Phase transitions

A. O. Lopes

Inst. Mat. - UFRGS

25 de fevereiro de 2015

Joint work:

Phase Transitions in One-dimensional Translation Invariant Systems: a Ruelle Operator Approach -
Cioletti and Lopes - Journal of Stat. Physics - 2015

Interactions, Specifications, DLR probabilities and the Ruelle Operator in the One-Dimensional Lattice - Cioletti and Lopes - Arxiv 2014

$\Omega = \{1, 2, ..., d\}^{\mathbb{N}}$ and the dynamics is given by the shift σ which acts on Ω.

Here $\sigma(x_0, x_1, x_2, ...) = (x_1, x_2, x_3, ...)$.

$\Omega = \{1, 2, ..., d\}^{\mathbb{N}}$ and the dynamics is given by the shift σ which acts on Ω.

Here $\sigma(x_0, x_1, x_2, ...) = (x_1, x_2, x_3, ...)$.

A potential is a continuous function $f : \Omega \to \mathbb{R}$ which describes the interaction of spins in the lattice $\mathbb{N}$. We have here d spins.

We denote by $\mathcal{M}(\sigma)$ the set of invariant probabilities measures (over the Borel sigma algebra of Ω) under σ.
The analysis of potentials $f : \{1, 2, ..., d\}^{\mathbb{Z}} \to \mathbb{R}$ is reduced via coboundary to the above case.

Definition (Pressure)

For a continuous potential $f : \Omega \to \mathbb{R}$ the Pressure of f is given by

$$P(f) = \sup_{\mu \in \mathcal{M}(\sigma)} \left\{ h(\mu) + \int_{\Omega} f \, d\mu \right\},$$

where $h(\mu)$ denotes the Shannon-Kolmogorov entropy of μ.

Definition

A probability measure $\mu \in \mathcal{M}(\sigma)$ is called an equilibrium state for f if

$$h(\mu) + \int_\Omega f \, d\mu = P(f).$$

Notation: μ_f for the equilibrium state for f.

Definition

A probability measure $\mu \in \mathcal{M}(\sigma)$ is called an equilibrium state for f if

$$h(\mu) + \int_\Omega f \, d\mu = P(f).$$

Notation: μ_f for the equilibrium state for f.

If f is continuous there always exists at least one equilibrium state.

Definition

A probability measure $\mu \in \mathcal{M}(\sigma)$ is called an equilibrium state for f if

$$h(\mu) + \int_\Omega f \, d\mu = P(f).$$

Notation: μ_f for the equilibrium state for f.

If f is continuous there always exists at least one equilibrium state.

The existence of more than one equilibrium state is a possible meaning for phase transition. If f is Holder the equilibrium state μ_f is unique. In this case μ_f is positive on open sets and has no atoms.

Definition

A probability measure $\mu \in \mathcal{M}(\sigma)$ is called an equilibrium state for f if

$$h(\mu) + \int_\Omega f \, d\mu = P(f).$$

Notation: μ_f for the equilibrium state for f.

If f is continuous there always exists at least one equilibrium state.

The existence of more than one equilibrium state is a possible meaning for phase transition. If f is Holder the equilibrium state μ_f is unique. In this case μ_f is positive on open sets and has no atoms.

When $f : \Omega = \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$ is continuous and a certain $k \in \{1, 2, ..., d\}$ is such that the Dirac delta on $k^\infty \in \Omega$ is an equilibrium state for f we say that there exists magnetization.

Definition

Given a continuous function $f : \Omega \to \mathbb{R}$, consider the Ruelle operator (or transfer) $\mathcal{L}_f : C(\Omega) \to C(\Omega)$ (for the potential f) defined in such way that for any continuous function $\psi : \Omega \to \mathbb{R}$ we have $\mathcal{L}_f(\psi) = \varphi$, where

$$\varphi(x) = \mathcal{L}_f(\psi)(x) = \sum_{y \in \Omega;\, \sigma(y)=x} e^{f(y)}\, \psi(y).$$

Definition

Given a continuous function $f : \Omega \to \mathbb{R}$, consider the Ruelle operator (or transfer) $\mathcal{L}_f : C(\Omega) \to C(\Omega)$ (for the potential f) defined in such way that for any continuous function $\psi : \Omega \to \mathbb{R}$ we have $\mathcal{L}_f(\psi) = \varphi$, where

$$\varphi(x) = \mathcal{L}_f(\psi)(x) = \sum_{y\in\Omega;\, \sigma(y)=x} e^{f(y)}\, \psi(y).$$

Definition

The dual operator $\mathcal{L}_f^*$ acts on the space of probability measures. It sends a probability measure μ to a probability measure $\mathcal{L}_f^*(\mu) = \nu$ defined in the following way: the probability measure ν is unique probability measure satisfying

$$< \psi, \mathcal{L}_f^*(\mu) > = \int_\Omega \psi d\, \mathcal{L}_f^*(\mu) = \int_\Omega \psi\, d\nu = \int_\Omega \mathcal{L}_f(\psi) d\mu \; =< \mathcal{L}_f(\psi), \mu >,$$

for any continuous function ψ.

