Strongly aperiodic SFTs on the (discrete) Heisenberg group

(joint work with Ayse Sahin and Ilie Ugarcovici)

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Basic notations:

$\mathcal{A}$ some finite (discrete) **alphabet**

$G = \langle \mathcal{G} | \mathcal{R} \rangle$ a **finitely generated group** with $\mathcal{G} = \{g_1, \ldots, g_k\}$ a set of generators ($k \in \mathbb{N}$)
and $\mathcal{R}$ a set of relators (finite words in $\mathcal{G} \dot\cup \mathcal{G}^{-1}$ equal to the identity)

$\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ (left) **shift action** of G on the full shift $\mathcal{A}^G$

$(g, x) \mapsto \sigma_g(x)$ where $\forall h \in G$: $\big(\sigma_g(x)\big)_h := x_{g^{-1}h}$

G **subshifts**: $X \subseteq \mathcal{A}^G$ shift invariant, closed subset

given by a family of forbidden patterns $\mathcal{F} \subseteq \bigcup_{F \subsetneq G \text{ finite}} \mathcal{A}^F$ on finite shapes such that

$$\mathsf{X}_{\mathcal{F}} := \left\{ x \in \mathcal{A}^G \;\middle|\; \forall\, F \subsetneq G \text{ finite}: \; x|_F \notin \mathcal{F} \right\}$$

X is a G **SFT** $:\Longleftrightarrow \exists\, \mathcal{F} \subseteq \bigcup_{F \subsetneq G \text{ finite}} \mathcal{A}^F$ with $|\mathcal{F}| < \infty$ and $X = \mathsf{X}_{\mathcal{F}}$ (local rules)

X is a **nearest neighbor** G SFT $:\Longleftrightarrow \exists\, \mathcal{F} \subseteq \bigcup_{g \in \mathcal{G}} \mathcal{A}^{\{1,g\}}$ and $X = \mathsf{X}_{\mathcal{F}}$
(constraints along edges in the Cayley graph of G)

Example: The G hard core shift is obtained by $\mathcal{F} := \{11_{(1,g_i)} \mid 1 \leqslant i \leqslant k\}$

Aperiodicity

$X \subseteq \mathcal{A}^G$ a G subshift (closed, shift-invariant)

The **stabilizer** of $x \in X$ under the shift action: $\mathrm{Stab}_\sigma(x) := \{g \in G \mid \sigma_g(x) = x\} \leqslant G$

$x \in X$ is a **weakly periodic** point $:\Longleftrightarrow |\mathrm{Stab}_\sigma(x)| > 1$

$x \in X$ is a **strongly periodic** point $:\Longleftrightarrow [G : \mathrm{Stab}_\sigma(x)] < \infty$ (equivalently $|\mathrm{Orb}_\sigma(x)| < \infty$)

A G subshift X is called

weakly aperiodic $:\Longleftrightarrow \forall\, x \in X:\ [G : \mathrm{Stab}_\sigma(x)] = \infty \ \wedge\ \exists\, x \in X:\ |\mathrm{Stab}_\sigma(x)| = \infty$

strongly aperiodic $:\Longleftrightarrow \forall\, x \in X:\ |\mathrm{Stab}_\sigma(x)| = 1$ (no periodic behavior left at all)

Observation: For $G = \mathbb{Z}^2$, a (weakly) periodic point is already **doubly periodic** (i.e. has two non-colinear periods). Hence a weakly aperiodic $\mathbb{Z}^2$ SFT is already strongly aperiodic.

(The proof uses the pigeon hole principle and works only in co-dimension 1.)

However for $G = \mathbb{Z}^3$ there is already a difference (there exist weakly, not strongly aperiodic $\mathbb{Z}^3$ SFTs, e.g. full-$\mathbb{Z}$-extensions of (strongly) aperiodic $\mathbb{Z}^2$ SFTs).

(Non-)Existence of aperiodic SFTs — Undecidability of the tiling problem

An (incomplete) history of strongly aperiodic SFTs on different groups:

$G = \mathbb{Z}$: Every (non-empty) $\mathbb{Z}$ SFT has periodic points. The emptyness problem is **decidable**.

$G = \mathbb{Z}^2$: **Wang's conjecture** (every non-empty $\mathbb{Z}^2$ SFT contains periodic points) disproved by

60's: Berger (huge alphabet size, 20.000 symbols, case analysis, "computer" proof)

70's: **Robinson** (56 symbols, rigid nearly minimal construction with many interesting properties)

90's: Kari-Culik (13 symbols – smallest so far, nearest neighbor rules, less rigid, positive entropy).

$\Rightarrow$ Existence of those strongly aperiodic $\mathbb{Z}^2$ SFTs implies **undecidability** of the tiling problem.

$G = \mathbb{Z}^3$: Example of **strongly aperiodic** $\mathbb{Z}^3$ SFTs by Kari-Culik $\Rightarrow$ Undecidability.
(using Wang-cubes, $\mathbb{Z}^2$ Kari-Culik example and cellular automata techniques)

$G = \mathbb{H}^2$ (hyperbolic plane): **Undecidability** of tiling problem (Kari, Margenstern)
Examples of **strongly aperiodic** $\mathbb{H}^2$ SFTs (Goodman-Strauss, Kari)

Mozes' construction of strongly aperiodic SFTs on simple Lie groups (non-explicit, using rigidity result for certain Lie groups)

Question: What about (weakly and) strongly aperiodic SFTs on other (non-abelian) groups?
Which groups admit weakly or even strongly aperiodic SFTs?

Theorem [Cohen, 2014]: If the finitely generated group G has at least **two ends**, then there are no strongly aperiodic G-SFTs.

Examples: $\mathbb{Z}$, free groups on finitely many generators

Theorem [Cohen, 2014]: If G, H are two torsion-free, finitely presented groups, which are quasi-isometric, then the existence of a strongly aperiodic G-SFT is **equivalent** to the existence of a strongly aperiodic H-SFT.

Examples: Groups quasi-isometric to $\mathbb{Z}$: No strongly aperiodic SFTs.
Groups quasi-isometric to $\mathbb{Z}^2$ or $\mathbb{Z}^d$ $(d \geqslant 2)$: Strongly aperiodic SFTs.

