The group of linear isometries of the Gurarij space is extremely amenable

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Definitions and Notations

- $\mathbb{U}$ is the Urysohn space (1925): the unique (up to isometry) Polish space that is both universal and ultrahomogeneous.
- $\mathbb{G}$ is The Gurarij space (1965): the unique (up to isometry (Lusky, 1976, Kubis and Solecki, 2011)) separable Banach space with the following property: Given finite-dimensional normed spaces $E \subseteq F$, given $\varepsilon > 0$, and given an isometric linear embedding $\gamma : E \longrightarrow \mathbb{G}$ there exists an injective linear operator $\psi : F \longrightarrow \mathbb{G}$ extending γ and satisfying that:

$$(1-\varepsilon)\|x\| \leq \|\psi(x)\| \leq (1+\varepsilon)\|x\|$$

- The Gurarij space $\mathbb{G}$ is in some way the analogue of the Urysohn space $\mathbb{U}$ in the category of Banach spaces

Definitions and Notations

- A compact convex subset K of some locally convex space is called a (Choquet) *simplex* when every point $x \in K$ is the barycenter of a unique probability measure μ_x such that $\mu_x(\partial_e(K)) = 1$
- The *Poulsen simplex* $\mathbb{P}$ (1961) is the unique (up to affine homeomorphism) metrizable simplex whose extreme points $\partial_e\mathbb{P}$ are dense on it.

Motivation 1

Groups	Universal for Polish groups	Universal minimal flow
$\mathrm{Iso}(\mathbb{U})$	Uspenkij, 1990	$\{*\}$: Pestov, 2002
$\mathrm{Iso}_L(\mathbb{G})$	Ben Yaacov, 2012	$\{*\}$: This talk, 2015
$\mathrm{Homeo}(Q)$	Uspenkij, 1986	?
$\mathrm{Aut}(\mathbb{P})$	?	$\mathbb{P}$: This talk, 2015

Note: The universal minimal flow of $Homeo(Q)$ is not Q (Uspenkij, 2000)

Definition

Let G be a Hausdorff topological group.

- A G-flow is a continuous action of G on a compact Hausdorff space X. Notation: $G \curvearrowright X$.
- $G \curvearrowright X$ is *minimal* if it contains no proper subflows, i.e., there is no (non-$\emptyset$) compact G-invariant set other than X

remark

1. $G \curvearrowright X$ is minimal iff every orbit is dense in X :

$$\forall x \in X \ \overline{G.x} = X$$

2. Every G-flow X contains a minimal subflow $Y \subseteq X$.(Zorn's Lemma)

Universal minimal flow

Definition

$G \curvearrowright X$ is universal when:

$$\forall\, G \curvearrowright Y \text{ minimal } \exists \pi : X \longrightarrow Y$$

continuous, onto, and so that

$$\forall g \in G \ \forall x \in X,\ \pi(g.x) = g.\pi(x)$$

"Every minimal G-flow is a continuous image of $G \curvearrowright X$"

Folklore

Let G be a Hausdorff topological group. Then there is a unique G-flow that is both minimal and universal.

Notation: $G \curvearrowright M(G)$

Universal minimal flow

General question

Describe $G \curvearrowright M(G)$ explicitly when G is a "concrete" group.

Example: Pestov, 98

$Homeo_+(\mathbb{S}^1) \curvearrowright M(Homeo_+(\mathbb{S}^1))$ is the natural action $Homeo_+(\mathbb{S}^1) \curvearrowright \mathbb{S}^1$.

Extreme amenability

Definition

A topological group G is extremely amenable if every continuous action of G on a compact set K has a fixed point. i.e there is $\xi \in K$ such that $g.\xi = \xi$ for every $g \in G$.

Remark

G is extremely amenable iff $M(G)$ is a singleton.

Motivation 2

Veech, 1977

No locally compact group is extremely amenable

Always a good news to have a new example of such a group.

