $P(f) < \infty$. Then, for any $\beta > 0$, if $\lambda_\beta = e^{P(\beta f)}$ there exists a probability measure ν_β finite and positive in cylinders and a continuos function $h_\beta > 0$ such that $L^*_{\beta f}\nu_\beta = \lambda_\beta \nu_\beta$, $L_{\beta f} h_\beta = \lambda_\beta h_\beta$ and $\mu_\beta = h_\beta d\nu_\beta$ is a equilibirum measure for βf.

Let $f : \Sigma_A(\mathbb{N}) \to \mathbb{R}$ be a function.
The Ruelle-Perron-Frobenius operator ig given by:
$L_f : C_b(\Sigma_A(\mathbb{N})) \to C_b(\Sigma_A(\mathbb{N}))$

$$(L_f g)(x) = \sum_{i \in \mathbb{N}} e^{f(ix)} g(ix).$$

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$P(\beta f)$ is real analytic in the parameter β.

For each β, there exist only one equilibrium measure.

h_β is uniformly bounded away from zero and from infinity.

μ_β is a Gibbs measure.

Exactly as in the shifts with a finite number of symbols.

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Definition (Bowen definition)

An invariant measure μ *is called of Gibbs Measure for the potential* $f : \Sigma_A(\mathbb{N}) \to \mathbb{R}$ *when there exist constants* $C_1, C_2 > 0$ *such that*

$$C_1 \leq \frac{\mu[x_0 \ldots x_{n-1}]}{\exp(S_n f(x) - nP(f))} \leq C_2$$

$x \in [x_0 \ldots x_{n-1}]$.

Spoiler! See the talk of Artur Lopes about equivalent definitions for Gibbsianity.

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O. Sarig - Proc. of AMS - 03'.

Theorem

Let $\Sigma_{\mathbf{A}}(\mathbb{N})$ be a topologically mixing Markov subshift, $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ a function with summable variation and $P(f) < \infty$. If f has a Gibbs measure then $\Sigma_{\mathbf{A}}(\mathbb{N})$ is finitely primitive.

Ergodic Optimization

The main problem in *Ergodic Optimization* is to guarantee the existence and to describe the *maximizing measures* for the system, that is, to describe the set of probability measures m satisfying:

$$m(f) := \sup_{\mu \in \mathcal{M}_\sigma} \int f \, d\mu = \int f \, dm$$

where M_σ denotes the set of the σ-invariant borel probability measures and f is a fixed potential $f : X \to \mathbb{R}$.

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Main ex-conjecture

Roughly:

Generically in the space of Lipschitz potentials with X compact and T with suitable properties the maximizing measure is unique and supported in an periodic orbit.

Ground States are Generically a Periodic Orbit. (Gonzalo Contreras) Abstract. We prove that for an expanding transformation the maximizing measures of a generic Lipschitz function are supported on a single periodic orbit. (arxiv)

Important Reference: O. Jenkinson - *Ergodic Optimization* - DCDS 2006.

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compact versus non-compact

In the compact setting ($X = \Sigma_{\mathbf{A}}$, with a finite alphabet) maximizing measures always exist.

- When X is compact since the potential f is always assume continuous by compactness (of M_σ) there exists a probability measure ν in M_σ such that $m(f) := \sup_{\mu \in \mathcal{M}_\sigma} \int f \, d\mu = \int f \, d\nu$

The problem in this context it is to describe the support of this measures.

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noncompact case

Even the existence of these measures is a non trivial problem.

Ergodic optimization for noncompact dynamical systems.
- O. Jenkinson, R. D. Mauldin and M. Urbański - (DS-07')

Ergodic optimization for countable alphabet subshifts of finite type.
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Connection with Equilibrium States

Under suitable hypothesis on $\Sigma_{\mathbf{A}}$ and f we can prove that there exist equilibrium measures μ_β for all $\beta > 0$ and any zero-temperature accumulation point of the family $(\mu_\beta)_{\beta>0}$ is a maximizing measures for the potential f.

This statement is true in both settings: compact and noncompact.

In the noncompact setting we need to control the behavior of f *at infinity* . We say that f is *coercive* when:

$$\lim_{i\to\infty} \sup f|_{[i]} = -\infty\,,$$

This condition is satisfied when we have for example:

$$\sum_{i\in\mathbb{N}} \exp(\sup f|_{[i]}) < \infty.$$

The condition is usually imposed under the potential to use the Ruelle operator in the thermodynamic formalism, when the shift is BIP, this is equivalente to f has finite Pressure.

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When the matrix **A** is finitely primitive and f satisfies the last condition:

Theorem (O. Jenkinson, R. D. Mauldin and M. Urbański 05')

The family of Gibbs measures $(\mu_{\beta f})_{\beta \geq 1}$ has at list one weak accumulation point as $\beta \to \infty$. Any accumulation point μ is a maximizing measure for f, and $\lim_{\beta \to \infty} \int f d\mu_{\beta f} = \int f d\mu$.