Theorem [Jeandel, 2015]: If a finitely generated group G admits a strongly aperiodic G-SFT, then G has **decidable word problem**.

The discrete Heisenberg group (and its "powers")

The **discrete Heisenberg group** can be defined as

$$\Gamma := \langle x, y, z \mid xz = zx\,;\ yz = zy\,;\ z = xyx^{-1}y^{-1} \rangle\ .$$

It is isomorphic to the group of **upper-triangular** 3×3-matrices with **integer parameters**

$$\Gamma \cong \left\{ \left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right) \,\middle|\, x, y, z \in \mathbb{Z} \right\}$$

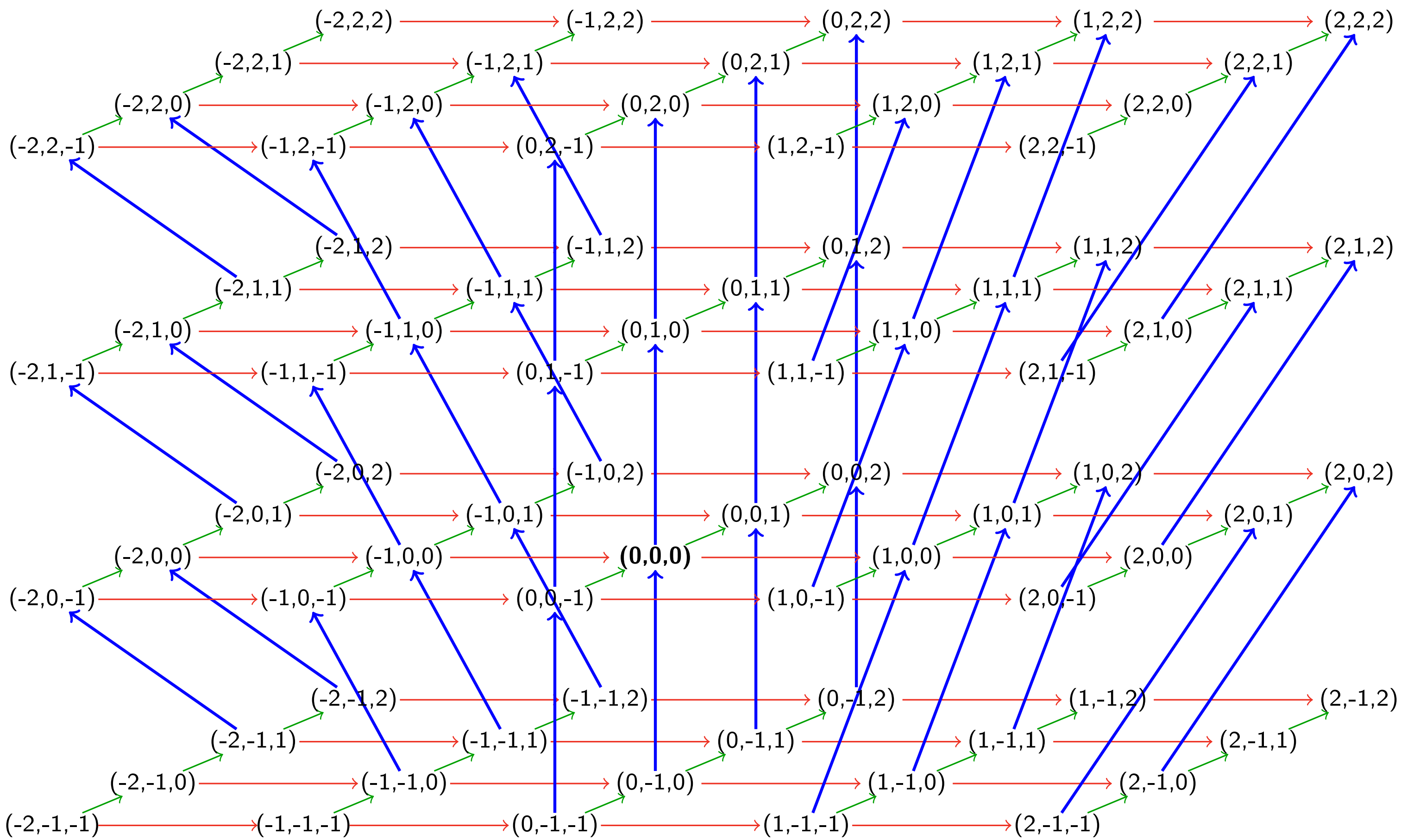
with ordinary (non-abelian) **matrix multiplication**

$$(x, y, z) \cdot (a, b, c) = (x + a\,,\ y + b\,,\ z + c + \underline{xb})\ .$$

The n**th-power** of the discrete Heisenberg group is defined as

$$\Gamma^{(n)} := \langle x, y, z \mid xz = zx\,;\ yz = zy\,;\ z^n = xyx^{-1}y^{-1} \rangle \qquad (n \in \mathbb{N})\ .$$

(Right) Cayley graph of the discrete Heisenberg group Γ



$$\Gamma := \langle x, y, z \mid xz = zx;\ yz = zy;\ z = xyx^{-1}y^{-1} \rangle \qquad \geqslant \Gamma^{(2)}$$ (every other $\langle x, z \rangle$-layer)

(Careful with non-abelian groups: Left shift action needs a right Cayley graph.)

Loosing periodic points in $\Gamma^{(n)}$ SFTs

Note: All powers of the Heisenberg group have two nice normal $\mathbb{Z}^2$ subgroups sitting inside.

Full $\Gamma^{(n)}$ shift: Stabilizer can be "anything". $\Rightarrow$ Lots of different periodic behavior.

