Some calculations concerning Talagrand's submeasure

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Theorem (Kalton and Roberts, 1983)

A submeasure μ is uniformly exhaustive if and only if there exists a measure λ that is equivalent to μ.
That is, $\mu(a_n) \to 0$ if and only if $\lambda(a_n) \to 0$, for all sequences $(a_n)_n$.

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    for all sequences $(A_n)_n$.
- Does every exhaustive measure $F : \mathfrak{A} \to X$ admit a control measure?

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- Can something similar be said about Talagrand's construction? (That is, can we take a sledgehammer to it!?)

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- For finite $X \subseteq \mathcal{M}$ where $X = \{(X_1, I_1, w_1), ..., (X_n, I_2, w_n)\}$ we adopt the following notation

$$w(\emptyset) = 0, \quad w(X) = \sum_{i=1}^{n} w_i, \quad \bigcup X = \bigcup_{i=1}^{n} X_i.$$

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- If $\mathcal{C} \subseteq \mathcal{M}$ then the function $\phi_{\mathcal{C}} : \mathfrak{B} \to \mathbb{R}$ defined by

$$\phi_{\mathcal{C}}(B) = \inf\{w(X) : X \subseteq \mathcal{C},\ X \text{ is finite and } B \subseteq \bigcup X\}$$

  is a submeasure (of course we need to see to it that there exists a finite $X \subseteq \mathcal{C}$ such that $\mathcal{T} \subseteq \bigcup X$).

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- Consider the associated submeasure
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- We will consider covers of $\mathcal{T}$ (and $[s]$) that have an easily calculable weight.

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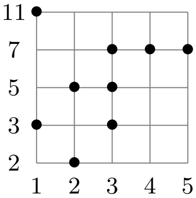

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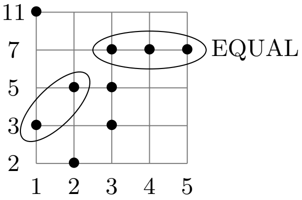

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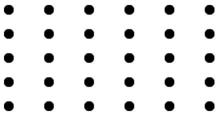

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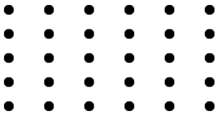

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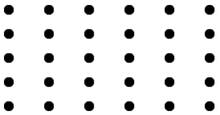

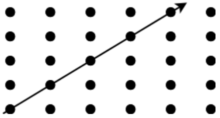

- Then $\bigcup_{i=1}^{6} X_i$, in the shape of a 'rectangle', properly covers $\mathcal{T}$.
- It turns out that this rectangular shape is common to all proper covers of $\mathcal{T}$.

Rectangles

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Lemma

Let $\{(X_i, I_i, w_i) : i \in I\} \subseteq \mathcal{D}$ *be a collection that properly covers* $\mathcal{T}$*. Then*

$$|\bigcup_{i \in I} I_i| \leq |I| - 1.$$

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- Recall that a *complete system of distinct representatives* for $\{I_i : i \in I\}$ (a CDR) is an injective function $F : I \to \bigcup_{i \in I} I_i$ such that $(\forall i \in I)(F(i) \in I_i)$, and that **Hall's marriage theorem** states that a CDR exists if and only if

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- If a CDR exists then $\bigcup_{i \in I} X_i$ will not cover $\mathcal{T}$.

Proof.

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- But then
$$|\bigcup_{i \in J \cup J'} I_i| \leq |J \cup J'| - 1 \text{ and } |J \cup J'| > |J|.$$

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- A similar analyse gives

$$\psi([s]) = \min\{2^{-m+1}, 2^{-m}\left(\frac{\beta(m)}{m}\right)^{\alpha(m)}\},$$

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- I don't know how to adapt these arguments to circumvent Talagrand's induction step(s).

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- The next submeasure to consider is now

$$\phi_{\mathcal{D}\cup\mathcal{E}}(B) = \inf\{w(X) : X \subseteq \mathcal{D} \cup \mathcal{E},\ X \text{ is finite and } B \subseteq \bigcup X\}.$$

  (...sigh).

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Thanks very much for your attention!