Examples

1. $\mathrm{Aut}(X,\mu)$: the group of all measure-preserving transformations of the standard Lebesgue measure space (X,μ), with the weak topology: the weakest topology making continuous every fonction $\mathrm{Aut}(X,\mu) \ni \tau \longmapsto \mu(A \cap \tau(A)) \in \mathbb{R}$(Giordano and Pestov, 2002)
2. $\mathrm{Aut}^\star(X,\mu)$: the group of all non-singular measure class preserving transformations of the standard Lebesgue measure space (X,μ), with the weak topology (Giordano and Pestov, 2007)
3. $\mathrm{Aut}(X,\mu)$ with the uniform topology: $d(\sigma,\tau) = \mu\{x \in X : \sigma(x) \neq \tau(x)\}$ is non-amenable. (Giordano and Pestov, 2002)

Giordano and Pestov question

Is the group $\mathrm{Aut}^{\star}(X, \mu)$ equipped with the uniform topology extremely amenable, or even amenable?

More examples

1. $\mathrm{Iso}(\mathbb{U})$ (Pestov, 2002)
2. In general:

Kechris, Pestov and Todorcevic, 2005

Suppose that $\mathbb{M}$ is a countable ultra homogeneous ordered structure. Then the automorphism group of $\mathbb{M}$ is extremely amenable if and only if $Age(\mathbb{M}) :=$ Finitely generated substructures of $\mathbb{M}$ has the Ramsey property.

(Approximate) Ramsey Properties

For the structures we are interested in, e.g. Gurarij space and Poulsen simplex, the *exact* Ramsey property is not the right notion to study the corresponding universal minimal flows. Instead, we need to deal with colourings of embeddings and *approximate*, not necessarily exact, Ramsey properties.
Similarly, there will not be ultra homogeneity but *approximate* ultra homogeneity.

(Approximate) Ramsey Properties

Definition

1. Given two Banach spaces X and Y, by an *embedding* from X into Y we mean a linear operator $T : X \longrightarrow Y$ such that $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$.
2. Let $\mathrm{Emb}(X, Y)$ be the collection of all embeddings from X into Y.
3. $\mathrm{Emb}(X, Y)$ is a metric space with the norm distance $d(T, U) := \|T - U\| := \sup_{x \in S_X} \|T(x) - U(x)\|$.

Bartošová , Lopez-Abad and M.

Given integers d, m and r, and given $\varepsilon > 0$, there exists $n = \mathbf{n}_\infty(d, m, r, \varepsilon)$ such that for every coloring $c : \mathrm{Emb}(\ell_\infty^d, \ell_\infty^n) \longrightarrow [r]$ there are $T \in \mathrm{Emb}(\ell_\infty^m, \ell_\infty^n)$ and $\widetilde{r} < r$ such that

$$T \circ \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m) \subseteq (c^{-1}\{\widetilde{r}\})_\varepsilon.$$

Definition

1. A finite dimensional space F is called *polyhedral* when the set of extremal points of its unit ball $\partial_e(B_F)$ is finite.
2. Given an integer d, let Pol_d be the class of all polyhedral spaces F such that $\#\partial_e(B_{F^*}) = 2d$.

Corollary 1

Given $d, m \in \mathbb{N}$, $r \in \mathbb{N}$ and $\varepsilon > 0$, there is $n = \mathbf{n_{pol}}(d, m, r, \varepsilon)$ such that for every $F \in \mathrm{Pol}_d$ every $G \in \mathrm{Pol}_m$ and every coloring $c : \mathrm{Emb}(F, \ell_\infty^n) \longrightarrow r$, there is $T \in \mathrm{Emb}(G, \ell_\infty^n)$and $\widetilde{r} < r$ such that

$$T \circ \mathrm{Emb}(F, G) \subseteq (c^{-1}\{\widetilde{r}\})_\varepsilon.$$

Ramsey Properties for finite dimensional

Given two finite dimensional spaces F and G and given $\theta \geq 1$. Let $\mathrm{Emb}_\theta(F, G) := \{T : F \longrightarrow G :$
T is an isomorphic embedding such that $\|T\|, \|T^{-1}\| \leq \theta\}$.