Full extension of strongly aperiodic $\mathbb{Z}^2$ SFT: Stabilizers of all points are cyclic groups and have to lie in the complement of the $\mathbb{Z}^2$ subgroup seeing the strongly aperiodic $\mathbb{Z}^2$ SFT.
$\Rightarrow$ weakly aperiodic $\Gamma^{(n)}$ SFTs

Restricted extension of Kari-Culik $\mathbb{Z}^2$ SFT: Stabilizers are cyclic groups and have to lie in a $\mathbb{Z}^2$ subgroup "perpendicular" to the $\mathbb{Z}^2$ subgroup seeing the Kari-Culik SFT.
$\Rightarrow$ weakly aperiodic $\Gamma^{(n)}$ SFT with more restrictive periodic directions

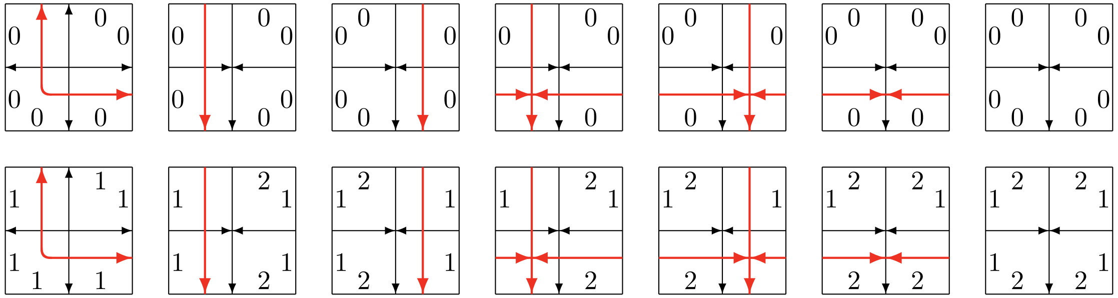
Question (Piantadosi)**:** Are there any strongly aperiodic SFTs on $\Gamma^{(n)}$? (difficulty: shear in $\Gamma^{(n)}$)

Theorem [Sahin, S., Ugarcovici, 2014]: For every $n \in \mathbb{N}$ there exist

- weakly aperiodic $\Gamma^{(n)}$ SFTs,
- weakly aperiodic $\Gamma^{(n)}$ SFTs with restricted set of periodic directions,
- strongly aperiodic $\Gamma^{(n)}$ SFTs. (explicit construction, alphabet size not too small, order of 200)

The $\mathbb{Z}^2$ Robinson SFT (strongly aperiodic)

The alphabet used in the **Robinson tilings** (displayed tiles can still be rotated giving a total of $4 \times 14 = 56$ symbols):



Nearest neighbor SFT rules: Square tiles are placed on $\mathbb{Z}^2$ edge to edge satisfying

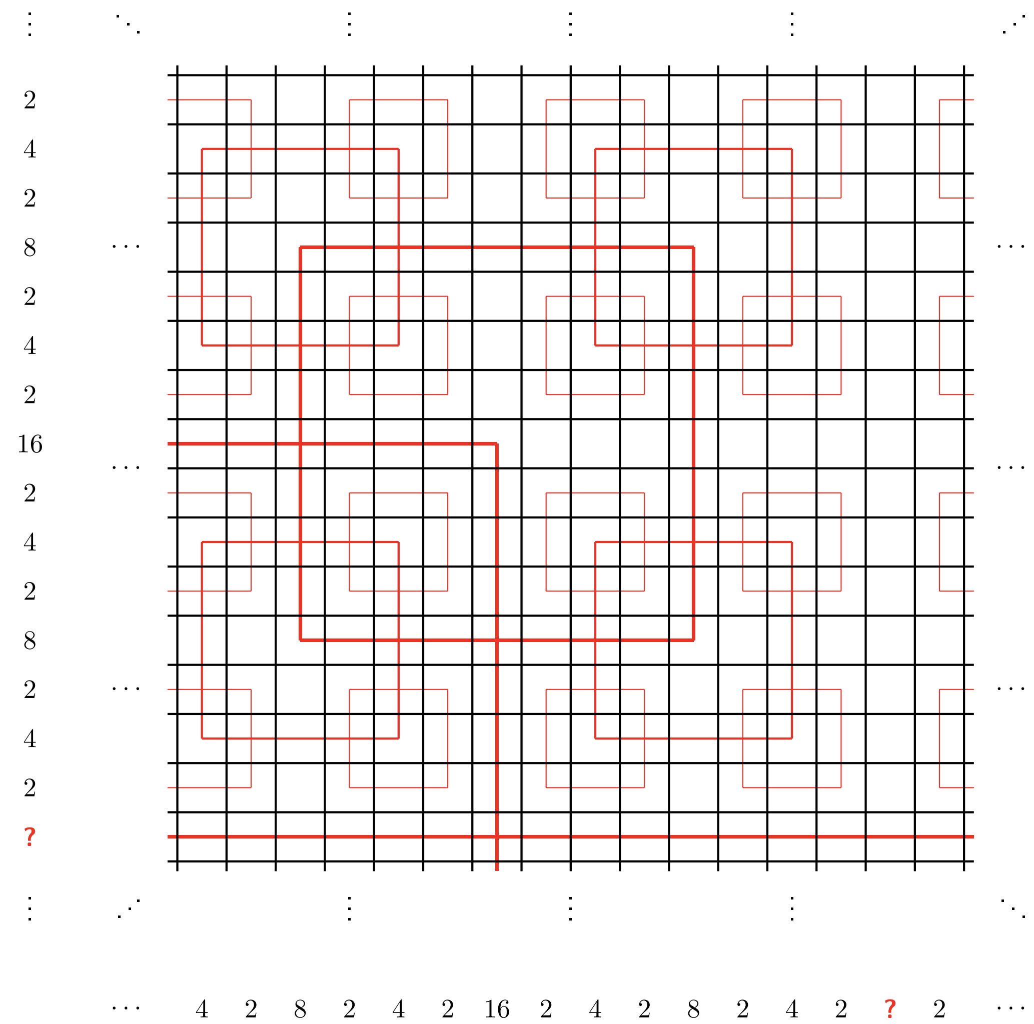
- Digits on either side of an edge have to sum to two.
- Black arrows have to meet head to tail.
- Red arrows have to meet head to tail.