Proof: Prohorov 's theorem and use that the measures $\mu_{\beta f}$ are Gibbs Measures.

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Proof: Prohorov 's theorem and use that the measures $\mu_{\beta f}$ are Gibbs Measures.

Naive idea:

Since the potential f decays to $-\infty$ when the symbols grow, we can restrict ourselves to periodic orbits whose symbols are all small.

Theorem (R. B. and R. Freire - ETDS (2014))

Let σ be the shift map on a transitive $\Sigma_{\mathbf{A}}(\mathbb{N})$ subshift and let be $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ be a function with bounded variation, coercive and $\sup f < \infty$. *Then, there is a finite set $\mathcal{A} \subset \mathbb{N}$ such that $\mathbf{A}|_{\mathcal{A}\times\mathcal{A}}$ is irreducible and*

$$m(f) = \sup_{\mu \in \mathcal{M}_\sigma(\Sigma_{\mathbf{A}}(\mathcal{A}))} \int f \, d\mu \, .$$

Furthermore, if ν is a maximizing measure, then

$$supp\, \nu \subset \Sigma_{\mathbf{A}}(\mathcal{A})\, .$$

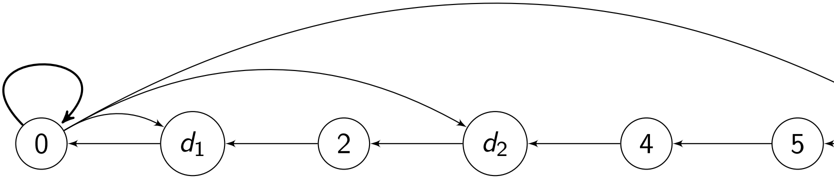

Figura : Example of Renewal shift.

Theorem (O. Sarig - CMP - 2001)

Let Σ be a Renewal Shift and f a locally Hölder potential such that $\sup f < \infty$. *Then there exists a constant $\beta_c \in (0, \infty]$ such that*

- *For $0 < \beta < \beta_c$ there exists an equilibrium probability measure μ_β corresponding to βf. For $t > \beta_c$ there is no equilibrium probability measures corresponding to tf;*
- *$P(\beta f)$ is real analytic on $(0, \beta_c)$ and linear on (β_c, ∞). At β_c, it is continuous but not analytic.*

Theorem (G. Iommi - 2007)

Let Σ be a Renewal Shift and f a locally Hölder potential such that $\sup f < \infty$. Then

- *For $\beta_c = \infty$, then there exists maximizing measures μ_β for f.*
- *If $\beta_c < \infty$, then there are no maximizing measures for f*

Here, $m(f) = \sup_{\mu \in \mathcal{M}_\sigma} \int f \, d\mu$ is the slope linear part of the pressure $P(\beta f)$.

Corollary: If you assume f coercive and locally Hölder potential then there is no phase transition in the Renewal Shift.

Definition:
A sub-action (for the potential f) is a function $u \in C^0(\mathbf{\Sigma})$ verifying $(f + u - u \circ \sigma)(\mathbf{x}) \leq m(f), \ \forall\, \mathbf{x} \in \mathbf{\Sigma}$.

Definition

A continuous function $V : \Sigma_A(\mathbb{N}) \to \mathbb{R}$ *is called calibrated sub-action to the potential* f *if for any* $x \in \Sigma_A(\mathbb{N})$ *there exist* $y \in \Sigma_A(\mathbb{N})$ *such that* $\sigma(y) = x$ *and*

$$V(x) = V(y) + f(y) - m(f).$$

$$M_f V(x) = \sup_{y \in \sigma^{-1}x} (V + f)(y) = \max_{1 \le j \le J} (V + f)(jx)$$

Jenkinson-Mauldin-Urbanski:
Existence of bounded and continuous V is equivalent to the existence of the maximizing measure in the of countable alphabet.

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Proposition

The family $V_\beta := \frac{1}{\beta} \log h_\beta$ is equicontinuous and uniformly bounded.

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Any accumulation point $V(x) := \lim_{\beta_i \to \infty} \frac{1}{\beta_i} \log h_{\beta_i}(x)$ is a calibrated for the potential f.

Uniformly in compacts.

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Two calibrated sub-actions differ by a constant.

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Main result:

Theorem (R.B., J. Mengue and E. Pérez)

If f is coercive, has summable variation, has an unique maximizing measure μ, finite pressure and $\Sigma_A(\mathbb{N})$ is finitely primitive then:

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \mu_\beta(C) = - \inf_{x\in C} I(x) \quad \text{for any } C = [x_0 \dots x_n].$$

where $I(x) = \sum_{n\geq 0} (V - V\circ\sigma - f + m(f)) \circ \sigma^n(x)$.

$I : \Sigma_A(\mathbb{N}) \to [0, +\infty]$ is lower semicontinuous and non-negative.
$x \in supp\ \mu \Rightarrow I(x) = 0$
$x \notin \Omega(f, \sigma) \Rightarrow I(x) > 0$

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