Definition

Let $f : \Omega \to \mathbb{R}$ be a continuous function. We call a probability measure ν a Gibbs probability for f if there exists a positive $\lambda > 0$ such that $\mathcal{L}_f^*(\nu) = \lambda\,\nu$. We denote the set of such probabilities by $\mathcal{G}^*(f)$.

Definition

Let $f : \Omega \to \mathbb{R}$ be a continuous function. We call a probability measure ν a Gibbs probability for f if there exists a positive $\lambda > 0$ such that $\mathcal{L}_f^*(\nu) = \lambda\,\nu$. We denote the set of such probabilities by $\mathcal{G}^*(f)$.

$\mathcal{G}^*(f) \neq \emptyset$.

Definition

Let $f : \Omega \to \mathbb{R}$ be a continuous function. We call a probability measure ν a Gibbs probability for f if there exists a positive $\lambda > 0$ such that $\mathcal{L}_f^*(\nu) = \lambda \nu$. We denote the set of such probabilities by $\mathcal{G}^*(f)$.

$\mathcal{G}^*(f) \neq \emptyset$.

Definition

If a continuous f is such $\mathcal{L}_f(1) = 1$ we say that f is normalized. Then, there exists μ (which is invariant) such that $\mathcal{L}_f^*(\mu) = \mu$. Any such μ is called *g*-measure associated to f. The $J : \Omega \to \mathbb{R}$ such that $\log J = f$ is called the Jacobian of μ. Moreover $h(\mu) = -\int logJ\, dmu$

Definition

Let $f : \Omega \to \mathbb{R}$ be a continuous function. We call a probability measure ν a Gibbs probability for f if there exists a positive $\lambda > 0$ such that $\mathcal{L}_f^*(\nu) = \lambda\,\nu$. We denote the set of such probabilities by $\mathcal{G}^*(f)$.

$\mathcal{G}^*(f) \neq \emptyset$.

Definition

If a continuous f is such $\mathcal{L}_f(1) = 1$ we say that f is normalized. Then, there exists μ (which is invariant) such that $\mathcal{L}_f^*(\mu) = \mu$. Any such μ is called *g*-measure associated to f. The $J : \Omega \to \mathbb{R}$ such that $\log J = f$ is called the Jacobian of μ. Moreover $h(\mu) = -\int logJ\, dmu$

Main property: if $f = \log J$ is Holder then given a continuous $b : \Omega \to \mathbb{R}$ we have that for any $x_0 \in \Omega$

$$\lim_{n \to \infty} \mathcal{L}_f^n(b)(x_0) = \int b d\mu.$$

The convergence is uniform on x_0.

very helpful: existence or not of a eigenfunction φ strictly positive. That is, existence on a main eigenvalue $\lambda > 0$ such that $\mathcal{L}_f(\varphi) = \lambda\varphi$ and $\varphi > 0$.

very helpful: existence or not of a eigenfunction φ strictly positive. That is, existence on a main eigenvalue $\lambda > 0$ such that $\mathcal{L}_f(\varphi) = \lambda\varphi$ and $\varphi > 0$.

For a Hölder potential f there exist a value $\lambda > 0$ which is a common eigenvalue for both Ruelle operator and its dual (and such $\varphi > 0$). The eigenprobability ν associated to λ is unique. This probability ν (which is unique) is a Gibbs state according to the above definition.

very helpful: existence or not of a eigenfunction φ strictly positive. That is, existence on a main eigenvalue $\lambda > 0$ such that $\mathcal{L}_f(\varphi) = \lambda\varphi$ and $\varphi > 0$.

For a Hölder potential f there exist a value $\lambda > 0$ which is a common eigenvalue for both Ruelle operator and its dual (and such $\varphi > 0$). The eigenprobability ν associated to λ is unique. This probability ν (which is unique) is a Gibbs state according to the above definition.

This eigenvalue λ is the spectral radius of the operator $\mathcal{L}_f$. If $\mathcal{L}_f(\varphi) = \lambda\varphi$ and $\mathcal{L}_f^*(\nu) = \lambda\nu$, then up to normalization (to get a probability measure) the probability measure $\mu = \varphi\,\nu$ is the equilibrium state for f.

very helpful: existence or not of a eigenfunction φ strictly positive. That is, existence on a main eigenvalue $\lambda > 0$ such that $\mathcal{L}_f(\varphi) = \lambda\varphi$ and $\varphi > 0$.

For a Hölder potential f there exist a value $\lambda > 0$ which is a common eigenvalue for both Ruelle operator and its dual (and such $\varphi > 0$). The eigenprobability ν associated to λ is unique. This probability ν (which is unique) is a Gibbs state according to the above definition.

This eigenvalue λ is the spectral radius of the operator $\mathcal{L}_f$. If $\mathcal{L}_f(\varphi) = \lambda\varphi$ and $\mathcal{L}_f^*(\nu) = \lambda\nu$, then up to normalization (to get a probability measure) the probability measure $\mu = \varphi\,\nu$ is the equilibrium state for f.

When there exists a positive continuous eigenfunction for the Ruelle operator (of a continuous potential f) it is unique. We remark that for a general continuous potential may not exist a positive continuous eigenfunction.

