Linear systems – the finite dimensional case

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Rio de Janeiro / October 2017

Outline

- Controlled and observed systems
- Stability
- Observability
- Controllablity
- Identifiability in parametric systems
- Compartmental systems
- Model reduction

October 3, 2017

Systems with controls and observer

System (A,B,C,D)

$$
\begin{aligned}
z' &= Az + Bu\\
y &= Cz + Du
\end{aligned}
$$

- $u \in \mathbb{R}^m$: input/control (usuallly constrained to a subset)
- $z \in \mathbb{R}^n$: state
- $y \in \mathbb{R}^k$: output/observation
- t_0 initial time, $z^0 := z(t_0)$ initial state (given or unknown)

$$\text{Control } u \in L_{1,\text{loc}}(\mathbb{R};\mathbb{R}^m) :=$$

$$\{u : \mathbb{R} \longrightarrow \mathbb{R}^m : u_{|[t_a,t_b]} \in L_1([t_a,t_b];\mathbb{R}^m) \text{ for all } t_a, t_b \in \mathbb{R}, t_a \leq t_b\}$$

For selfcontained considerations of controlled systems see for instance

J. Baumeister and A. Leitão
Introdução Teoria de Controle e Programacao Dinamica
IMPA Mathematical Publications, Euclides Project, 2008

Example: damped harmonic oszillator

Consider the differential equation

$$(*) \quad x'' + 2dx' + cx = u$$

of second order with observation $y = x$.
This describes a particle with mass 1 following Newton's law under the outer force $f := u$, the inner force cx and the friction force $r := 2dx'$.

Introducing new variables $x, v := x'$ this system can be reformulated as a system (A, B, C) as follows:

$$(**) \quad z' = Az + Bu\,,\; y = Cz$$

with

$$z = \begin{pmatrix} x \\ v \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 \\ -c & -2d \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Stability

Stability is a property which depends on the matrix A only.
Therefore, we consider just the equation for the state:

$$z' \;=\; Az$$

$$\text{State} \qquad z(t) = \exp(A(t - t_0))z^0\,,\; t \geq t_0$$

This can easily be verified. Clearly one must know the definition of a matrix exponential.

Stability of A or of the system above is the question concerning the long-time behavior of solutions. Engineers are interested in answers to this question (You may consider the harmonic oszillator as a model for damper in a car operating with a spring and a hydraulic damping).
A special type of long-time behavior shows a solution z which is at rest:

$$z(t) = z(t_0) \text{ for all } t \geq t_0\,.$$

This implies $z'(t) = \theta$ for all $t \geq t_0$ and this implies

$$Az^0 = \theta$$

Stability-1

Consider the more general system

$$z' \quad = \quad f(z)$$

Definition

(1) *Each point $\overline{z}$ with $f(\overline{z}) = \theta$ is called a* **critical point**

(2) *A critical point $\overline{z}$ is* **stable** *if every solution which starts nearby of $\overline{z}$ stays nearby.*

(3) *A critical point $\overline{z}$ is* **asymptotically stable** *if it is stable and if every solution which starts nearby of $\overline{z}$ converges for $t \to \infty$ to $\overline{z}$.*

Fact

Consider the linear system $z' = Az$ and $\overline{z} = \theta$. Then

- *$\overline{z}$ is stable if $Re(\lambda) \leq 0$ for every eigenvalue λ of A.*
- *$\overline{z}$ is asymptotically stable if $Re(\lambda) < 0$ for every eigenvalue λ of A. The converse does not hold!*

Example: damped harmonic oszillator-1

The eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2d \end{pmatrix}$$

are given as

$$\lambda_\pm = -d \pm \sqrt{d^2 - c}$$

Clearly, this implies that the harmonic oscillator is asymptotical stable if $d > 0$.
Set $\omega := \sqrt{|c - d^2|}$. Then we have for the general solution x of the damped harmonic oszillator (a_1, a_2 are free „amplitudes“):

- $d = c$ (**critical damping**) $x(t) = \exp(-dt)(a_1 + a_2 t)$.
- $d < c$ (**underdamped case**)
  $x(t) = \exp(-dt)(a_1 \cos(\omega t) + a_2 \sin(\omega t))$.
- $d > c$ (**overdamped case**)
  $x(t) = \exp(-dt)(a_1 \exp(-\omega t) + a_2 \exp(\omega t))$.