Corollary 2: Ingredient 1

Given finite normed dimensional spaces F and G, an integer r, numbers $\theta > 1$ and $\varepsilon > 0$, there exists $n = \mathbf{n_{fd}}(F, G, r, \theta, \varepsilon)$ such that for every coloring $c : \mathrm{Emb}_{\theta^2}(F, \ell_\infty^n) \longrightarrow r$, there are $\widetilde{T} \in \mathrm{Emb}_\theta(G, \ell_\infty^n)$ and $\widetilde{r} < r$ such that

$$\widetilde{T} \circ \mathrm{Emb}_\theta(F, G) \subseteq (c^{-1}\{\widetilde{r}\})_{\theta^2 - 1 + \varepsilon}.$$

Extreme amenability of $Iso_L(\mathbb{G})$: Pestov's criteria

Let G be a group acting on X and (Y, d) a metric space.

Definition

A function $f : X \longrightarrow (Y, d)$ is *finitely left oscillation stable* when for every finite subset $F \subseteq X$ and every $\varepsilon > 0$ there is $g \in G$ such that $\operatorname{osc}(f \restriction gF) \leq \varepsilon$.

Pestov: Ingredient 2

For a topological group G, the following are equivalent.

1. G is extremely amenable
2. Every bounded real-valued left uniformly continuous function f on G is finitely left oscillation stable.
3. Every bounded real-valued right uniformly continuous function f on G is finitely right oscillation stable.

E.A. of $Iso_L(\mathbb{G})$: Approximate ultra homogeneity

W. Kubis and S. Solecki: Ingredient 3

Let $X \subseteq \mathbb{G}$ be a subspace of finite dimension, $\theta > 1$ and let $\gamma \in \mathrm{Emb}_\theta(X, \mathbb{G})$. Then there exists $g \in G = Iso_L(\mathbb{G})$ such that $\|g \restriction X - \gamma\| \leq \theta - 1$.

Hints of the proof

1. Let $f : G \longrightarrow \mathbb{R}$, $\varepsilon > 0$, and $F \subseteq G$ be as Ingredient 2.
2. There are n_0 and $\delta > 0$ such that $\|g \restriction X_{n_0} - h \restriction X_{n_0}\| \leq \delta \implies |f(g) - f(h)| \leq \dfrac{\varepsilon}{4} \forall g, h \in G$.
3. Let $Y := \langle \bigcup_{\sigma \in F} \sigma(X_{n_0}) \rangle$ is a finite dimensional subspace of $\mathbb{G}$.
4. Fix a finite $\varepsilon/4$-net $\mathcal{N}$ of the image of f and an isometry $\theta : \ell_\infty^n \longrightarrow X_n$.
5. Denote $n := \mathbf{n_{fd}}(X_{n_0}, Y, \#\mathcal{N}, (1 + \delta/4)^{1/2}, \delta/4)$.
6. given $\varphi \in \mathrm{Emb}_{1+\delta/4}(X_{n_0}, \ell_\infty^n)$, by Ingredient 3, let $g_\varphi \in G$ be such that $\|\theta \circ \varphi - g_\varphi \restriction X_{n_0}\| \leq \frac{\delta}{4}$

More Hints of the proof

1. Define
$$c_0 : \mathrm{Emb}_{1+\frac{\delta}{4}}(X_{n_0}, \ell_\infty^n) \longrightarrow \#\mathcal{N}$$
as follows: given $\varphi \in \mathrm{Emb}_{1+\delta/4}(X_{n_0}, \ell_\infty^n)$, let $c_0(\varphi) \in \mathcal{N}$ be such that $|c_0(\varphi) - f(g_\varphi)| \leq \frac{\varepsilon}{4}$
2. By ingredient 1, there exists $\gamma \in \mathrm{Emb}_{(1+\delta/4)^{1/2}}(Y, \ell_\infty^n)$ and $\eta \in \mathcal{N}$ such that $\gamma \circ \mathrm{Emb}_{(1+\delta/4)^{1/2}}(X_{n_0}, Y) \subseteq (c_0^{-1}\{\eta\})_{\frac{\delta}{2}}$.
3. Using again ingredient 3, we find $\overline{g} \in G$ such that $\|\overline{g} \restriction Y - \theta \circ \gamma\| \leq (1 + \frac{\delta}{4})^{\frac{1}{2}} - 1 \leq \frac{\delta}{4}$.
4. $\mathrm{osc}(f \restriction \overline{g}F) \leq \varepsilon$

Remark

The Extreme amenability of the Polish group $Iso_L(\mathbb{G})$ is in fact, via the approximate ultra homogeneity of $\mathbb{G}$, equivalent to the ingredient 1, i.e. the Ramsey Property for finite dimensional normed spaces.