For a fixed potential f consider a real parameter $\beta = 1/T$, where T is temperature. Then, $p(\beta) = P(\beta f)$ is a real function.

For a fixed potential f consider a real parameter $\beta = 1/T$, where T is temperature. Then, $p(\beta) = P(\beta f)$ is a real function.
If f is Holder then $p(\beta)$ is real analytic on β. Moreover,

$$\frac{d\,p(\beta)}{d\beta} = \int f\,d\mu_{\beta\,f}.$$

where $\mu_{\beta\,f}$ is the (unique) equilibrium state for βf.

For a fixed potential f consider a real parameter $\beta = 1/T$, where T is temperature. Then, $p(\beta) = P(\beta f)$ is a real function.
If f is Holder then $p(\beta)$ is real analytic on β. Moreover,

$$\frac{d\,p(\beta)}{d\beta} = \int f\, d\mu_{\beta\, f}.$$

where $\mu_{\beta\, f}$ is the (unique) equilibrium state for βf.
Possible meanings for phase transition: there exists a critical value β_c such that

1. The function $p(\beta) = P(\beta\, f)$ is not analytic at $\beta = \beta_c$
2. There are more than one equilibrium state, that is, at least two probability measures maximizing $h(\mu) + \beta_c \int_\Omega f\, d\mu$.
3. The dual of Ruelle operator has more than one eigenprobability for the potential $\beta_c f$. We denote $\mathcal{G}^*$
4. There exist more than one DLR (to be defined later) probability for the potential $\beta_c f$. We denote $\mathcal{G}^{DLR}$
5. There is more than one Thermodynamic Limit probability (to be defined later) for the potential $\beta_c\, f$. We denote $\mathcal{G}^{TL}$.

Decay of correlation of exponential type (for a large class of observable functions φ) occurs for the equilibrium probability of a Hölder potential. That is: $\int \varphi(\sigma^n(x)\,)\,(\,\varphi(x) \;\; - \int \varphi\, d\mu) \;\; d\mu(x) \sim C\ \theta^{-n}$ with $\theta < 1$, when $n \to \infty$

Decay of correlation of exponential type (for a large class of observable functions φ) occurs for the equilibrium probability of a Hölder potential. That is: $\int \varphi(\sigma^n(x)) \left(\varphi(x) - \int \varphi \, d\mu \right) d\mu(x) \sim C\, \theta^{-n}$ with $\theta < 1$, when $n \to \infty$

By the other hand, in some cases where there is phase transition (not Hölder), for the equilibrium probability (at the transition temperature) one gets polynomial decay of correlation.

Decay of correlation of exponential type (for a large class of observable functions φ) occurs for the equilibrium probability of a Hölder potential. That is: $\int \varphi(\sigma^n(x))\,(\,\varphi(x) \;-\int \varphi\, d\mu)\;\; d\mu(x) \sim C\,\theta^{-n}$ with $\theta < 1$, when $n \to \infty$

By the other hand, in some cases where there is phase transition (not Hölder), for the equilibrium probability (at the transition temperature) one gets polynomial decay of correlation.

That is for some φ we have $\int \varphi(\sigma^n(x))\,(\,\varphi(x) \;-\int \varphi\, d\mu)\;\; d\mu(x) \sim C\, n^{-\rho}$ with $\rho > 0$, when $n \to \infty$.

The Double Hofbauer Model.

We will define $g : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ which is continuous but not Holder.

We define two infinite collections of cylinder sets given by

$$L_n = \overline{\underbrace{000...0}_{n}1} \quad \text{and} \quad R_n = \overline{\underbrace{111...1}_{n}0}, \quad \text{for all } n \geq 1.$$

The Double Hofbauer Model.

We will define $g : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ which is continuous but not Holder.

We define two infinite collections of cylinder sets given by

$$L_n = \overline{\underbrace{000...0}_{n}1} \quad \text{and} \quad R_n = \overline{\underbrace{111...1}_{n}0}, \quad \text{for all } n \geq 1.$$

Fix two real numbers $\gamma > 1$ and $\delta > 1$, satisfying $\delta < \gamma$.

We define $g = g_{\gamma,\delta} : \Omega \to \mathbb{R}$ in the following way: for any $x \in \Omega$

$$g(x) = \begin{cases} -\gamma \log \frac{n}{n-1}, & \text{if } x \in L_n, \text{ for some } n \geq 2; \\ -\delta \log \frac{n}{n-1}, & \text{if } x \in R_n, \text{ for some } n \geq 2; \\ -\log \zeta(\gamma), & \text{if } x \in L_1; \\ -\log \zeta(\delta), & \text{if } x \in R_1; \\ 0, & \text{if } x \in \{1^\infty, 0^\infty\}, \end{cases}$$

where $\zeta(s) = \sum_{n \geq 1} 1/n^s$.