Facts: These rules force

- **hierarchical structure** of nested squares of side length 2^k for $k \in \mathbb{N}$,
- **crosses** appear exactly **at the corners** of those squares,
- crosses in rows (and columns) of a valid tiling appear with **period** 2^k and the sequence of those periods forms a **Toeplitz sequence**

$$\ldots, 2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, \ldots.$$

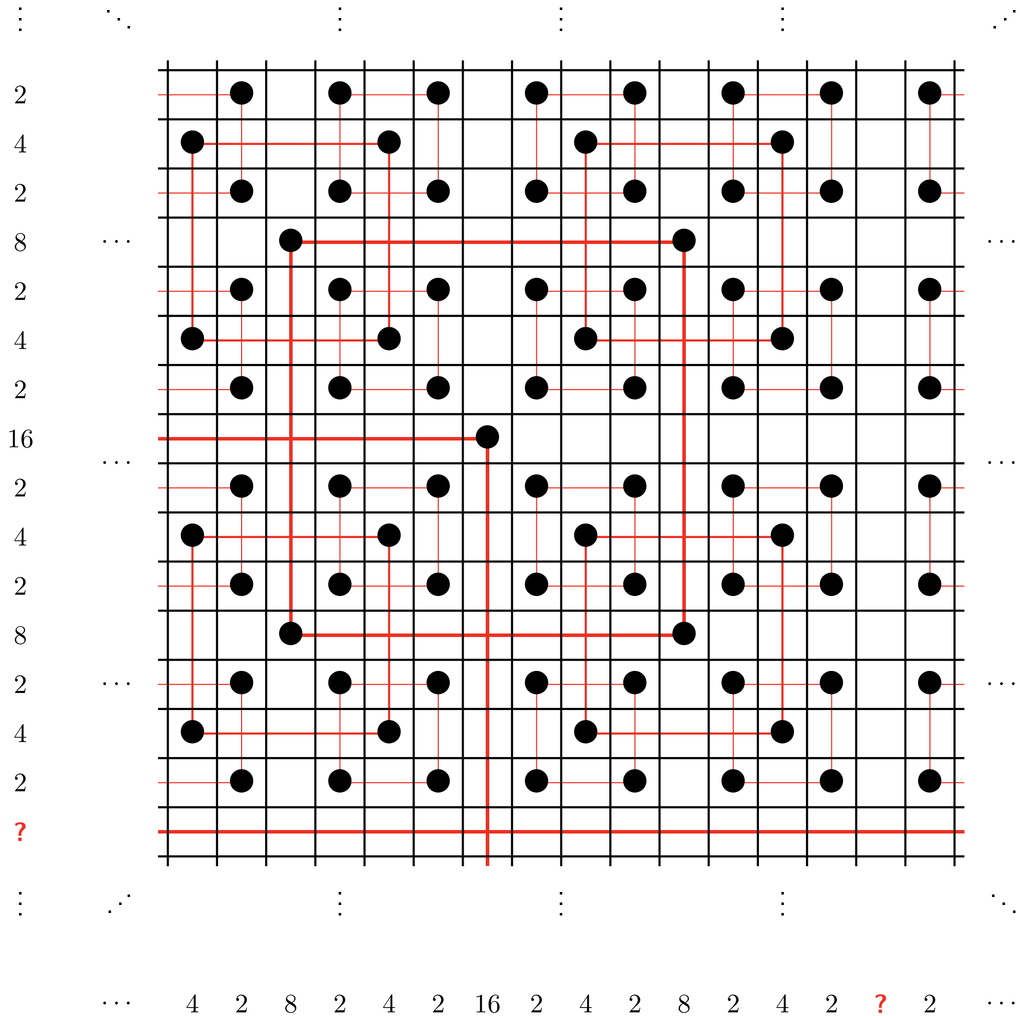
Part of a point in the $\mathbb{Z}^2$ Robinson SFT (note the hierarchical structure)



Period of crosses in rows resp. columns recorded along the left resp. bottom edge.

(regular points vs. (un-)broken exceptional points)

Part of a point in the $\mathbb{Z}^2$ Robinson SFT (note the hierarchical structure)



Period of crosses in rows resp. columns recorded along the left resp. bottom edge.

(regular points vs. (un-)broken exceptional points)

Step 1: Robinson rules in the $\langle x, z\rangle$-layers of $\Gamma^{(2)}$

The $\langle x, z\rangle$-cosets in $\Gamma^{(2)}$ are isomorphic to $\mathbb{Z}^2$.

Force Cayley graph edges in x- **and** z-**direction** to respect the (symmetric) nearest neighbor **Robinson rules** given by the $\mathbb{Z}^2$ Robinson SFT.
$\Rightarrow$ **Every** $\langle x, z\rangle$-**coset sees a Robinson point**.
Hence there are **no periodic directions within the** $\langle x, z\rangle$-**subgroup**.

Clearly we **need some restrictions along the** y-**edges** as well.
(Otherwise we would get periodic behaviour like in a full-extension.)

