Dynamic Asymptotic Dimension

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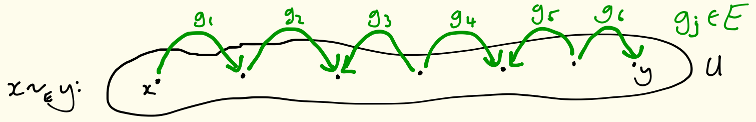

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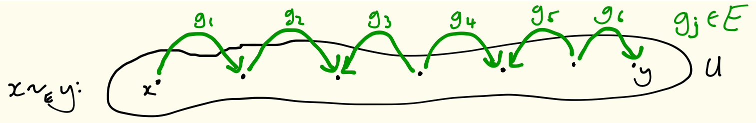

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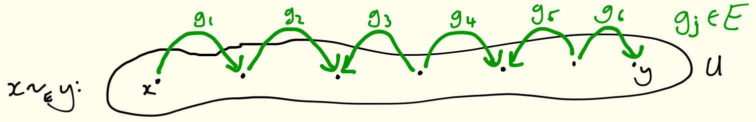

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$$\sup_{x \in U} |[x]_E| < \infty.$$

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Definition

The *dynamic asymptotic dimension* of $G \circlearrowright X$ is the smallest $d \in \mathbb{N}$ with the following property.
For any finite subset $E \subseteq G$, there is an open cover

$$X = U_0 \cup U_1 \cup \cdots \cup U_d$$

of X by sets that are small for E.

1. Examples

2. Small subalgebras

3. Applications - structure and K-theory

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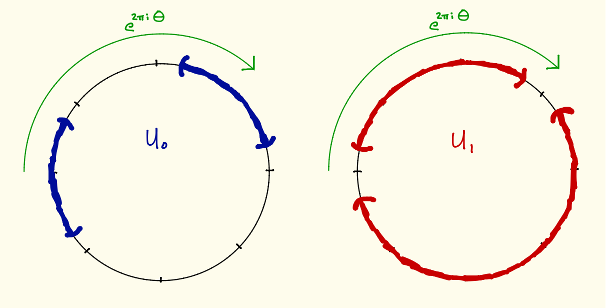

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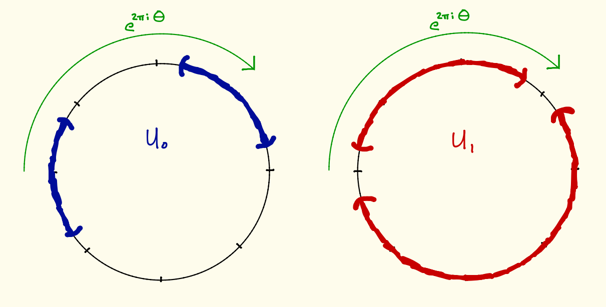

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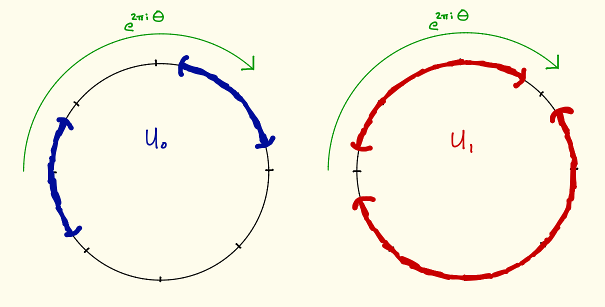

More generally:

Theorem

$\mathbb{Z} \circlearrowright X$ free, minimal action on compact space. Then d.a.d.$(\mathbb{Z} \circlearrowright X) = 1$.

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$$\mathcal{U} = \mathcal{U}_0 \sqcup \cdots \sqcup \mathcal{U}_d$$

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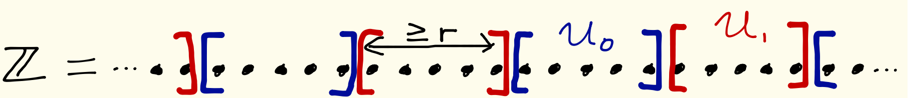

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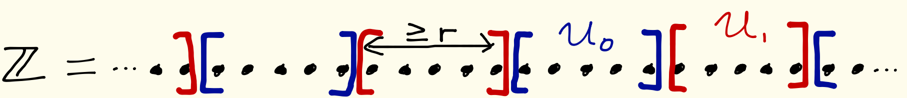

Example: $\mathrm{asdim}(\mathbb{Z}^d) = d$.

Examples of groups with finite asymptotic dimension:

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Corollary (essentially due to Rørdam and Sierakowski)

Any G with finite asymptotic dimension admits a free, minimal action on the Cantor set with finite dynamic asymptotic dimension.