(Baraviera-Leplaideur-Lopes) Stoch. Dyn (2012).
We define $H : \Omega = \{0,1\}^{\mathbb{N}} \to \Omega$ by:

$$H((\underbrace{0,...,0}_{c_1},\underbrace{1,...,1}_{c_2}\underbrace{0,...,0}_{c_3},1,...)) = (\underbrace{0,...,0}_{2c_1},\underbrace{1,...,1}_{c_2}\underbrace{0,...,0}_{c_3},1,\ldots),$$

and

$$H((\underbrace{1,\ldots,1}_{c_1},\underbrace{0,\ldots,0}_{c_2}\underbrace{1,\ldots,1}_{c_3},1,\ldots)) = (\underbrace{1,\ldots,1}_{2c_1},\underbrace{0,\ldots,0}_{c_2}\underbrace{1,\ldots,1}_{c_3},1,\ldots).$$

We define the renormalization operator $\mathcal{R}$ in the following way: given the potential $V_1 : \Omega \to \mathbb{R}$ we get $V_2 = \mathcal{R}(V_1)$ where

$$V_2(x) = V_1(\sigma(\,(H(x)))\,) + V_1(H(x)).$$

It is easy to see that for γ and δ fixed the corresponding double Hofbauer potential g is fixed for $\mathcal{R}$.

This g is not normalized but there exist an explicit expression for the leading eigenfunction φ associated to the main eigenvalue 1. In this case $\phi = g + \varphi - \varphi \circ \sigma$ is normalized.

This g is not normalized but there exist an explicit expression for the leading eigenfunction φ associated to the main eigenvalue 1. In this case $\phi = g + \varphi - \varphi \circ \sigma$ is normalized.
The function $p(\beta) = P(\beta\, g)$ is not analytic at $\beta_c = 1$. We can show the exact expression for $p(\beta)$ when $\beta \sim 1$.

Theorem

In the case $2 > \gamma > \delta > 1$*, we have*

$$p(\beta) = C\,(1-\beta)^{\alpha} + \textit{high order terms.}$$

In the case $3 > \gamma > \delta > 2$*, we have*

$$p(\beta) = A_1(1-\beta) + C_1\,(1-\beta)^{\alpha}(1+o(1)).$$

Since $p(\beta) = 0$ *for* $\beta > 1$ *there is a lack of analyticity of the pressure* $p(\beta)$ *at* $\beta = 1$*.*
In the case $2 > \gamma > \delta > 1$*, we have lack of differentiability at* $\beta = 1$*.*

There are more than one equilibrium state for g, more explicitly, at least three ergodic probability measures maximizing $h(\mu) + \int_\Omega g \, d\mu$, when $\gamma, \delta > 2$.

There are more than one equilibrium state for g, more explicitly, at least three ergodic probability measures maximizing $h(\mu) + \int_\Omega g \, d\mu$, when $\gamma, \delta > 2$.

Two of them are δ_{0^∞} and δ_{1^∞} which represents magnetization at temperature $T = 1$.

There are more than one equilibrium state for g, more explicitly, at least three ergodic probability measures maximizing $h(\mu) + \int_\Omega g\, d\mu$, when $\gamma, \delta > 2$.

Two of them are δ_{0^∞} and δ_{1^∞} which represents magnetization at temperature $T = 1$.

The dual of Ruelle operator has one eigenprobability for g.

There are more than one equilibrium state for g, more explicitly, at least three ergodic probability measures maximizing $h(\mu) + \int_\Omega g \, d\mu$, when $\gamma, \delta > 2$.

Two of them are δ_{0^∞} and δ_{1^∞} which represents magnetization at temperature $T = 1$.

The dual of Ruelle operator has one eigenprobability for g.

There exist more than one DLR probability measure for the potential g.

There are more than one equilibrium state for g, more explicitly, at least three ergodic probability measures maximizing $h(\mu) + \int_\Omega g \, d\mu$, when $\gamma, \delta > 2$.

Two of them are δ_{0^∞} and δ_{1^∞} which represents magnetization at temperature $T = 1$.

The dual of Ruelle operator has one eigenprobability for g.

There exist more than one DLR probability measure for the potential g.

There is more than one Thermodynamic Limit for g (to be defined later). This is obtained via the Renewal Theorem.

There are more than one equilibrium state for g, more explicitly, at least three ergodic probability measures maximizing $h(\mu) + \int_\Omega g\, d\mu$, when $\gamma, \delta > 2$.

Two of them are δ_{0^∞} and δ_{1^∞} which represents magnetization at temperature $T = 1$.

The dual of Ruelle operator has one eigenprobability for g.

There exist more than one DLR probability measure for the potential g.

There is more than one Thermodynamic Limit for g (to be defined later). This is obtained via the Renewal Theorem.

We can present the exact parameter ρ which describes the polynomial decay of correlation $n^{-\rho}$ for the observable $I_{\overline{0}}$,, when $\gamma, \delta > 2$. This is obtained via the Renewal Theorem.

DLR Probabilities

Let $\mathcal{B}$ denote the Borel sigma-algebra on $\Omega = \{0,1\}^{\mathbb{N}}$ and $\mathcal{X}_n = \sigma^{-n}(\mathcal{B})$, that is, the σ-algebra generated by the random variables $X_n, X_{n+1}, \ldots$ on the Bernoulli space, where $X_n(x) = x_n$ for all $x \in \Omega$.

