Parametric partial differential equations

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Outline

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- Parabolic equation
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Numerical differentiation

(Numerical) differentiation of data will play a role twice in this part.

Forward/Direct Problem:
With a continuous function $x : [0,1] \longrightarrow \mathbb{R}$ *compute*

$$y(t) := \int_0^t x(s)ds\,,\ t \in [0,1]\,.$$

Inverse Problem: *Given a function* $y : [0,1] \longrightarrow \mathbb{R}$ *determine* $x := y'$.

Suppose that instead of the continuous function $y : [0,1] \longrightarrow \mathbb{R}$ a „measured“ function $y^\varepsilon : [0,1] \longrightarrow \mathbb{R}$ is available only. We assume:

$$|y^\varepsilon(t) - y(t)| \le \varepsilon \text{ for all } t \in [0,1]\,.$$

It is reasonable to try to reconstruct the derivative $x := y'$ of y at $\tau \in (0,1)$ by

$$x^{\varepsilon,h}(\tau) := D_h y^\varepsilon(\tau) := \frac{y^\varepsilon(\tau+h) - y^\varepsilon(\tau)}{h}$$

Numerical differentiation-1

We obtain

$$\begin{aligned}|x^{\varepsilon,h}(\tau) - x(\tau)| \quad &\leq \quad \Big|\frac{y(\tau+h)-y(\tau)}{h} - x(\tau)\Big| \\ &\quad + \Big|\frac{(y^\varepsilon - y)(\tau+h) - (y^\varepsilon - y)(\tau)}{h}\Big|\,.\end{aligned}$$

Under the assumption that the unknown solution x is continuously differentiable we have

$$\frac{y(\tau+h)-y(\tau)}{h} - x(\tau) = \frac{1}{2}y''(\eta)h \quad \text{for some } \eta \in [0,1]\,.$$

When we know a bound (a priori bound/source condition)

$$|x'(t)| \leq E \text{ for all } t \in [0,1]\,,$$

(actually $|x'(t)| \leq E$ in a neighborhood of τ) then we conclude

$$|x^{\varepsilon,h}(\tau) - x(\tau)| \leq \frac{1}{2}hE + 2\frac{\varepsilon}{h}$$

Numerical differentiation-1

Choose

$$h := h_{\text{opt}} := 2\sqrt{\frac{\varepsilon}{E}}$$

Then

$$|x^{\varepsilon,h(\varepsilon)}(\tau) - x(\tau)| \leq 2\sqrt{E\,\varepsilon}\,.$$

The choice of h_{opt} is in contrast to the well posed case. Here one should choose h as small as possible.

If the imprecise is preciser Kirchgraber, Kirsch, Stoffer, 2001

The result above means that the inverse of integration is Hölder continuous with exponent $\frac{1}{2}$.

- How to improve this result?
- Use symmetric differential quotients

Numerical differentiation-2

$$x^{\varepsilon,h}(\tau) := D_h^s y^\varepsilon(\tau) := \frac{y^\varepsilon(\tau+h) - y^\varepsilon(\tau-h)}{2h}$$

where the discretization parameter $h \neq 0$ has to be chosen such that $\tau \pm h \in [0,1]$. Then under the assumption that the unknown solution x is twice continuously differentiable we have

$$\frac{y(\tau+h) - y(\tau-h)}{2h} - x(\tau) = \frac{1}{12} y'''(\eta) h^2 \quad \text{for some } \eta \in [0,1]\,.$$

When we know a bound (a priori bound)

$$|x''(t)| \leq E \text{ for all } t \in [0,1]\,,$$

then we conclude

$$|x^{\varepsilon,h}(\tau) - x(\tau)| \leq \frac{1}{12} h^2 E + \frac{\varepsilon}{h}$$

and we obtain in an analogous manner to the above argumentation

$$|x^{\varepsilon,h(\varepsilon)}(\tau) - x(\tau)| \leq E^{\frac{1}{3}} \varepsilon^{\frac{2}{3}}\,.$$

Numerical differentiation-3

The conditions

$$x \text{ continuosly differentiable and } |x'(t)| \le E, t \in [0,1]\,,$$

$$x \text{ twice continuosly differentiable and } |x''(t)| \le E, t \in [0,1]\,,$$

are **source-type conditions**, respectively. But this can be made rigourous in Hilbert space only. We sketch the idea.

$$A : L_2[0,1] \longrightarrow L_2[0,1]\,,\; A^* : L_2[0,1] \longrightarrow L_2[0,1]$$

with

$$A^*(w)(s) = -\int_s^1 w(r)\,dr\,,\; s \in [0,1]\,,$$

Therefore

$$x = A^*(w) \text{ or } x = A^*A(w)$$

implies the differentiability conditions in the a priori bound and a requirement

$$\|w\|_{L_2[0,1]} \le E$$

completes the a priori bound above (but in the Hilbert space setting).

