On maximality of bounded groups on Banach spaces and on the Hilbert space

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FADYS, February 2015

Terminology[1]

The results presented here are joint work with Christian Rosendal, from the University of Illinois at Chicago.

In this talk all spaces are complete, all Banach spaces are unless specified otherwise, separable, infinite dimensional, and, for expositional ease, assumed to be complex.

[1] The author acknowledges the support of FAPESP, process 2013/11390-4

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Introduction: Mazur's rotations problem

Definition

- $\mathrm{Isom(X)}$ *is the group of linear surjective isometries on a Banach space* X*.*
- *The group* $\mathrm{Isom(X)}$ *acts* ***transitively*** *on the unit sphere* S_X *of* X *if for all* x, y *in* S_X*, there exists* T *in* $\mathrm{Isom}(X)$ *so that* $Tx = y$*.*

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The group $\mathrm{Isom(H)}$ acts transitively on any Hilbert space H.

Conversely if $\mathrm{Isom(X)}$ acts transitively on a Banach space X, must it be linearly isomorphic? isometric to a Hilbert space?

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Answers:

(a) if $\dim X < +\infty$: YES to both

(b) if $\dim X = +\infty$ is separable: ???

(c) if $\dim X = +\infty$ is not separable: NO to both

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Proof.

(a) $X = (\mathbb{R}^n, \|.\|)$. Choose an inner product $< ., . >$ such that $\|x_0\| = \sqrt{< x_0, x_0 >}$ for some x_0. Define

$$[x, y] = \int_{T \in \mathrm{Isom}(X, \|.\|)} < Tx, Ty > dT,$$

This a new inner product for which the T still are isometries, and $\|x\| = \sqrt{[x, x]}$, since holds for x_0 and by transitivity. $\square$

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Proof.

(b) Prove that for $1 \leq p < +\infty$, the orbit of any norm 1 vector in $L_p([0,1])$ under the action of the isometry group is dense in the unit sphere.

Then note that any ultrapower of $L_p([0,1])$ is a non-hilbertian space on which the isometry group acts transitively. $\square$

So we have the next unsolved problem which appears in Banach's book "Théorie des opérations linéaires", 1932.

Problem (Mazur's rotations problem, first part)

If $X, \|.\|$ is separable and transitive, must X be hilbertian (i.e. isomorphic to the Hilbert space)?

Problem (Mazur's rotations problem, second part)

Assume $X, \|.\|$ is hilbertian and transitive, must X be a Hilbert space?

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This question divides into two unsolved problems

(a) If $(X, \|.\|)$ is separable and transitive, must $\|.\|$ be uniformly convex?

(b) If $(X, \|.\|)$ is separable, uniformly convex, and transitive, must it be hilbertian?

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At this point it is only known that in (a) X must be strictly convex (F. - Rosendal 2015), and that if e.g. X^* is separable or X is a separable dual, then X has to be uniformly convex (Cabello-Sanchez 1997).

Problem (Mazur's rotations problem, second part)

Assume $(X, \|.\|)$ *is hilbertian and transitive, must* $(X, \|.\|)$ *be a Hilbert space?*

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Of course if $G = \mathrm{Isom}\,(X, \|.\|)$ is unitarizable, i.e. a unitary group in some equivalent Hilbert norm $\|.\|'$ on X, then by transitivity $\|.\|'$ will be a multiple of $\|.\|$ and so $(X, \|.\|)$ will be a Hilbert space.

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So one part of Mazur's problem is related to the question of which bounded representations on the Hilbert space are unitarizable, i.e. which bounded subgroups of $\mathrm{Aut}(\ell_2)$ are unitarizable.

Observation: Mazur and Dixmier

Theorem (Day-Dixmier, 1950)

Any bounded representation of an amenable topological group on the Hilbert space is unitarizable.

By Ehrenpreis and Mautner (1955) this does not extend to all (countable) groups.

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Observation

If $(X, \|.\|)$ is hilbertian, and $\mathrm{Isom}(X, \|.\|)$ acts transitively on $S_{X,\|.\|}$, and is amenable, then $(X, \|.\|)$ is a Hilbert space.

Contents

Principles of renorming theory

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- smoother: e.g. $x \mapsto \|x\|$ must have differentiability properties,
- more symmetric: i.e. the norm induces more isometries.

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- more symmetric: i.e. the norm induces more isometries.

Let us concentrate on the second aspect.

Introduction: transitive and maximal norms

In 1964, Pełczyński and Rolewicz looked at Mazur's rotations problem and defined properties of a given norm $\|.\|$.
In what follows $\mathcal{O}_{\|.\|}(x)$ represents the orbit of the point x of X, under the action of the group $\mathrm{Isom}(X, \|.\|)$, i.e.
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Definition

Let X be a Banach space and $\|.\|$ an equivalent norm on X. Then $\|.\|$ is

(i) *transitive if $\forall x \in S_X$, $\mathcal{O}_{\|.\|}(x) = S_X$.*

(ii) *quasi transitive if $\forall x \in S_X$, $\mathcal{O}_{\|.\|}(x)$ is dense in S_X.*

(iii) *maximal if there exists no equivalent norm $|||.|||$ on X such that $\mathrm{Isom}(X, \|.\|) \subsetneq \mathrm{Isom}(X, |||.|||)$ with proper inclusion.*

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Of course $(i) \Rightarrow (ii)$, and also $(ii) \Rightarrow (iii)$ (Rolewicz).