DLR Probabilities

Let $\mathcal{B}$ denote the Borel sigma-algebra on $\Omega = \{0,1\}^{\mathbb{N}}$ and $\mathcal{X}_n = \sigma^{-n}(\mathcal{B})$, that is, the σ-algebra generated by the random variables $X_n, X_{n+1}, \ldots$ on the Bernoulli space, where $X_n(x) = x_n$ for all $x \in \Omega$.

Definition

Given a potential ϕ we say that a probability measure m is a DLR probability for ϕ if for all $n \in \mathbb{N}$ and any cylinder set $\overline{x_0 x_1 \ldots x_{n-1}}$, we have m-almost every $z = (z_0, z_1, z_2, ...)$ that

$$\mathbb{E}_m(I_{\overline{x_0 x_1 \ldots x_{n-1}}} \,|\, \mathcal{X}_n)(z) = \frac{e^{\phi(z)+\phi(\sigma(z))+\ldots+\phi(\sigma^{n-1}(z))}}{\sum_{y \text{ such that } \sigma^n(z)=\sigma^n(y)} e^{\phi(y)+\phi(\sigma(y))+\ldots+\phi(\sigma^{n-1}(y))}}.$$

DLR Probabilities

Let $\mathcal{B}$ denote the Borel sigma-algebra on $\Omega = \{0,1\}^{\mathbb{N}}$ and $\mathcal{X}_n = \sigma^{-n}(\mathcal{B})$, that is, the σ-algebra generated by the random variables $X_n, X_{n+1}, \ldots$ on the Bernoulli space, where $X_n(x) = x_n$ for all $x \in \Omega$.

Definition

Given a potential ϕ we say that a probability measure m is a DLR probability for ϕ if for all $n \in \mathbb{N}$ and any cylinder set $\overline{x_0 x_1 \ldots x_{n-1}}$, we have m-almost every $z = (z_0, z_1, z_2, \ldots)$ that

$$\mathbb{E}_m(I_{\overline{x_0 x_1 \ldots x_{n-1}}} \,|\, \mathcal{X}_n)(z) = \frac{e^{\phi(z)+\phi(\sigma(z))+\ldots+\phi(\sigma^{n-1}(z))}}{\sum_{y \text{ such that } \sigma^n(z)=\sigma^n(y)} e^{\phi(y)+\phi(\sigma(y))+\ldots+\phi(\sigma^{n-1}(y))}}.$$

The set of all DLR probabilities for ϕ is denoted by $\mathcal{G}^{DLR}(\phi)$. In general this set is not unique. DLR probabilities do not have to be invariant for the shift.

We assume that $\phi = \log J$ is normalized.
In this case we have the simple condition

$$\mathbb{E}_m(I_{\overline{x_0 x_1 \ldots x_{n-1}}} \mid \mathcal{X}_n)(z) = e^{\phi(z)+\phi(\sigma(z))+\ldots+\phi(\sigma^{n-1}(z))}$$

We assume that $\phi = \log J$ is normalized.
In this case we have the simple condition

$$\mathbb{E}_m(I_{\overline{x_0 x_1 \ldots x_{n-1}}} \,|\, \mathcal{X}_n)(z) = e^{\phi(z)+\phi(\sigma(z))+\ldots+\phi(\sigma^{n-1}(z))}$$

If $\mathcal{L}^*_{\log J}(m) = m$ then for any continuous function $f : \Omega \to \mathbb{R}$ we have that

$$\mathbb{E}_m(f \,|\, \mathcal{X}_n)(x) = \mathcal{L}^n_{\log J}(f)(\sigma^n(x)).$$

We assume that $\phi = \log J$ is normalized.
In this case we have the simple condition

$$\mathbb{E}_m(I_{\overline{x_0 x_1 \dots x_{n-1}}} \,|\, \mathcal{X}_n)(z) = e^{\phi(z)+\phi(\sigma(z))+\dots+\phi(\sigma^{n-1}(z))}$$

If $\mathcal{L}^*_{\log J}(m) = m$ then for any continuous function $f : \Omega \to \mathbb{R}$ we have that

$$\mathbb{E}_m(f \,|\, \mathcal{X}_n)(x) = \mathcal{L}^n_{\log J}(f)(\sigma^n(x)).$$

Suppose that J is positive and continuous. If $\mathcal{L}^*_{\log J}(m) = m$ then,
$\mathcal{G}^*(\log J) \subset \mathcal{G}^{DLR}(\log J)$.

We assume that $\phi = \log J$ is normalized.
In this case we have the simple condition

$$\mathbb{E}_m(I_{\overline{x_0 x_1 \ldots x_{n-1}}} \,|\, \mathcal{X}_n)(z) = e^{\phi(z)+\phi(\sigma(z))+\ldots+\phi(\sigma^{n-1}(z))}$$

If $\mathcal{L}^*_{\log J}(m) = m$ then for any continuous function $f : \Omega \to \mathbb{R}$ we have that

$$\mathbb{E}_m(f \,|\, \mathcal{X}_n)(x) = \mathcal{L}^n_{\log J}(f)(\sigma^n(x)).$$

Suppose that J is positive and continuous. If $\mathcal{L}^*_{\log J}(m) = m$ then, $\mathcal{G}^*(\log J) \subset \mathcal{G}^{DLR}(\log J)$.