Parabolic equation

The heat equation in the open subset Ω of $\mathbb{R}^d$:

$$D_t u - \Delta u = f$$

$L := \Delta$ is the Laplace operator. More general:

$$D_t u + Lu = f \qquad (\#)$$

where L is a partial differential operator in divergence form:

$$Lu = -\sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u) - \sum_{j=1}^{d} \partial_j(b^j u) + cu$$

L is called here the *abc*-operator of $(\#)$ and $a = (a^{ij}), b = (b^j), c$ its coefficients which may depend on the space variable $\xi \in \Omega \subset \mathbb{R}^d$ and time $t \in (0, T)$; $L = L_{a,b,c}$. D_t is the time derivative (in the classical sense).

The *stationary case* $(D_t u = \theta)$ leads to a generalized diffusion equation:

$$Lu = f \qquad (\#\#)$$

Parabolic equation-1

The equation

$$D_t u + Lu = f \qquad (\#)$$

may be used to describe transport of material/energy/probability in a body/domain Ω^3 .

- a conductivity/diffusion coefficient/... constant
- b advection/convection/... constant
- c reaction/... constant

The equation $(\#)$ is called an **autonomous equation** if all coefficients a, b, c do not depend on time.

Initial boundary initial value problem

In order to become a well-posed problem one has to add side conditions:

(IBVP)

$$\begin{aligned} D_t u + Lu &= f \\ Ru &= g \textit{ on } [0,T] \times \partial\Omega \text{ (} \textit{boundary conditions}) \\ u(0) &= u^0 \textit{ in } \Omega \text{ (} \textit{initial condition}) \end{aligned}$$

T is called the final time and R is called a boundary operator:

- **Dirichlet condition** $Ru = u$.
- **Neumann condition** $R_{a,b}u = -\sum_{i=1}^{d} \nu_i (\sum_{j=1}^{d} a^{ij} \partial_j u + b^i u)$ where ν is the outer normal of $\partial\Omega$. (The boundary $\partial\Omega$ has to be smooth)

Parametric view

In the next part of this lecture:

Assumptions and conditions for well-posedness of (IBVP)

Under these assumptions we have for all admissible parameter triple (a, b, c) a uniquely determined solution $u = u(a, b, c)$ of

(IBVP)(a,b,c)

$$\begin{aligned} D_t u + L_{a,b,c} u &= f \\ R_{a,b} u &= g \textit{ on } [0, T] \times \partial\Omega \textit{ boundary conditions} \\ u(0) &= u^0 \textit{ in } \Omega \textit{ initial condition} \end{aligned}$$

belonging to a certain solution space. Hence, we have a **parameter-to-solution map**

$$(a, b, c) \longmapsto S(a, b, c) \textit{ where } u := S(a, b, c) \textit{ solves (IBVP)(a,b,c)}$$

Identification problems

Identification Problem

Try to invert the solution map !

Too ambitious!

$a/b/c$–problems/one-dimensional (toy-problems)

a-problem: *Find* a *under the assumptions* $b \equiv 0$, $c \equiv 0$.

b-problem: *Find* b *under the assumptions* $a \equiv 1$, $c \equiv 1$.

c-problem: *Find* c *under the assumptions* $a \equiv 1$, $b \equiv 0$.

A stationary a-problem

Determination of a diffusion coefficient

$$(*) \left\{ \begin{array}{rcl} -(az_\xi)_\xi & = & f \; in \; \Omega := (0,1) \\ a(0)z_\xi(0) & = & g_0, z(1) \; = \; 0 \end{array} \right.$$

Determine a from the state z (and the data g_0, f)

Integration of the equation with respect to ξ yields the formula

$$a(\xi)z_\xi(\xi) = g_0 - \int_0^\xi f(s)\,ds\,,\; \xi \in \Omega := [0,1]$$

Problems

(1) $a(\xi) =?$ if $z_\xi(\xi) = 0$. Avoidance: $g_0 - \int_0^\xi f(s)ds > 0, \xi \in [0,1]$

(2) $a(\xi) =?$ if $z_\xi(\xi)$ is small. Stability problem.