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For $U \overset{\text{open}}{\subseteq} X$ and $E \overset{\text{finite}}{\subseteq} G$ define

$$\begin{aligned} C^*(U; E) :=& C^*(f_1 u_g f_2 \mid f_1, f_2 \in C_0(U), \ g \in E) \\ \subseteq& C(X) \rtimes_r G. \end{aligned}$$

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If U is small for E, then $C^*(U; E)$ is 'nice' (e.g. subhomogeneous, with explicit primitive ideal space...).

Slightly more explicitly, for $U \overset{\text{open}}{\subseteq} X$, define $U^{(m)} := \{x \in U \mid |[x]_E| = m\}$.

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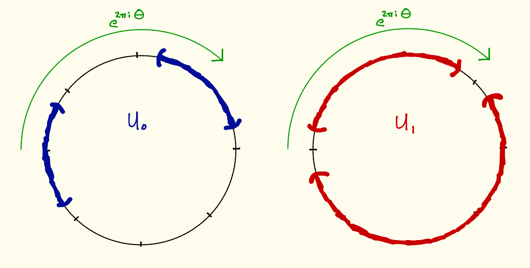

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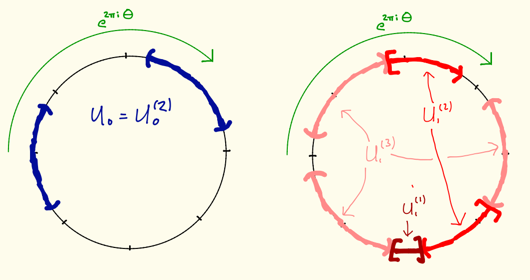

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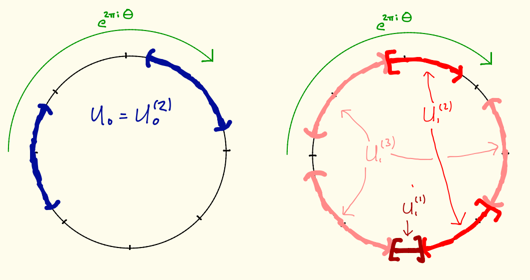

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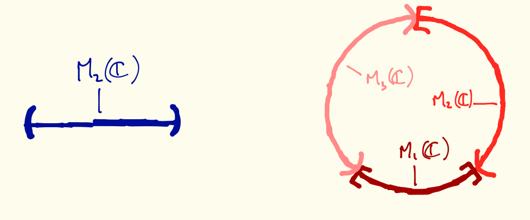

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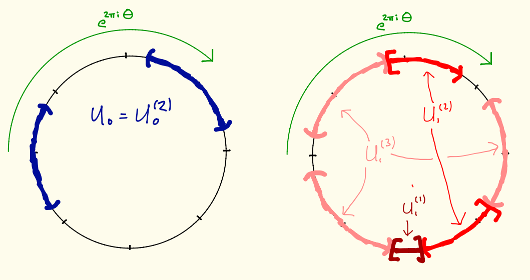

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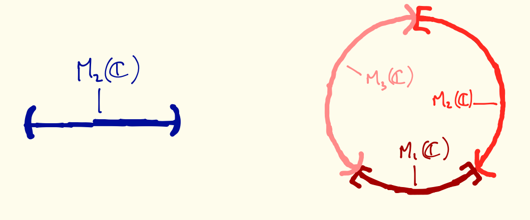

(always look sort of like this).

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The *nuclear dimension* of A is the smallest $d \in \mathbb{N}$ with the following property.

For any $\epsilon > 0$ and $\mathcal{F} \overset{\text{finite}}{\subseteq} A$, there are finite dimensional C^*-algebras $B_0, \ldots, B_d$ and c.c.p. maps

$$A \xrightarrow{\psi_i} B_i \xrightarrow{\phi_i} A$$

such that ϕ_i preserves orthogonality, and such that

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- The nuclear dimension of $C(X)$ equals the covering dimension of X.
- Nuclear dimension has been important in the circle of ideas around Elliott's classification program ...

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$$nucdim(C(X) \rtimes_r G) + 1 \leqslant (dim(X) + 1)(d.a.d.(G \circlearrowright X) + 1)$$

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2. *(Toms-Winter)* $\mathbb{Z} \circlearrowright X$ *free and minimal. Then* $nucdim(C(X) \rtimes_r \mathbb{Z}) \leqslant 2dim(X) + 1$.

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2. *(Toms-Winter) $\mathbb{Z} \curvearrowright X$ free and minimal. Then $nucdim(C(X) \rtimes_r \mathbb{Z}) \leqslant 2dim(X) + 1$.*
3. *Any G admits free and minimal $G \curvearrowright X$, X the Cantor set, with $nucdim(C(X) \rtimes_r G) \leqslant asdim(G)$.*

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'Sort-of' get a Mayer-Vietoris sequence:

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As $C^*(U_0; E)$ etc. have 'computable' K-theory, can compute $K_*(C(X) \rtimes_r G)$.

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... but the technique - using controlled K-theory, and decomposition into almost ideals - is more elementary, and works in the purely algebraic setting ...