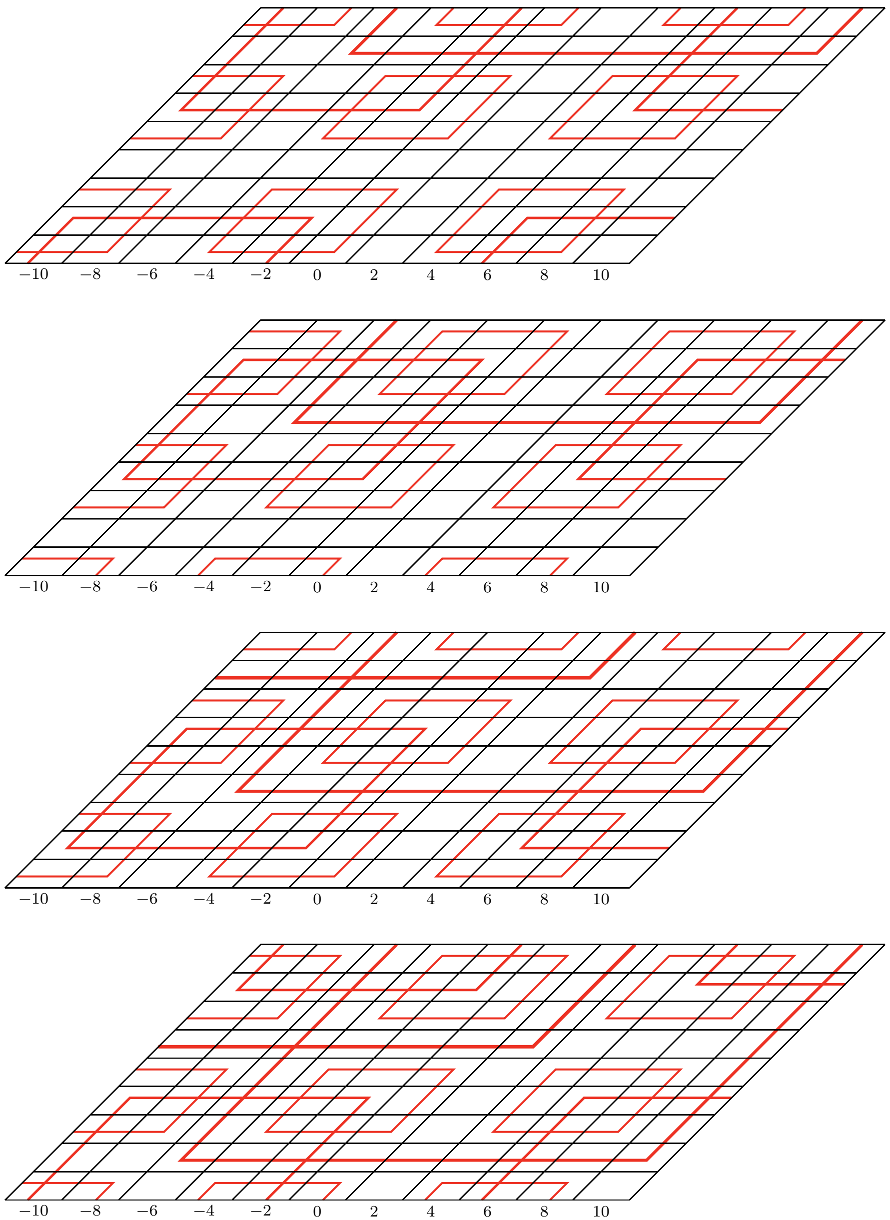
This is the tricky part! (Loose rules allow for periodic points, too rigid rules imply emptyness.)

Force Cayley graph edges in y-**direction** to respect the **"crosses have to propagate"** rule:

If site $g \in \Gamma^{(2)}$ sees a cross, then site gy has to see a cross as well.
If site $g \in \Gamma^{(2)}$ sees an arm, then site gy has to see an arm as well.

$\Rightarrow$ This forces **crosses to be "aligned"**, in particular periods of crosses along rows in the z-direction have to coincide going one layer up/down.

A Robinson point in each $\langle x, z \rangle$-coset of $\Gamma^{(2)}$



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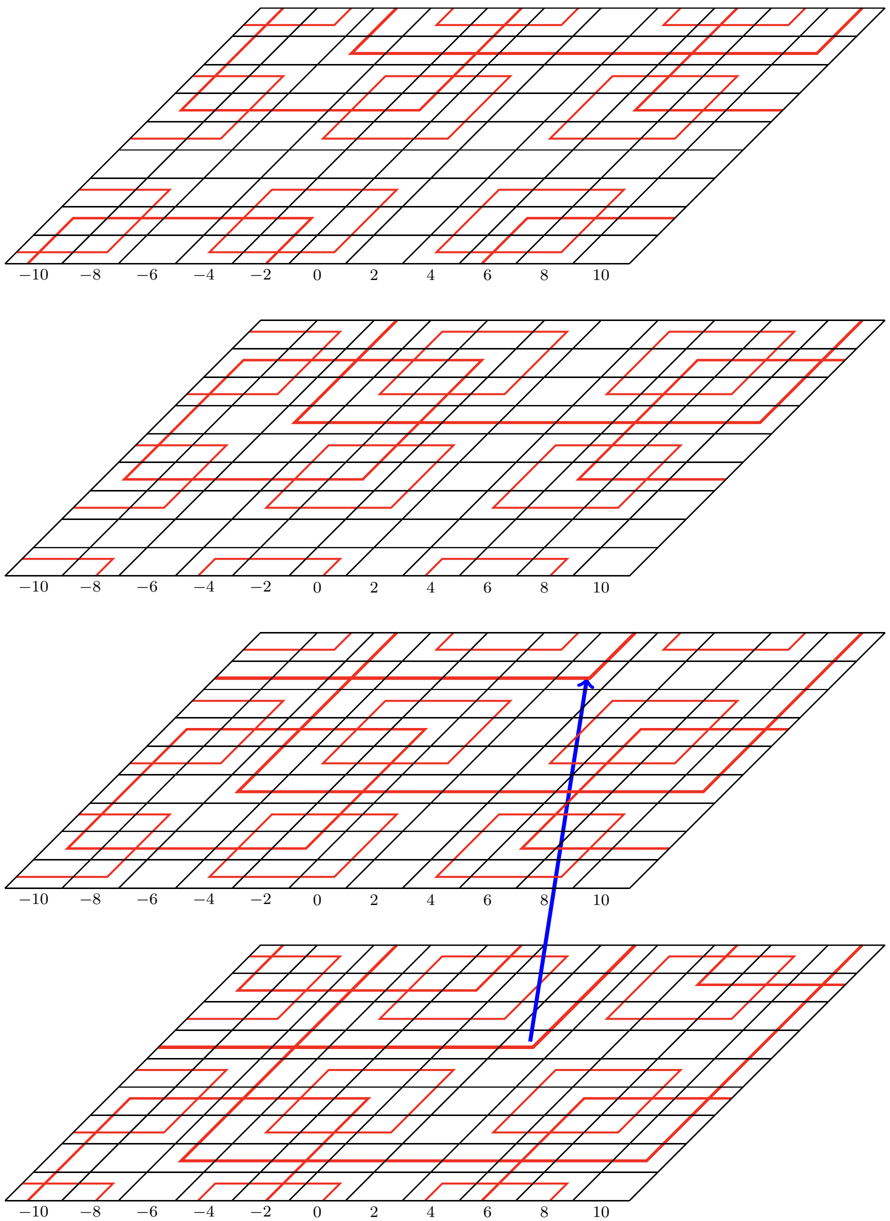
Checking the "crosses have to propagate"-rule by blue y-edges