The Poulsen simplex $\mathbb{P}$

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The universal minimal flow of the group $\mathrm{Aut}(\mathbb{P})$ of affine homeomorphisms on $\mathbb{P}$ with the compact-open topology is $\mathbb{P}$.

Hints of the proof: Universality

Proposition

The stabilizer G_p of any extreme point $p \in \mathbb{P}$ is extremely amenable.

Hints

1. Every metrizable simplex is the inverse limit of some system $(\Delta_n, \rho_n)_n$. Where Δ_n is the positive part of the unit ball of ℓ_1^{n-1}. (Lazar and Lindenstrauss)
2. Let $\mathrm{Epi}(K, L)$ be the collection of all affine continuous mappings from K onto L.
3. As consequence of the approximate Ramsey property for the positive embedding between ℓ_∞^n's, we have:

Ramsey property for finite simplex

Theorem

Given integers d, m and r, $p \in \partial_e(\Delta_d)$, $q \in \partial_e(\Delta_m)$, and given $\varepsilon > 0$ there exist an integer $n = \mathbf{n_{Simpl,0}}(d, m, r, \varepsilon)$ such that for every coloring $c : \mathrm{Epi}((\Delta_n, t), (\Delta_d, p)) \longrightarrow r$, there is $\gamma \in \mathrm{Epi}((\Delta_n, t), (\Delta_m, q))$ and $r_0 < r$ such that

$$\mathrm{Epi}((\Delta_m, q), (\Delta_d, p)) \circ \gamma \subseteq (c^{-1}\{r_0\})_\varepsilon. \tag{1}$$

The corresponding quotient G/G_p is precompact and its completion is G-homeomorphic to $\mathbb{P}$

Hints of the proof: minimality

Minimality: Consequence of the following:

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Every non-trivial metrizable simplex for which the natural action of its group of affine homeomorphisms is minimal is affinely homeomorphic to the Poulsen simplex.

More consequences: Ramsey properties for finite metric spaces

1. Let $\mathcal{F}(M,p)$ be the *Lipschitz Free space* over the pointed metric space (M,p).
2. If M is a finite metric space, then $\mathcal{F}(M)$ is a polyhedral space.

Bartošová , Lopez-Abad and M.: Ingredient 1′

For every finite metric spaces M and N, $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite metric space P such that for every coloring $c : \mathrm{Emb}(M,P) \longrightarrow r$, there exists $\sigma \in \mathrm{Emb}(N,P)$ and $\bar{r} < r$ such that

$$\sigma \circ \mathrm{Emb}(M,N) \subseteq (c^{-1}(\bar{r}))_{\varepsilon}.$$

Alternative Proof of Pestov's Theorem on $\mathbb{U}$

Changing in the proof of the extreme amenability of $\mathrm{Iso}(\mathbb{G})$

1. Ingredient 1 by ingredient $1'$
2. keeping Ingredient 2, and
3. Ingredient 3 by the Ultrahomogeneity of $\mathbb{U}$

we obtain an alternative proof of

Pestov

The group $\mathrm{Iso}(\mathbb{U})$ is extremely amenable.

Some questions

Group	Lévy group	Automatic continuity
$\mathrm{Iso}(\mathbb{U})$	Pestov, 2005	Sabok, 2014
$\mathrm{Iso}_L(\mathbb{G})$	?	?
$\mathrm{Homeo}(Q)$	No	?
$\mathrm{Aut}(\mathbb{P})$	No	?

Gromov and Milman

1. An increasing sequence (G_n) of compact subgroups of G, equipped with their Haar probability measures μ_n, is a Lévy sequence if for every open $V \ni e$ and every sequence $A_n \subseteq K_n$ of measurable sets such that $\liminf_{n\to\infty} \mu_n(A_n) > 0$, we have that $\lim_{n\to\infty} \mu_n(VA_n) = 1$.
2. G is a Lévy group if it has a Lévy sequence of compact subgroups whose union is dense in G.