There are examples of continuous potentials $\phi = \log J$ such that there is more than one ergodic probability μ in $\mathcal{G}^*(\log J)$.
Quas - ETDS (1996)

We assume that $\phi = \log J$ is normalized.
In this case we have the simple condition

$$\mathbb{E}_m(I_{\overline{x_0 x_1 \ldots x_{n-1}}} \,|\, \mathcal{X}_n)(z) = e^{\phi(z)+\phi(\sigma(z))+\ldots+\phi(\sigma^{n-1}(z))}$$

If $\mathcal{L}^*_{\log J}(m) = m$ then for any continuous function $f : \Omega \to \mathbb{R}$ we have that

$$\mathbb{E}_m(f \,|\, \mathcal{X}_n)(x) = \mathcal{L}^n_{\log J}(f)(\sigma^n(x)).$$

Suppose that J is positive and continuous. If $\mathcal{L}^*_{\log J}(m) = m$ then, $\mathcal{G}^*(\log J) \subset \mathcal{G}^{DLR}(\log J)$.

There are examples of continuous potentials $\phi = \log J$ such that there is more than one ergodic probability μ in $\mathcal{G}^*(\log J)$.
Quas - ETDS (1996)

In this case one get phase transition in the DLR sense.
This also happens for the Double Hofbaeur model, but...

Thermodynamic Limit probabilities - the role of the boundary condition

Thermodynamic Limit probabilities - the role of the boundary condition

Fix an $y \in \Omega$ - the boundary condition.
For a given $n \in \mathbb{N}$ consider the probability measure on Ω so that for any Borel F, we have

$$\mu_n^y(F) = \frac{1}{Z_n^y} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} 1_F(x) \exp(-(f(x) + f(\sigma(x)) + ... + f(\sigma^{n-1}(x) \,))$$

where Z_n^y is a normalizing factor called partition function given by

$$Z_n^y = \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} \exp(-(f(x) + f(\sigma(x)) + ... + f(\sigma^{n-1}(x) \,)).$$

Thermodynamic Limit probabilities - the role of the boundary condition

Fix an $y \in \Omega$ - the boundary condition.
For a given $n \in \mathbb{N}$ consider the probability measure on Ω so that for any Borel F, we have

$$\mu_n^y(F) = \frac{1}{Z_n^y} \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(y)}} 1_F(x) \exp(-(\, f(x) + f(\sigma(x)) + ... + f(\sigma^{n-1}(x)\ ))$$

where Z_n^y is a normalizing factor called partition function given by

$$Z_n^y = \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(y)}} \exp(-(\, f(x) + f(\sigma(x)) + ... + f(\sigma^{n-1}(x)\ )).$$

In the Ruelle Operator formalism:

$$\mu_n^y(F) = \frac{\mathcal{L}_f^n\,(1_F)\,(\sigma^n(y))}{\mathcal{L}_f^n\,(1)\,(\sigma^n(y))} \quad \text{or} \quad \mu_n^y = \frac{1}{\mathcal{L}_f^n(1)(\sigma^n(y))}[\,(\mathcal{L}_f)^*\,]^n\,(\delta_{\sigma^n(y)}).$$

Definition

Consider $f : \Omega \to \mathbb{R}$. For a fixed $y \in \Omega$ any weak limit of the subsequences $\mu^{y}_{n_k,}$, when $k \to \infty$ is called Thermodynamic Limit probability with boundary condition y. Now we consider the collection of all the Thermodynamic Limits varying $y \in \Omega$ and take the closed convex hull of this collection. This set is denoted by $\mathcal{G}^{TL}(f)$.

Definition

Consider $f : \Omega \to \mathbb{R}$. For a fixed $y \in \Omega$ any weak limit of the subsequences $\mu^y_{n_k,}$, when $k \to \infty$ is called Thermodynamic Limit probability with boundary condition y. Now we consider the collection of all the Thermodynamic Limits varying $y \in \Omega$ and take the closed convex hull of this collection. This set is denoted by $\mathcal{G}^{TL}(f)$.

Proposition Suppose that $f = \log J$ is continuous. Then, $\mathcal{G}^{TL}(f) = \mathcal{G}^{DLR}(f)$.

Definition

Consider $f : \Omega \to \mathbb{R}$. For a fixed $y \in \Omega$ any weak limit of the subsequences $\mu^y_{n_k}$, when $k \to \infty$ is called Thermodynamic Limit probability with boundary condition y. Now we consider the collection of all the Thermodynamic Limits varying $y \in \Omega$ and take the closed convex hull of this collection. This set is denoted by $\mathcal{G}^{TL}(f)$.

Proposition Suppose that $f = \log J$ is continuous. Then, $\mathcal{G}^{TL}(f) = \mathcal{G}^{DLR}(f)$.

If f is Hölder it is known that for any fixed y we have $\lim_{n\to\infty} \mathcal{L}_f^n(I_{[a]})(y) = \mu([a])$, where μ is the fixed point for the operator $\mathcal{L}_f^*$. (which is the equilibrium state for $\log J$) and $[a]$ is any cylinder set.