(3) Numerical differentiation of the data z is necessary.

(4) The dependence of a on the data z is nonlinear.

A stationary a-problem-1

A stability estimate asks for the continuity of the inverse of the parameter to solution mapping:

$$z \longmapsto a \text{ where } z \text{ solves } (\star)$$

The following assumptions allow a fast illustration of the mentioned problems above.

Assumptions:

- $a(0)$ *is known.*
- $f \in C[0,1]$, $g_0 - \int_0^\xi f(s)ds > 0$ *for all* $s \in [0,1]$.
- *Admissible parameters:*
  $Q_{ad} := \{a \in C[0,1] : a(\xi) \geq \alpha, \xi \in [0,1], a(0) = a^*\}$
- *Admissible states:*
  $Z_{ad} := \{z \in C^2[0,1] : |z_{\xi\xi}(\xi)| \leq \kappa, z_\xi(\xi) \geq \gamma, \xi \in [0,1]\}$

The parameter α, κ, γ are some a priori given positive constants.

A stability estimate-3

Let a_1, a_2 admissible parameters and let z^1, z^2 be the associated admissible states. Under the assumption above we have the stability estimate

$$\|a^1 - a^2\|_{L_2[0,1]} \leq \gamma^{-2} \left(|g_0| + \int_0^1 |f(s)|)\, ds \right) \sqrt{2\kappa}\, \|z^1 - z^2\|^{\frac{1}{2}}_{L_2[0,1]}$$

Proof:

We have:

$$a^i(\xi) = \frac{1}{z^i_\xi(\xi)} \left(g_0 - \int_0^\xi f(s) \right) ds \,,\, \xi \in \Omega := [0,1], i = 1, 2, ,$$

$$a^1(\xi) - a^2(\xi) = \frac{z^2_\xi(\xi) - z^1_\xi(\xi)}{z^1_\xi(\xi) z^2_\xi(\xi)} \left(g_0 - \int_0^\xi f(s)\, ds \right) .$$

A stability estimate-4

Hence

$$\int_0^1 |a^1(\xi) - a^2(\xi)|^2 \, ds \le c \int_0^1 (z_\xi^1(s) - z_\xi^2(s))^2 \, ds$$

where $c := \gamma^{-4} \left(|g_0| + \int_0^1 |f(s)|) \, ds \right)^2$. Now we obtain by using the assumption that $a(0)$ is known as a^*:

$$\begin{aligned}
\int_0^1 (z_\xi^1(s) - z_\xi^2(s))^2 \, ds &= \int_0^1 (z_{\xi\xi}^1(s) - z_{\xi\xi}^2(s))(z^1(s) - z^2(s))^2 \, ds \\
&\le \left(\int_0^1 (z_{\xi\xi}^1(s) - z_{\xi\xi}^2(s))^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^1 (z^1(s) - z^2(s))^2 \, ds \right)^{\frac{1}{2}}
\end{aligned}$$

This implies the result.

The continuity modulus of the solution to parameter map as a mapping from $L_2[0,1]$ nach $L_2[0,1]$ is not to bad: we have Hölder continuity with exponent $\frac{1}{2}$.

A stationary c-problem

Data: $\Omega = (0, \pi), a = 1, b = 0, f = 2,$

(BVP)

$$
\begin{aligned}
-z_{\xi\xi} + cz &= 2\,, \quad in\ (0,\pi)\,,\\
z_\xi(0) = z_\xi(\pi) &= 0
\end{aligned}
$$

Solution: $c^0 := 2, z^0 := 1$

$$
\begin{aligned}
c^n(\xi) &:= \frac{2 - \cos((n+1)\xi)}{1 + \frac{1}{(n+1)^2}\cos((n+1)\xi)}\\
z^n(\xi) &:= 1 + \frac{1}{(n+1)^2}\cos((n+1)\xi)\,,\ \xi \in (0,\pi), n \in \mathbb{N}\,.
\end{aligned}
$$