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Examples of (i): ℓ_2, of (ii): $L_p(0,1)$, of (iii): ℓ_p.

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Observation

Note that (iii) means that $\mathrm{Isom}(X, \|.\|)$ *is a maximal bounded subgroup of* $\mathrm{Aut}(X)$.

Questions (Wood, 1982)

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Question (Deville-Godefroy-Zizler, 1993)

Does every uniformly convex Banach space admit an equivalent quasi-transitive norm?

Recent solutions to Wood and DGZ problems

Theorem (F. - Rosendal, 2013)

There exists a separable uniformly convex Banach space X without an equivalent maximal norm. Equivalently there is no maximal bounded subgroup of automorphisms on X.

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Theorem (Dilworth - Randrianantoanina, 2014)

Let $1 < p < +\infty, p \neq 2$. Then

- *ℓ_p does not admit an equivalent quasi-transitive norm.*
- *there exists a bounded group of isomorphisms on ℓ_p which is not contained in any maximal one.*

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Question

Let $1 < p < +\infty, p \neq 2$. Does $L_p([0,1])$ admit a transitive norm?

An idea of the proof of our result

Theorem (F. - Rosendal, 2013)

There exists a separable uniformly convex Banach space X without an equivalent maximal norm. Equivalently $\mathrm{Aut}(X)$ *does not have a maximal bounded subgroup.*

First note a fact about ”finite-dimensional” isometries.

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First note a fact about "finite-dimensional" isometries.

Proposition (Cabello-Sanchez, 1997)

If X is separable and the group of isometries which are finite rank perturbations of Id acts transitively, then X is isometric to the Hilbert space.

On the HI spaces of Gowers-Maurey (1993), see also Argyros-Haydon (2011), one may prove that all isometries are of the form $\lambda Id + F$,

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Theorem

Let X be separable, reflexive HI space, and let G be a bounded group of isomorphisms on X. Then

(a) *If all G-orbits are relatively compact then G acts nearly trivially on X, meaning there is a G-invariant decomposition*

$$X = F \oplus H,$$

F finite-dimensional, G acts by multiple of the identity on H.

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We also note that

- if G acts nearly trivially on X infinite dimensional, then G is properly included in some bounded subgroup of $\mathrm{Aut}(X)$,
- there exist (uniformly convex) HI spaces without Sc. basis.

So finally we have the implications:

Theorem

- *there exists a separable, uniformly convex HI space X without a Schauder basis,*
- *every bounded group of isomorphisms on X acts almost trivially on X,*
- *no such group is maximal bounded in $\mathrm{Aut}(X)$,*
- *X does not carry any equivalent maximal norm.*

Alaoglu-Birkhoff and Jacobs - de Leeuw - Glicksberg

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Our results extend to general results on ”small” subgroups of isometries on any separable reflexive space.
Alaoglu - Birkhoff theorem (1940) and Jacobs - de Leeuw - Glicksberg theorems (1960s) relate, for reflexive X:

- isometric representations of groups on X, or representations of semi-groups as semi-groups of contractions on X, to
- decompositions of X as direct sums of closed subspaces,

Theorem

Let X be a separable reflexive space and $G \subset GL(X)$ be bounded. Then X admits the G-invariant decompositions:

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(a) *(Alaoglu - Birkhoff type decomposition)*

$$X = H_G \oplus (H_{G^*})^{\perp},$$

where $H_G = \{x \in X : Gx = \{x\}\}$,
$H_{G^*} = \{\phi \in X^* : G\phi = \{\phi\}\}$,
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(b) *(Jacobs - de Leeuw - Glicksberg type decomposition)*

$$X = K_G \oplus (K_{G^*})^\perp,$$

where $K_G = \{x \in X : \overline{Gx} \text{ is compact}\}$,
$K_{G^*} = \{\phi \in X^* : \overline{G\phi} \text{ is compact}\}$, *and furthermore*
$K_{G^*}^\perp = \{x \in X : x \text{ furtive, i.e. } \exists T_n \in G : T_n x \to^w 0\}$.

Using and reproving Alaoglu-Birkhoff and Jacobs - de Leeuw - Glicksberg decompositions and a bit of theory of Polish groups:

Theorem

Let X be separable, reflexive Banach space and G be a bounded group of isomorphisms on X of the form $Id + F$, F finite range, which is SOT-closed in $GL(X)$. Then

a) *if all G-orbits are relatively compact then G acts nearly trivially on X,*
b) *if some G-orbit is not relatively compact then X has a G-invariant complemented subspace with a Schauder basis.*

Contents

Hilbert

If a bounded representation π of a group G on ℓ_2 is unitarizable then of course $\pi(G)$ extends to a transitive and therefore maximal bounded subgroup of $\mathrm{Aut}(\ell_2)$, which is conjugate to the unitary group.