That is,

$$\lim_{n\to\infty} \mu_n^y = \mu.$$

Definition

Consider $f : \Omega \to \mathbb{R}$. For a fixed $y \in \Omega$ any weak limit of the subsequences $\mu^y_{n_k}$, when $k \to \infty$ is called Thermodynamic Limit probability with boundary condition y. Now we consider the collection of all the Thermodynamic Limits varying $y \in \Omega$ and take the closed convex hull of this collection. This set is denoted by $\mathcal{G}^{TL}(f)$.

Proposition Suppose that $f = \log J$ is continuous. Then, $\mathcal{G}^{TL}(f) = \mathcal{G}^{DLR}(f)$.

If f is Hölder it is known that for any fixed y we have $\lim_{n\to\infty} \mathcal{L}^n_f(I_{[a]})(y) = \mu([a])$, where μ is the fixed point for the operator $\mathcal{L}^*_f$. (which is the equilibrium state for $\log J$) and $[a]$ is any cylinder set.

That is,

$$\lim_{n\to\infty} \mu^y_n = \mu.$$

In the case of the Double Hofbauer there are points where $J = 0$.

Here we take the potential J which is the normalization for the double Hofbauer g. That is $\log J = g + \log \varphi - \varphi \circ \sigma$ and φ is the eigenfunction.

Here we take the potential J which is the normalization for the double Hofbauer g. That is $\log J = g + \log \varphi - \varphi \circ \sigma$ and φ is the eigenfunction.

Renewal Equation: given a sequence $a : \mathbb{N} \to \mathbb{R}$ and a probability measure p defined on $\mathbb{N}$ we can ask whether exists or not another sequence $A : \mathbb{N} \to \mathbb{R}$ satisfying the following associated Renewal Equation: for all $q \in \mathbb{N}$

$$A(q) = [A(0)p_q + A(1)p_{q-1} + A(2)\, p_{q-2} + ... + A(q-2)p_2 + A(q-1)p_1] + a(q).$$

Here we take the potential J which is the normalization for the double Hofbauer g. That is $\log J = g + \log \varphi - \varphi \circ \sigma$ and φ is the eigenfunction.

Renewal Equation: given a sequence $a : \mathbb{N} \to \mathbb{R}$ and a probability measure p defined on $\mathbb{N}$ we can ask whether exists or not another sequence $A : \mathbb{N} \to \mathbb{R}$ satisfying the following associated Renewal Equation: for all $q \in \mathbb{N}$

$$A(q) = [A(0)p_q + A(1)p_{q-1} + A(2)\, p_{q-2} + ... + A(q-2)p_2 + A(q-1)p_1] + a(q).$$

If $M = \sum_{q=1}^{\infty} q\, p_q$ then

$$\lim_{q\to\infty} A(q) = \frac{\sum_{q=1}^{\infty} a(q)}{M}.$$

One important feature of the Renewal Theorem is that we get the limit value of $A(q)$, as $q \to \infty$, without knowing the explicit values of the $A(q)$. In our case $\sum p(n) = \sum \frac{n^{-\gamma}}{\zeta(\gamma)} \sum \frac{n^{-\delta}}{\zeta(\delta)}$ in the Double Hofbauer.

We show that:

Proposition: For the Double Hofbauer model

$$\lim_{q\to\infty} \mu_q^{0^\infty}([0]) = 1 \text{ and } \lim_{q\to\infty} \mu_q^{1^\infty}([0]) = 0.$$

We show that:

Proposition: For the Double Hofbauer model

$$\lim_{q\to\infty} \mu_q^{0^\infty}([0]) = 1 \text{ and } \lim_{q\to\infty} \mu_q^{1^\infty}([0]) = 0.$$

Proposition: For any periodic points y and $z \in \Omega$ (being not the fixed points) we have

$$\lim_{q\to\infty} \mu_q^{y}([0]) = \lim_{q\to\infty} \mu_q^{z}([0]).$$

We want to investigate the Thermodynamic Limit

$$\lim_{q\to\infty} \mu_q^y([a]) = \lim_{q\to\infty} \mathcal{L}_{\log J}^q(I_{[a]})(\sigma^q(y)),$$

for a point y in the Bernoulli space Ω and for a cylinder set $[a]$.

We want to investigate the Thermodynamic Limit

$$\lim_{q\to\infty} \mu_q^y([a]) = \lim_{q\to\infty} \mathcal{L}_{\log J}^q(I_{[a]})(\sigma^q(y)),$$

for a point y in the Bernoulli space Ω and for a cylinder set $[a]$.
We consider the case $[a] = [0]$ and $y = 0\,1^\infty$. The main point is to estimate $\lim_{q\to\infty} \mathcal{L}_{\log J}^q(I_{[0]}(0\,1^\infty))$.