A stationary c-problem-1

Then

$$\begin{aligned} \|z^n - z^0\|_{L_2(0,\pi)} &\leq \frac{\kappa}{(n+1)^2}\,, \\ \|z^n - z^0\|_{H^1(0,\pi)} &\leq \frac{\kappa}{n+1}\,, \\ \|c^n\|_{L_2(0,\pi)} &\leq \kappa\,,\ n \in \mathbb{N}\,. \end{aligned}$$

But

$$\|c^n - c^0\|_{L_2(0,\pi)} \geq 2\sqrt{2\pi}\,,\ n \in \mathbb{N},$$

$c \in L_2(0,\pi)$ does not depend continuously on the solution $z \in H_1(0,\pi)$
Moreover, the sequence $(c^n)_{n\in\mathbb{N}}$ is not bounded in $H_1(0,\pi)$

Oszillation of $(c^n)_{n\in\mathbb{N}}$ should be surpressed by an a priori bound for the derivative of the admissible parameters!

A stationary a/c-problem

Data: $\Omega = (0,\pi), b = 0, f(\xi) := -1 + \xi^2,$

(IBVP)

$$\begin{aligned} -(az_\xi)_\xi + cz &= -1 + \xi^2\,, \;\; in\ (0,\pi)\,, \\ a(0)z_\xi(0) = 0, z(\pi) &= 1 + \pi^2 \end{aligned}$$

Solutions:
$a^0 := 1, c^0 := 1, z^0(\xi) := 1 + \xi^2$
$a^1 := 2, c^1 := 1 + 2/(1 + \xi^2), z^1(\xi) := 1 + \xi^2$

No identifiability of two parameters. Two experiments may help!

J. Baumeister and K. Kunisch
Identifiability and stability of a two–parameter estimation problem
Applicable Analysis 23, 1991

T. Hein and M. Meyer
Simultaneous identification of independent parameters in elliptic equations – numerical studies
J. Inverse and Ill-posed Problems 16.5, 2008

Identification of a drift-term

What is an **option** at the finance market?

It is a **contract** that gives the owner the right to buy or sell a specified amount of a particular underlying asset of a fixed price within a fixed period of time.

The **option price** u is determined by a number of factors, including the underlying stock price x, the current time t, the maturity date T, the exercise price K, the risk–free interest rate R, the dividend yield on the stock D and finally the local stock volatility σ, that measures the riskiness of the stock.

Volatility is defined as the instantaneous time–independent variance of expected stock returns. It is the only parameter that cannot be observed on the market directly. It is assumed (mostly) constant but in reality it depends on the stock price, which changes over time; see below.

Modelling (Black-Merton-Scholes):

Let the option price $z := z(\cdot, \cdot; K, T)$ satisfy the *Black–Scholes* partial differential equation

$$\frac{\partial z}{\partial \tau} + \frac{1}{2}\xi^2\sigma(\xi)^2\frac{\partial^2 z}{\partial \xi^2} + (R - D)\xi\frac{\partial z}{\partial \xi} - Rz = 0\,,\ \xi > 0, \tau \in (t_0, T)\,,$$

subject to the boundary condition

$$z(0, \tau) = 0\,,\ \tau \in (t_0, T)$$

and the final condition

$$z(\xi, T) = \max(0, \xi - K)\,,\ \xi > 0\,.$$

The parameter $\sigma(\cdot)^2$ has to be determined from the current market prices

$$z(\cdot, \cdot; K, T)\,, K \in \mathcal{K}, T \in \mathcal{T}\ ,$$

where $\mathcal{K}$ is a subset of strikes and $\mathcal{T}$ is a subset of maturity times.

Black-Merton-Scholes/forward problem

Let $D := 0$ and set for a fixed pair (K, T)

$$y := \ln(S/K), \tau := T-t, a(y,\tau) := \frac{1}{2}\sigma(S,t)^2\,,\ c(y,\tau) := z(S,t;K)\,.$$

Then

$$\begin{aligned}
\frac{\partial c}{\partial \tau} - a(y,\tau)\frac{\partial^2 c}{\partial y^2} - (r - a(y,\tau))\frac{\partial c}{\partial y} - rc &= 0\,,\ y \in \mathbb{R}, \tau \in (0,T)\,. \\
c(-\infty,\tau) = 0\,,\ \lim_{y\to\infty}(c(y,\tau) - Ke^y) &= 0\,,\ \tau \in (0,T)\,, \\
c(y,0) &= K(e^y - 1)^+\,,\ y \in \mathbb{R}\,.
\end{aligned}$$

This problem has an analytical solution.

