Random walks on random graphs

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Metric spaces of graphs

$\mathcal{G}$ denotes the set of locally finite, connected, rooted graphs G; $V(G)$ vertex set, $E(G)$ edge set, root vertex r.

d_G graph distance on G, i.e. $d_G(v, w)$ equals smallest number of edges in a path (in G) connecting v and w.

The ball $B_G(v; R)$ of radius R centred at $v \in V(G)$ is the subgraph of G spanned by vertices at graph distance at most R from v. Denote $B_G(r; R) = B_G(R)$.

(Ultra)metric d on $\mathcal{G}$ defined by

$$d(G, G') = \inf\{\tfrac{1}{R+1} \mid B_G(R) = B_{G'}(R)\}$$

$(\mathcal{G}, d)$ is a complete separable metric space.

- The set $\mathcal{T}$ of rooted locally finite trees is a closed subset of $\mathcal{G}$.

- Similar notions, if $\mathcal{G}$ is replaced by the set $\bar{\mathcal{G}}$ of planar graphs.

Random graphs

A random (planar) graph is a probability measure μ on $\mathcal{G}$ (resp. $\bar{\mathcal{G}}$).

Random planar trees

Let $\mathcal{T}_N$ be the set of planar rooted trees of size N, i.e. with N edges, and with root of degree 1, and
let $p_0, p_1, p_2, \ldots$ be branching probabilities fulfilling

$$\sum_{n=0}^{\infty} p_n = 1 \quad \text{and} \quad \sum_{n=0}^{\infty} n p_n = 1$$

.

Define the probability measure μ_N on $\mathcal{T}$ supported on $\mathcal{T}_N$ by

$$\mu_N(\tau) = \frac{1}{Z_N} \prod_{v \in V(\tau) \setminus \{r\}} p_{\sigma_v - 1}\,, \quad \tau \in \mathcal{T}_N\,,$$

where σ_v denotes the degree of v in τ and

$$Z_N = \sum_{\tau \in \mathcal{T}_N} \prod_{v \in V(\tau) \setminus \{r\}} p_{\sigma_v - 1}\,.$$

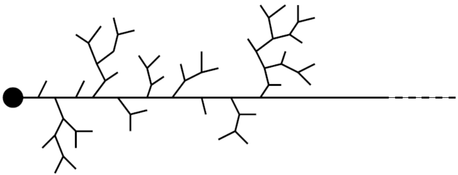
Theorem 1(Generic random trees) If the radius of convergence ρ of the generating function $\sum_{n=0}^{\infty} p_n \zeta^n$ satisfies $\rho > 1$ then the weak limit

$$\mu = \lim_{N \to \infty} \mu_N$$

exists and is supported on the set $\mathcal{S}$ of infinite rooted trees with a single spine (infinite linear path $v_0, v_1, v_2, \ldots$ starting at the root $r = v_0$).

D, Jonsson, Wheater 2007

More general results including non-generic trees by Jansson 2012.



A tree with a single spine

Characterization of μ:

A tree in $\mathcal{S}$ is determined by a sequence $L_1, \ldots, L_{l_i}$ of left branches and a sequence $R_1, \ldots, R_{k_i}$ of right branches rooted at v_i for each $i = 1, 2, 3, \ldots$ These are finite rooted trees which are independently and identically distributed for given $(l_i, r_i), i = 1, 2, 3, \ldots$ with distribution

$$\nu(\tau) = \prod_{v \in V(\tau) \setminus \{r\}} p_{\sigma_v - 1} \, .$$

Moreover the pairs (l_i, k_i) are independent and identically distributed according to

$$\mu\{(l_i, k_i) = (l, k)\} = p_{l+k+1} \, , \quad k, l = 0, 1, 2, 3 \ldots$$

Special cases:

a) The uniform infinite planar rooted tree (UIPT) is obtained for $p_n = 2^{-n-1}$, in which case

$$\prod_{v \in V(\tau)\setminus\{r\}} p_{\sigma_v - 1} = 2 \cdot 4^{-|\tau|}\,.$$

b) The incipient infinite percolation cluster on a regular m-ary tree is obtained for

$$p_n = \binom{m-1}{n} q^n (1-q)^{m-1-n}\,, \quad n = 0, 1, \ldots, m-1\,.$$

The uniform infinite planar triangulation (UIPTri)

Let $\mathcal{P}_N \subset \bar{\mathcal{G}}$ denote the set of triangulations of S^2 with N vertices and a root edge (rr'). Let ν_N^t be the uniform measure supported on $\mathcal{P}_N$, i.e.

$$\nu_N^t(T) = \frac{1}{|\mathcal{P}_N|} = \frac{1}{2}\frac{(N-2)!(3N-5)!}{(4N-9)!}\,, \quad T \in \mathcal{P}_N\,.$$

Tutte 1962

Theorem 2(Angel & Schramm, 2002) The weak limit $\nu^t = \lim_{N\to\infty} \nu_N^t$ exists as a probability measure on $\bar{\mathcal{G}}$ supported on the set on infinite triangulations of the plane with one end.