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Let π be a non-unitarizable representation of a group G on ℓ_2.

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Let π be a non-unitarizable representation of a group G on ℓ_2.

- if $\pi(G)$ is included in some maximal bounded group, then there exists a maximal non-Hilbert norm on ℓ_2. Then we should ask whether it can be quasi-transitive or transitive;

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Let π be a non-unitarizable representation of a group G on ℓ_2.

- if $\pi(G)$ is included in some maximal bounded group, then there exists a maximal non-Hilbert norm on ℓ_2. Then we should ask whether it can be quasi-transitive or transitive;
- if not then $\pi(G)$ cannot provide a negative solution to the second half of Banach-Mazur problem, and Wood's second question should be reformulated as:

Question

Does there exist a Banach space on which any bounded group of isomorphisms is included in a maximal one?

Hilbert

Assume $\pi : G \to \mathrm{Aut}(\ell_2)$ is non-unitarizable.
Note that if we define

$$|||x||| = \sup_{g \in G} \|\pi(g)x\|_2,$$

then we obtain a uniformly convex space (Bader, Furman, Gelander and Monod 2007), which is linearly isomorphic (but not isometric) to ℓ_2, and on which G acts by isometries.

So we may and shall use general results about uniformly convex Banach spaces.

We obtain restrictions on how $\pi(G)$ may be extended to a (maximal) bounded group.

Proposition (F. Rosendal 2015)

Suppose that $\lambda\colon \Gamma \longrightarrow \mathcal{U}(\mathcal{H})$ is an irreducible unitary representation of a group Γ on a separable Hilbert space $\mathcal{H}$ and $d\colon \Gamma \longrightarrow \mathcal{B}(\mathcal{H})$ is an associated non-inner bounded derivation. Suppose that $G \leqslant GL(\mathcal{H} \oplus \mathcal{H})$ is a bounded subgroup leaving the first copy of $\mathcal{H}$ invariant and containing $\lambda_d[\Gamma]$. Then the mappings $G \longrightarrow \mathrm{GL}(\mathcal{H})$ defined by

$$\begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \mapsto u \quad \text{and} \quad \begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \mapsto v$$

are `sot`*-isomorphisms between G and the respective images in $\mathrm{GL}(\mathcal{H})$.*

Example

(see Ozawa, Pisier, ...) Let T be the Cayley graph of F_∞ with root e.

Let $L\colon \ell_1(T) \longrightarrow \ell_1(T)$ be the "Left Shift", i.e. the bounded linear operator satisfying $L(1_e) = 0$ and $L(1_s) = 1_{\hat{s}}$ for $s \neq e$, where $\hat{s}$ is the predecessor of s.

Let for every $g \in \mathrm{Aut}(T)$,

$$d(g) = \lambda(g)L - L\lambda(g)$$

on $\ell_1(T)$. Check that $d(g)$ extends to a continuous operator on $\ell_2(T)$, so d defines a bounded derivation associated to λ, which, however, is not inner, meaning that

$$g \mapsto \begin{pmatrix} \lambda(g) & d(g) \\ 0 & \lambda(g) \end{pmatrix}$$

is a bounded non unitarizable represensation on the Hilbert.

Theorem (F. Rosendal 2015)

Let d be the derivation defined above and suppose that $G \leqslant \mathrm{GL}(\ell_2(T) \oplus \ell_2(T))$ is a bounded subgroup leaving the first copy of $\ell_2(T)$ invariant and containing $\lambda_d[\mathrm{Aut}(T)]$. Then there is a continuous homogeneous map $\psi\colon \ell_2(T) \longrightarrow \ell_2(T)$ for which

$$L^* + \psi\colon \ell_2(T) \longrightarrow \ell_\infty(T) \quad \text{and} \quad L - \psi\colon \ell_1(T) \longrightarrow \ell_2(T)$$

commute with $\lambda(g)$ for $g \in \mathrm{Aut}(T)$ and so that every element of G is of the form

$$\begin{pmatrix} u & u\psi - \psi v \\ 0 & v \end{pmatrix}$$

for some $u, v \in \mathrm{GL}(\ell_2(T))$.
Finally, the following mappings are sot*-isomorphisms:*

$$\begin{pmatrix} u & u\psi - \psi v \\ 0 & v \end{pmatrix} \mapsto u \qquad \text{and} \qquad \begin{pmatrix} u & u\psi - \psi v \\ 0 & v \end{pmatrix} \mapsto v$$

Conclusion

The following questions remain open:

Question

Show that $L_p(0,1), 1 < p < +\infty, p \neq 2$ does not admit an equivalent transitive norm.

Question

Find a non-unitarizable, maximal bounded, subgroup of $\mathrm{Aut}(\ell_2)$.

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