Black-Merton-Scholes/forward problem-1

A transformation back to the S-t–variables gives the solution of the Black-Scholes equation:

$$C(S,t) = S\mathcal{N}(d_+(\sigma)) - Ke^{-r(T-t)}\mathcal{N}(d_-(\sigma))\,,$$

where

$$\mathcal{N}(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \exp(-\frac{s^2}{2})ds\,,\ a \in \mathbb{R}$$

$$d_\pm(\sigma) = \frac{\ln(\frac{S}{K}) + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\,,\ \sigma \geq 0\,;$$

here $\mathcal{N}$ denotes the cumulative normal distribution function.

Suppose that we have a model to list the option prices as function of the parameters already introduced:

$$C \,:\, (0,\infty)\times(0,T) \ni (S,t) \;\longmapsto\; C(S,t;K,T,r,\sigma) \in \mathbb{R}$$

An inverse problem of finance

Given $S_*, t_*, r, T_{\max}$, a set $\mathcal{K} \subset (0,\infty)$ and a subset $\mathcal{T} \subset (t_*, T_{\max})$
Determine σ from $C(S_*, t_*; K, T, r, \sigma)$, $K \in \mathcal{K}, T \in \mathcal{T}$.

Identification of a drift-term/references

I. Bouchouev and V. Isakov
The inverse problem of option pricing
Inverse Problems 11, 1997

T. Hein
Analytische und numerische Studien zu inversen Problemen der Optionspreisbildung
Thesis, TU Chemnitz, 2003

A. De Cezaro
On a parabolic inverse problem arising in Finance: convex and iterative regularization
Thesis, IMPA, 2010

J. Zubelli
Inverse problems and stochastic volatility models
IMPA, 2014

H. Egger and H.W. Engl
Tihonov regularization applied to the inverse problem of option pricing: convergence analysis and rates
Inverse Problems 21, 2007

Identification of a drift-term/Inverse Problem

The Black-Scholes formula worked well before the huge crashes 1987 and 1989, but since then, it has been observed that market prices contradict the model. Mainly, the assumptions in the model concerning the volatility are in doubt. There are strategies to remedy this observation:

- *Implied volatility*
- *Stochastic volatility*
- *Local volatility*

See for instance

J. Zubelli
Inverse problems and stochastic volatility models
IMPA, 2014

J. Baumeister
Inverse Problems in Finance
In: Recent Developments in Computational Finance: Foundations, Algorithms and Applications, World Scientific, 2012

Estimation of the volatility/Dupires method

Dupire observed in 1994 that there is rather direct way to solve the inverse problems of finance.
Given $S_* > 0, t_* > 0$, one introduces a new variable U as follows:

$$U(K,T) = U(K,T;S_*,t_*,r,\sigma) := C(S_*,t_*;K,T,r,\sigma)\,.$$

Then one has for the volatility

$$\frac{1}{2}\sigma(K,T)^2 = \frac{\dfrac{\partial U}{\partial T}(K,T) + rK\dfrac{\partial U}{\partial K}(K,T)}{K^2\dfrac{\partial^2 U}{\partial K^2}(K,T)}\,,\; K \in (0,\infty), T \in (t_*,\infty)\,.$$

Despite its simplicity, this approach has severe practical shortcommings which reflects the ill-posedness of the problem.
Nevertheless, one can use this formula to compute approximations for the volatility.

Estimation of the volatility/Dupires method-1

We assume that the following market data are available:

- Expiration dates $T_1, \dots, T_N$.
- Traded options with strike prices $K_{i1}, \dots, K_{im}$, for each expiration date T_i.
- Market prices V_{ij} for an option with expiration date T_i and strike price K_{ij}.

To evaluate Dupire's formula we need to differentiate these discrete data with respect to the strike variable K and the maturity time T. Since the times T are „very discrete“ differentiaton with respect to T is easier to implement than the differentiation with respect K.

B. Dupire
Pricing with a smile
Risk Magazine, 1994