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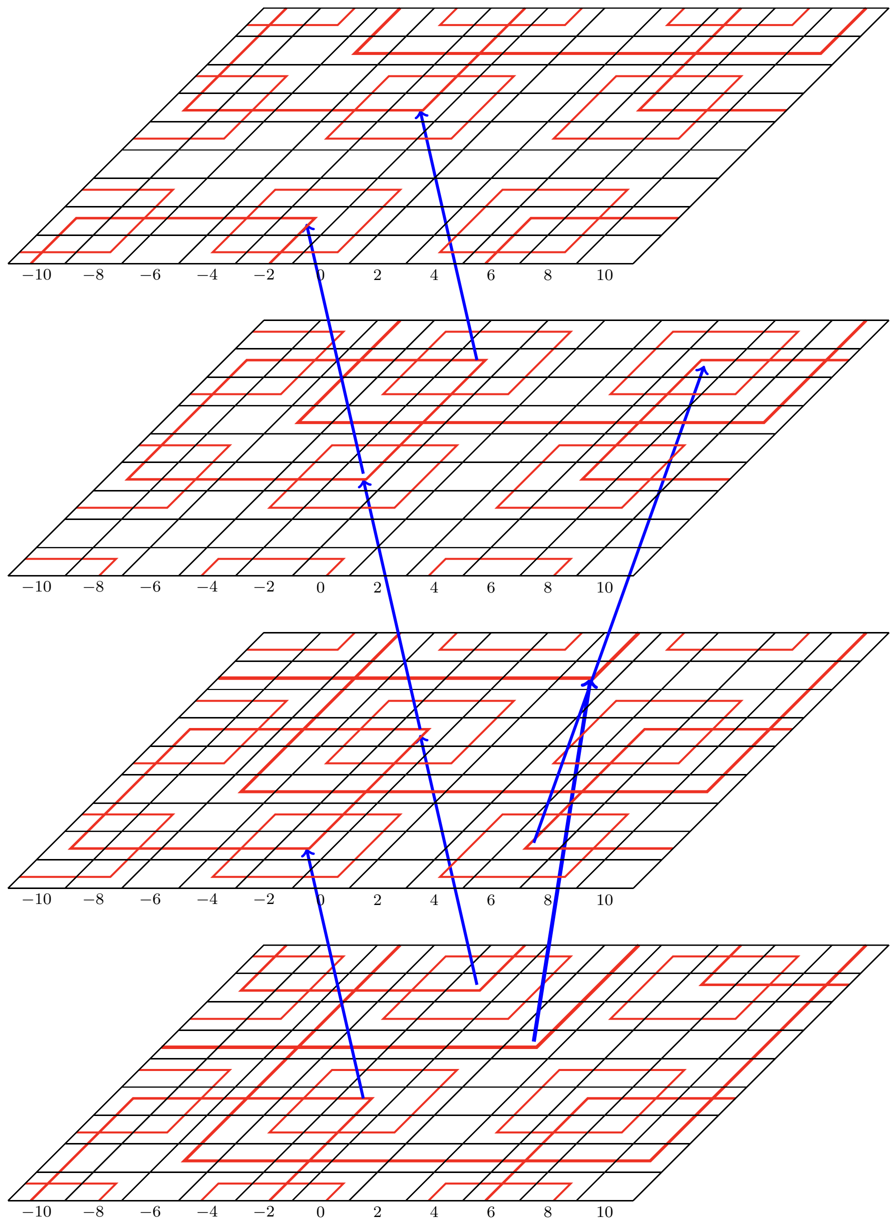

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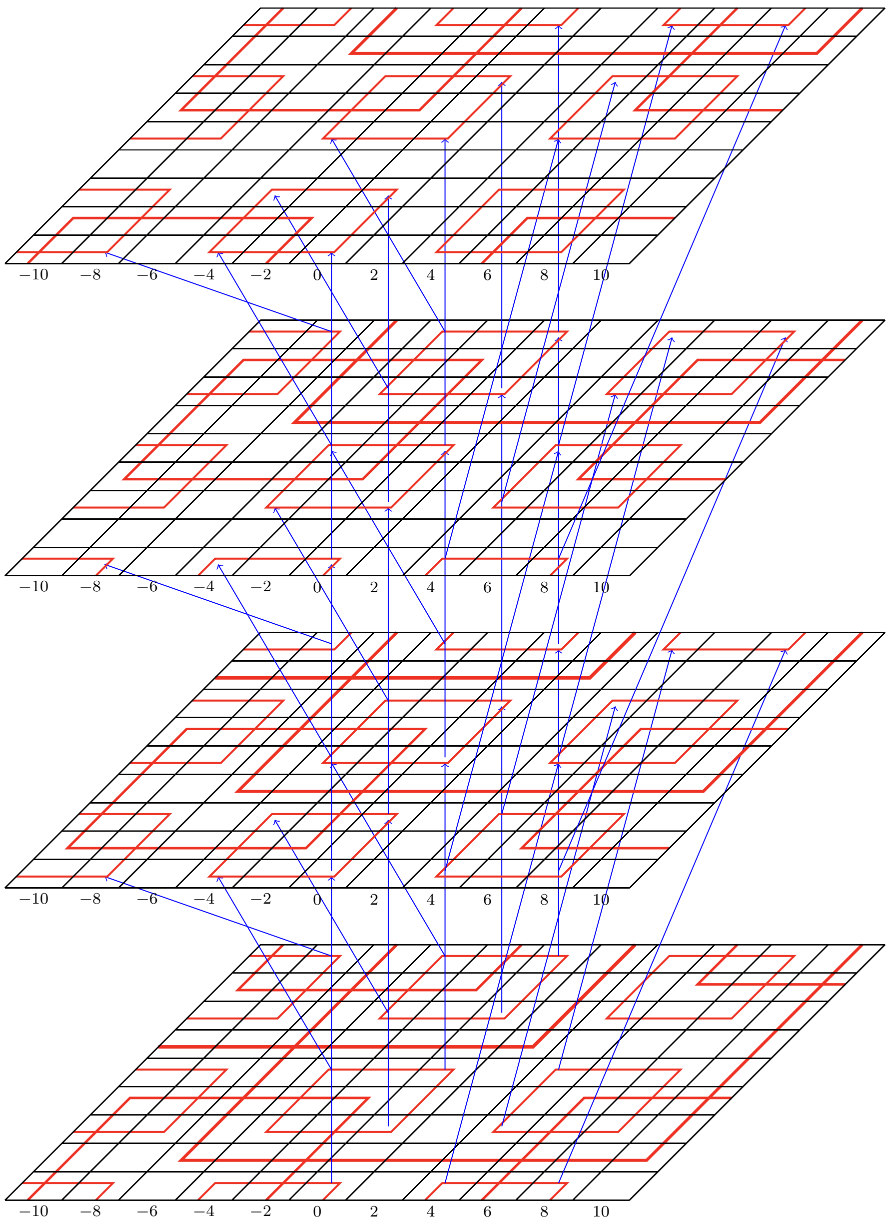