Observability

$$\begin{aligned} z' &= Az \\ y &= Cz \end{aligned}$$

$$\begin{array}{rl} \text{State} & z(t) = \exp(A(t-t_0))z^0\,,\ t \geq t_0 \\ \text{Observation} & y(t) = C\exp(A(t-t_0))z^0\,,\ t \geq t_0 \end{array}$$

$$y(t_0) = Cz^0, y'(t_0) = Cx'(t_0) = CAz^0\,, \dots, y^{(n-1)}(t_0) = CA^{n-1}z^0\,.$$

Equivalent conditions: (Caley-Hamilton Theorem!)

- The system (A, Θ, C) is observable, i.e. the initial state z^0 can be determined uniquely from the observation y.
- The observability matrix $O(A,C)$ has full rank, i.e. $\mathrm{rank}(O(A,C)) = n$ where

$$O(A,C)^t := \begin{pmatrix} C & CA & \cdots & CA^{n-1} \end{pmatrix}\,.$$

Example: damped harmonic oszillator-2

In this example we have

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2d \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and therefore

$$O(A, C) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and we conclude:

The damped harmonic oszillator is observable.

Controllability

$$z' = Az + Bu$$

State $$z(t) = \exp(A(t-t_0))z^0 + \int_{t_0}^{t} \exp(A(t-s))Bu(s)\,ds$$
$$t \in [t_0, \infty)\,.$$

Controllability means that for each $t_1 > t_0$ and $z^1 \in \mathbb{R}^n$ there exists a control $u \in L_1(t_0, t_1; \mathbb{R}^m)$ with $z(t_1) = z^1$, i.e.

$$z(t_1) = z^1 = \exp(A(t_1 - t_0))z^0 + \int_{t_0}^{t_1} \exp(A(t_1 - s))Bu(s)\,ds\,,$$

or

Controllability-1

$$\exp(-At_1)z(t_1) - \exp(-At_0)z^0 = \int_{t_0}^{t_1} \exp(-As)Bu(s)\,ds\,,$$

$$\exp(-At_1)z(t_1) - \exp(-At_0)z^0 = C(A,B)\begin{pmatrix} \int_{t_0}^{t_1} \alpha_0(s)u(s)\,ds \\ \vdots \\ \int_{t_0}^{t_1} \alpha_{n-1}(s)u(s)\,ds \end{pmatrix}\,,$$

where

$$C(A,B) := \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

is the so called **controllability matrix** of the system (A,B).

Controllability and minimal systems

Fact

Consider the system (A, B). Then the following conditions are equivalent:

- *The system (A, B) is controllable, i.e. for each $t_1 > t_0$ and each $z^1 \in \mathbb{R}^n$ there exists $u \in L_1(t_0, t_1; \mathbb{R}^m)$ such that the solution z satisfies $z(t_1) = z^1$.*
- *The observability matrix $C(A, B)$ has full rank, i.e. $rank(C(A, B)) = n$.*

Definition

A system (A, B, C) is **minimal** *iff it is both controllable and observable.*

Example: damped harmonic oszillator-3

In this example we have

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2d \end{pmatrix} ,\; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and therefore

$$C(A,B) = \begin{pmatrix} 1 & 0 \\ 0 & -c \end{pmatrix}$$

and we conclude:

The damped harmonic oszillator is controllable if $c \neq 0$.

Moreover, after all:

The damped harmonic oszillator is a minimal system if $c \neq 0$.

Minimal systems

Definition

A real square matrix is called a **Hurwitz matrix** *if all its eigenvalues have negative real part.*

Fact

Let (A, B, C) be minimal and let A be a Hurwitz matrix. Then

$$(*) \;\; AW_C + W_C A^t = -BB^t \,,\; W_O A^t + AW_O = -C^t C$$

where

$$\begin{aligned} W_C &:= \int_0^\infty \exp(As) BB \exp(A^t s)\, ds \\ W_O &:= \int_0^\infty \exp(A^t s) C^t C \exp(As)\, ds \end{aligned}$$

Moreover, W_C, W_C are symmetric and positive definite matrices.