The uniform infinite planar quadrangulation (UIPQ)

Let $\mathcal{Q_N} \subset \bar{\mathcal{G}}$ denote the set of quadrangulations of S^2 with N vertices and a root edge (rr'). Let ν_N^q be the uniform measure supported on $\mathcal{Q_N}$, i.e.

$$\nu_N^q(T) = \frac{1}{|\mathcal{Q_N}|} = \frac{1}{2}\frac{N!(N+2)!}{(2N)!}3^{-N}\,, \quad Q \in \mathcal{Q_N}\,.$$

Theorem 3(Chassaing & D, 2003) The weak limit $\nu^q = \lim_{N\to\infty} \nu_N^q$ exists as a probability measure on $\bar{\mathcal{G}}$ supported on the set on infinite quadrangulations of the plane with one end.

Is obtained from a bijective correspondence between $\mathcal{Q}_N$ and well-labelled trees of size $N+2$. (G. Schaeffer 1998)

Further work by Krikun 2008 and Ménard 2010.

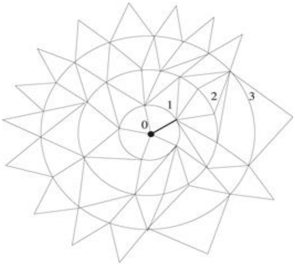
The uniform infinite causal triangulation (UICT)

Let $\mathcal{CT}_N \subset \bar{\mathcal{G}}$ denote the set of causal triangulations with $2N$ triangles and a root edge (rr'). Let ν_N^c be the uniform measure supported on $\mathcal{CT}_N$, i.e.

$$\nu_N^c(T) = \frac{1}{|\mathcal{CT}_N|}\,, \quad T \in \mathcal{CT}_N\,.$$

Theorem 4 The weak limit $\nu^c = \lim_{N\to\infty} \nu_N^c$ exists as a probability measure on $\bar{\mathcal{G}}$ supported on the set on infinite causal triangulations of the plane.

(D, Jonsson, Wheater, 2010)



A causal triangulation

Structure of random graphs

Definition The Hausdorff dimension d_h of a connected (infinite) graph G is defined by

$$d_h = \lim_{R\to\infty} \frac{\ln |E(B_G(v;R))|}{R}\,, \tag{3.1}$$

provided the limit exists (independent of v).

Spectral dimension

Simple random walk on G: Define p_G on finite walks $\omega = (\omega_0, \omega_1, \ldots, \omega_m)$ on G by

$$p_G(\omega) = \prod_{i=0}^{m-1} \sigma_{\omega_i}^{-1}\,.$$

p_G defines a probability distribution p_G^m on walks of fixed length m and fixed initial vertex v.

The probability for a walk (of length m) starting at v to end at w is

$$q_G(m; v, w) = \sum_{\omega: v \to w, |\omega| = m} p_G(\omega)\,.$$

Define the corresponding *cumulated probability*

$$Q_G(n; v, w) = \sum_{m=0}^{n} q_G(m; v, w)\,, \quad n = 0, 1, 2, \ldots$$

G is *recurrent* if $Q_G(n; v, v) \to \infty$ as $n \to \infty$. Otherwise G is *transient*.

Definition The spectral dimension of a recurrent (connected) graph G is

$$d_s = 2 - 2 \lim_{n \to \infty} \frac{\ln Q_G(n; v, v)}{\ln n}\,.$$

Some results

Theorem 5 For any generic random tree it holds that

$$d_h = 2 \quad \text{and} \quad d_s = \frac{4}{3}\,.$$

DJW 2007

Barlow & Kumagai 2006 for percolation case.

Theorem 6 For the UIPTri and UIPQ it holds that $d_h = 4$ almost surely and in average.

Angel & Schramm 2002 for UIPTri

Chassaing & D 2003 for UIPQ

Theorem 7 The UICT is almost surely recurrent.

DJW 2010

Theorem 8 The UIPTri and UIPQ are almost surely recurrent.

Gurel-Gurevich & Nachmias 2012

Further developments

- Statistical systems on planar random graphs.

Examples:
1) The Ising model on a random tree.

(D & Napolitano 2014)

2) The Ising model on a planar quadrangulation. Matrix model techniques in grand canonical ensemble.

(V. Kazakov 1986)

- Higher dimensional random triangulations or random complexes.

- Scaling limits of random graphs.