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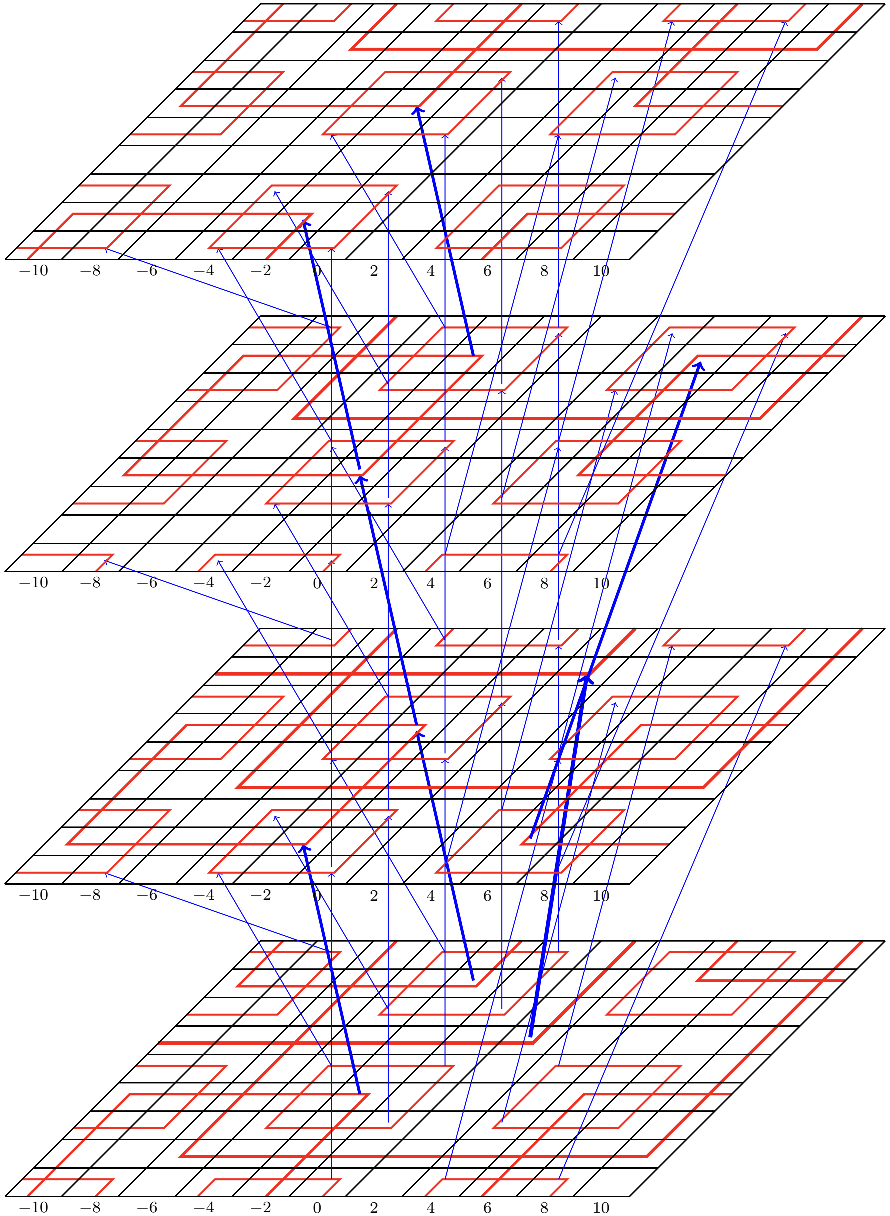


Step 2: **"Coset magic"** implies non-emptyness (check the $\Gamma^{(2)}$ SFT is not empty)

The **shear** in a z-row of $\Gamma^{(2)}$ is an even number and grows linearly (by 2) with the distance from the $(x = 0)$-coset. (Site $g = (x, y, z) \in \Gamma^{(2)}$ is connected via a y-edge to site $gy = (x, y+1, z+2x)$.)

This is were "magically" the **Toeplitz structure of the periods of crosses** fits together with the **shear**. (see previous slide)

This is easily checked for **exceptional Robinson points**, where there is a z-row with a **unique cross** which has to be sheared along from each $\langle x, z\rangle$-coset to the next.
(For regular points we have to use a **compactness argument** looking at larger and larger finite patterns.)
$\Rightarrow$ **Our $\Gamma^{(2)}$ SFT is non-empty.**

Lemma: When the **positions of the cross-symbols** in a $\mathbb{Z}^2$ Robinson point are known, all finite (i.e. complete) **squares can be reconstructed**.

This Lemma **gives us control** about all admissible configurations in our $\Gamma^{(n)}$ SFT:
Seeing a Robinson point in a $\langle x, z\rangle$-coset, the "crosses have to propagate"-rule forces all crosses in the $\langle x, z\rangle$-cosets above and below, thus **determining "uniquely"** the Robinson point in those cosets.
$\Rightarrow$ **Our $\Gamma^{(2)}$ SFT has a rigid structure, but points still have one periodic direction.**

Step 3: **Destroying all remaining (weakly) periodic points**

Now look at the **positions of crosses in a fixed $\langle y, z\rangle$-coset** of our constructed points. They form **"slanted" stripes** of slope $2x$ and width 2^k (for some $k \in \mathbb{N}$).

Using local rules implement **synchronized binary counters** which run in those strips, starting from 0, increasing by $+1$ going to the next $\langle x, z\rangle$-coset above and resetting to 0 after 2^{2^k}-steps.