Proof:
If (A, B, C) is minimal and if A is a Hurwitz matrix then the gramians W_C, W_O exist. We conclude the identity $(*)$ for W_C as follows, the argumentation for W_O is the same. The integrals exists due the assumption that A is a Hurwitz matrix. Moreover we have

$$AW_C + W_C A^t = \int_0^\infty \frac{d}{dt}(\exp(As)BB^t \exp(A^t s))\, ds = -BB^t\,.$$

Clearly, W_C, W_C are symmetric and positive definite matrices.

Parametric systems

System $(A(p), B(p), C(p))$

$$\begin{aligned} z' &= A(p)z + B(p)u \\ y &= C(p)z \end{aligned}$$

$p \in \mathcal{P}_{\mathrm{ad}}$ (unknown) parameter, $u(t) \in \Omega$, Ω open neighborhood of θ .

Observation $y(t) = C\exp(A(p)t)z^0 + \int_0^t C\exp(A(p)(t-s))B(p)u(s)\,ds$

Definition

$p \in \mathcal{P}_{ad}$ *is* **identifiable by experiments** *in* $[0, T]$ *if*

$$C\exp(A(p)t)z^0 + \int_0^t C\exp(A(p)(t-s))B(p)u(s)\,ds =$$

$$= C\exp(A(q)t)z^0 + \int_0^t C\exp(A(q)(t-s))B(q)u(s)\,ds, t \in [0, T],$$

for all admissible controls u *implies* $p = q$.

Parametric systems-1

Suppose $z^0 = \theta$ and let $p, q \in \mathcal{P}_{\text{ad}}$. Since each control may take values in an open set we are lead to condition

$$C \exp(A(p)t)B(p) = C \exp(A(q)t)B(q), t \in [0, T] \,.$$

Therefore, we should consider for $q \in \mathcal{P}_{\text{ad}}$ the quantities

$$Y_j(q) := C(q)A(q)^j B(q) \,, \, j = 0, 1, \ldots \,.$$

Lemma

$z^0 = \theta,\ p, q \in \mathcal{P}_{ad}$. *Equivalent conditions:*

(1) $Y_j(p) = Y_j(q) \,, \, j = 0, 1, \ldots \,.$

(2) $Y_j(p) = Y_j(q) \,, \, j = 0, 1, \ldots, 2n - 1 \,.$

Proof:

The Caley-Hamilton Theorem plays the important role. For each $q \in \mathcal{P}_{\text{ad}}$ we obtain numbers $\alpha_0(q), \ldots, \alpha_{n-1}(q)$ with

$$A(q)^n = \sum_{i=0}^{n-1} \alpha_i(q) A(q)^i \,.$$

For a complete proof see

J. Baumeister
Stable solution of inverse problems
Vieweg, 1987

Theorem

$z^0 = \theta$, $p \in \mathcal{P}_{ad}$. Equivalent conditions:

(1) *p is identifiably.*

(2) *For all $q \in \mathcal{P}_{ad}$ there exists $j \in \{0, \ldots, 2n-1\}$ with $Y_j(q) \neq Y_j(p)$.*

Parametric system – Example

$$
\begin{aligned}
z_1' &= -(p_1+p_2)z_1 + p_2 z_2 + u, z_1(0) = 0,\\
z_2' &= p_2 z_1 - p_3 z_2, z_2(0) = 0,\\
y &= z_1
\end{aligned}
$$

$$
\mathcal{P}_{\mathrm{ad}} := \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 : p_i \geq 0, i = 1, 2, 3\}\,.
$$

$$
\begin{aligned}
Y_0(p) &= 1\,,\ Y_1(p) \ = \ -(p_1+p_2)\\
Y_2(p) &= (p_1+p_2)^2 + p_2^2\\
Y_3(p) &= -(p_1+p_2)(p_1^2 + 2p_1p_2 + 3p_2^2) - p_2^2 p_3
\end{aligned}
$$

Result Each $p \in \mathcal{P}_{\mathrm{ad}}$ with $p_2 > 0$ is identifiable.

In the general case: Apply symbolic computation!

Compartmental systems

Compartment model: Describes a number of compartments, each containing distinct, well mixed material. Compartments exchange material with each other following certain rules.

Applications Biology, medicine, physiology, ...