We want to investigate the Thermodynamic Limit

$$\lim_{q\to\infty} \mu_q^y([a]) = \lim_{q\to\infty} \mathcal{L}_{\log J}^q(I_{[a]})(\sigma^q(y)),$$

for a point y in the Bernoulli space Ω and for a cylinder set $[a]$.
We consider the case $[a]=[0]$ and $y = 0\,1^\infty$. The main point is to estimate $\lim_{q\to\infty} \mathcal{L}_{\log J}^q(I_{[0]}(0\,1^\infty))$.
For any $q \geq 2$

$$\mathcal{L}_{\log J}^q(I_{[0]})(01\ldots) = \frac{(q-1)^{-\gamma}}{\zeta(\gamma)}\,\mathcal{L}_{\log J}^1(I_{[0]})(10\ldots)+$$
$$\frac{(q-2)^{-\gamma}}{\zeta(\gamma)}\,\mathcal{L}_{\log J}^2(I_{[0]})(10\ldots)+\ldots+\frac{3^{-\gamma}}{\zeta(\gamma)}\,\mathcal{L}_{\log J}^{q-3}(I_{[0]})(10\ldots)+$$
$$+\frac{2^{-\gamma}}{\zeta(\gamma)}\,\mathcal{L}_{\log J}^{q-2}(I_{[0]})(10\ldots)+\frac{1}{\zeta(\gamma)}\,\mathcal{L}_{\log J}^{q-1}(I_{[0]})(10\ldots)\,+\,\frac{(q+1)^{-\gamma}r(q+1)}{\zeta(\gamma)}.$$

and moreover

We want to investigate the Thermodynamic Limit

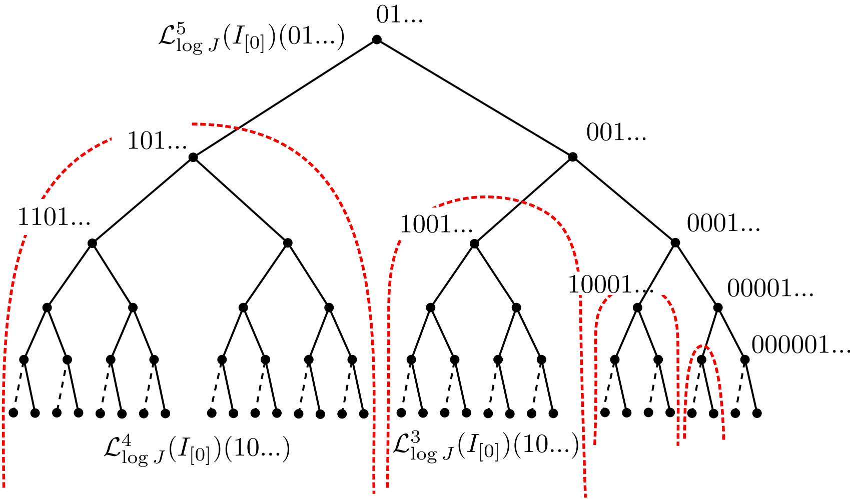
$$\lim_{q\to\infty} \mu_q^y([a]) = \lim_{q\to\infty} \mathcal{L}_{\log J}^q(I_{[a]})(\sigma^q(y)),$$

for a point y in the Bernoulli space Ω and for a cylinder set $[a]$.
We consider the case $[a]=[0]$ and $y=0\,1^\infty$. The main point is to estimate $\lim_{q\to\infty} \mathcal{L}_{\log J}^q(I_{[0]}(0\,1^\infty))$.
For any $q \geq 2$

$$\mathcal{L}_{\log J}^q(I_{[0]})(01\ldots) = \frac{(q-1)^{-\gamma}}{\zeta(\gamma)} \mathcal{L}_{\log J}^1(I_{[0]})(10\ldots) +$$
$$\frac{(q-2)^{-\gamma}}{\zeta(\gamma)} \mathcal{L}_{\log J}^2(I_{[0]})(10\ldots) + \ldots + \frac{3^{-\gamma}}{\zeta(\gamma)} \mathcal{L}_{\log J}^{q-3}(I_{[0]})(10\ldots) +$$
$$+\frac{2^{-\gamma}}{\zeta(\gamma)} \mathcal{L}_{\log J}^{q-2}(I_{[0]})(10\ldots) + \frac{1}{\zeta(\gamma)} \mathcal{L}_{\log J}^{q-1}(I_{[0]})(10\ldots) + \frac{(q+1)^{-\gamma} r(q+1)}{\zeta(\gamma)}.$$

and moreover

$$\mathcal{L}_{\log J}^q(I_{[0]})(10...) = \frac{1}{\zeta(\delta)} q^{-\delta} + \frac{1}{\zeta(\delta)}(q-1)^{-\delta} \mathcal{L}_{\log J}^1(I_{[0]})(01...) + \ldots$$
$$+\frac{1}{\zeta(\delta)} 3^{-\delta} \mathcal{L}_{\log J}^{q-3}(I_{[0]})(01..) + \frac{2^{-\delta}}{\zeta(\delta)} \mathcal{L}_{\log J}^{q-2}(I_{[0]})(01..) + \frac{1}{\zeta(\delta)} \mathcal{L}_{\log J}^{q-1}(I_{[0]})(010..).$$


01...
$\mathcal{L}^5_{\log J}(I_{[0]})(01\ldots)$
101...
001...
1101...
1001...
0001...
10001...
00001...
000001...
$\mathcal{L}^4_{\log J}(I_{[0]})(10\ldots)$
$\mathcal{L}^3_{\log J}(I_{[0]})(10\ldots)$