$\Rightarrow$ As the width of those strips is arbitrarily large in each point, this **breaks periodicity**. (similar to proving aperiodicity of a hierarchical $\mathbb{Z}^2$ SFT)

$\Rightarrow$ The $\Gamma^{(2)}$ SFT with those counters is **strongly aperiodic**.

Step 4: Final adjustment to get a strongly aperiodic $\Gamma^{(n)}$ SFT for each $n \in \mathbb{N}$

Put our strongly aperiodic $\Gamma^{(2)}$ SFT in **any power** of the Heisenberg group (spacing its $\langle x, z\rangle$-coset configurations accordingly and filling in the remaining cosets with symbols that just propagate information).

$\Rightarrow$ **There exist strongly aperiodic $\Gamma^{(n)}$ SFTs for every $n \in \mathbb{N}$.**

Semi-direct products of type $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ (a.k.a. abelian-by-cyclic groups)

Let A be a 2×2 matrix over the non-negative integers.

The **semi-direct product** $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ can be seen as an **HNN-extension**:

$$\mathbb{Z}^2 \rtimes_A \mathbb{Z} = \langle t, a_1, a_2 \,|\, a_1 a_2 = a_2 a_1\,,\ t a_1 t^{-1} = a_1^{A_{1,1}} a_2^{A_{2,1}}\,,\ t a_2 t^{-1} = a_1^{A_{1,2}} a_2^{A_{2,2}} \rangle$$

Note: Still all those groups have $\mathbb{Z}^2$ sitting inside.

Examples:

$A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$: $\mathbb{Z}^2 \rtimes_A \mathbb{Z} \cong \mathbb{Z}^3$ (strongly aperiodic SFTs by Kari-Culik's Wang cubes)

$A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$: $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is the **Flip-group** (doable)

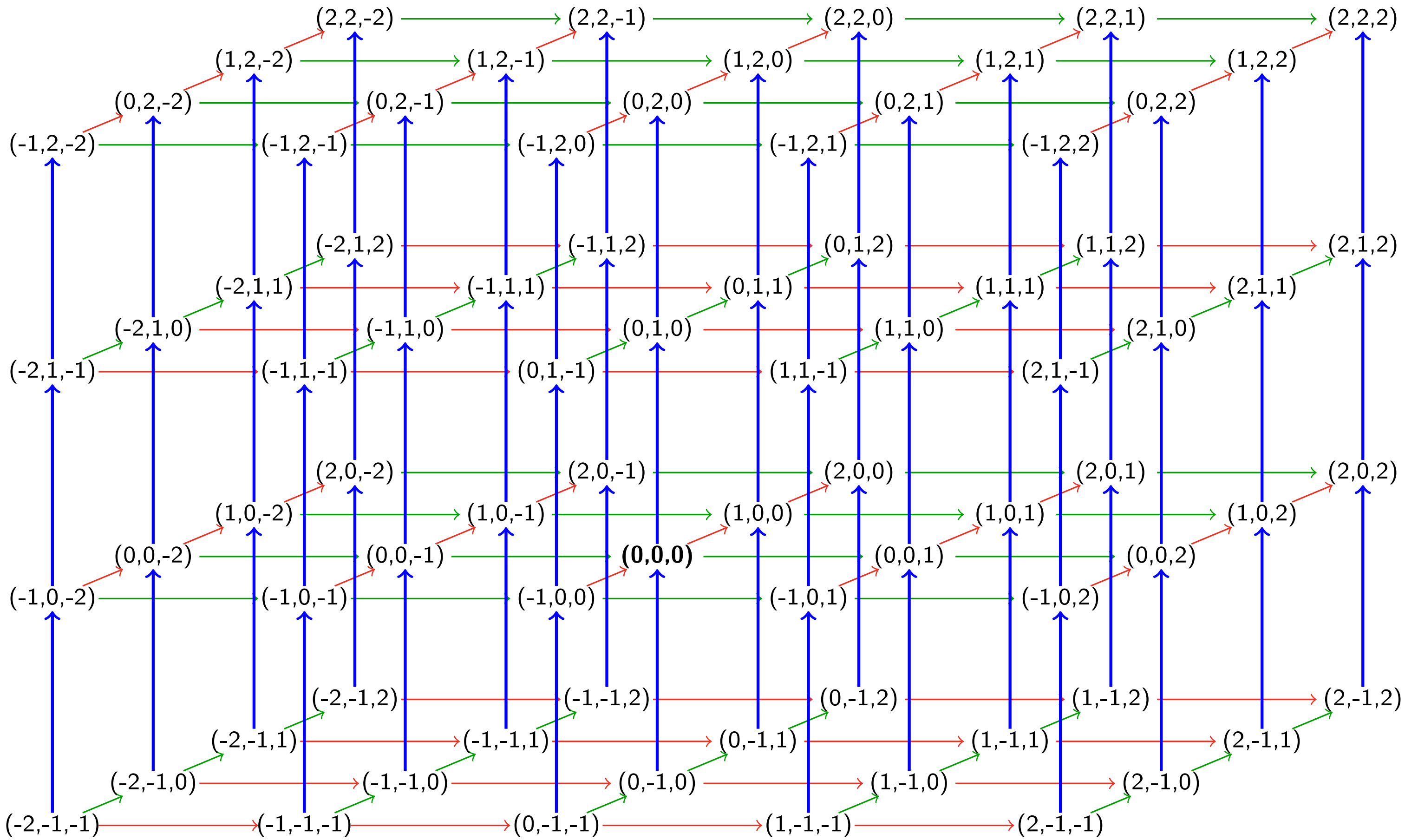
$A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$: $\mathbb{Z}^2 \rtimes_A \mathbb{Z} \cong \Gamma$ is the **discrete Heisenberg group** (doable)

$A = \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$: $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is the n-th power of the Heisenberg group (doable)

$A = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$: $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is a "2-dimensional" Baumslag-Solitar group (open)

$A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$: $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is the **Sol group** (particular solvable Lie group) (open)

(Right) Cayley graph of the Flip-(x, z) group (its square is isomorphic to $\mathbb{Z}^3$)



$$\mathrm{Flip} := \mathbb{Z}^2 \rtimes_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)} \mathbb{Z} \cong \langle x, y, z \mid xz = zx;\ yx = zy;\ yz = xy \rangle$$