D.H. Anderson
Compartmental modelling and tracer kinetics
Notes Biomathematics, 1983

J.A. Jacquez
Compartmental Analysis in Biology and Medicine
University of Michigan Press, 1996,

K.W. Little
Environmental Fate and Transport Analysis with Compartmental Modelling
CRC Press, 2012

G.G. Walter
Compartmental modeling with networks
Springer, 1999

Compartmental systems-1

Modelling

- Compartments $C_0, C_1, \dots, C_n$
- C_0 outer enviroment/external world
- f_{ij} flow rate from compartment i to compartment j
- $z_i(t)$ quantity of material in the ith compartment at time t
- $z(t) := (z_1(t), \dots, z_n(t))$ state of the system at time t

Rule $z_i' =$ rate of inflow - rate of outflow
Compartmental equations

$$z_i' = \sum_{j=1, j\neq i}^{n} f_{ij} - \sum_{j=0, j\neq i}^{n} f_{ij} + v_i \, , \; i = 1, \dots, n \, .$$

$v_i(t)$ input to the ith compartment from the outer enviroment.
Fractional transfer coefficients: $p_{ij} := f_{ij} z_j^{-1}$

Compartmental systems-2

System $z' = A(p)z + v$
Observation $y = Cz$

Inverse problem in compartmental theory

Given the observation y
Determine each coefficient p_{ij} *of the matrix* A

Model reduction

Model order reduction reduction tries to capture the essential features of a structure/model in a smaller approximation.

Pioneering steps and tools:
Fourier approximation, Lancos and Arnoldi in matrix theory, finite elements, wavelets

Here the focus is on finite dimensional models.

W.H. Schilders and H.A. van der Vorst and J. Rommes
Model reduction: Theory, research aspects and applications
Springer 2008

A.C. Antoulas
Approximation of large-scale dynamical systems
SIAM 2005

Model reduction-1

Model:

$$(P)\ \begin{cases} z' &= f(z,u) \\ y &= g(z,u) \end{cases}$$

$z \in \mathbb{R}^n$ state, $u \in U \subset \mathbb{R}^m$ control, $y(t) \in \mathbb{R}^k$ observation, f, g given.

Model reduction: Find

$$(P_r)\ \begin{cases} x_r' &= f_r(x_r,u) \\ y_r &= g_r(x_r,u) \end{cases}$$

where $x(t) \in \mathbb{R}^r, u(t) \in \mathbb{R}^m, y_r \in \mathbb{R}^l, f_r, g_r$ given and $r \leq n, l \leq k$. We consider controls with a time horizon T, i.e. the system is considered in the time interval $[0, T]$.

Model reduction-2

Goal:
Find (P_r) in such a way that the following requirement is met:
There is a bound κ_r (complexity-misfit) such that

$$\|y - y_r\| \leq \kappa_r \|u\|, u \in U, \text{ and } \kappa_r \uparrow 0 \text{ if } r \uparrow n\,.$$

r, l are the crucial numbers.

Here, we assumed that the space of controls is endowed by a norm.

The misfit is due to the approximation order of the reducedmodel. There are many reduction methods. We sketch below a method for linear systems.

Model reduction-3

Consider the following systems:

Model: $M := \left(\begin{array}{c|c} A & B \\ \hline C & \Theta \end{array}\right)$ Reduced model: $M_r := \left(\begin{array}{c|c} A_r & B_r \\ \hline C_r & \Theta \end{array}\right)$

What is a method to come from (A, B, C) to (A_r, B_r, C_r), and what are the features which should preserved, which goals should achieved? We sketch only the steps which lead to a decomposition.

Model reduction-4

Fact

If (A, B, C) is minimal and if A is a Hurwitz matrix then there exists a diagonal matrix G such that

$$AG + GA^t = -BB^t \,,\; A^t G + GA = -C^t C \,,\; W_C = W_O = G\,.$$

Let

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq \sigma_{r+1} \geq \cdots \geq \sigma_n$$

be the singular values of G where we assume that $\sigma_r \gg \sigma_{r+1}$.
Then we decompose A, B, C as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} ,\; B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} ,\; C = \begin{pmatrix} C_1 & C_2 \end{pmatrix} .$$

Now the reduced form is found as

$$M_r := \left(\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & \Theta \end{array} \right)$$