Functionalanalytic tools and linear equations

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Outline

- Banach and Hilbert spaces revisited
- Gelfand triple
- Linear equations
- Lax-Milgram Lemma
- The Hilbert Uniqueness Method (HUM)
- Linear equations/stability
- Method of Tikhonov

Banach spaces

All vector spaces $\mathcal{H}, \mathcal{X}, \mathcal{Y}, \ldots$ are real vector spaces with null vector θ

Let $\mathcal{X}$ be a real vector space.

A mapping $\|\cdot\|_{\mathcal{X}} = \|\cdot\| : \mathcal{X} \longrightarrow \mathbb{R}$ is a norm iff $\|\cdot\|$ is definite, homogeneous and satisfies the triangle inequality

Let $(\mathcal{X}, \|\cdot\|)$ be normed space.

- Open balls $B_r(x)$ and closed balls $\overline{B}_r(x)$ are convex sets.
- $\mathcal{X}$ is a Banach space or complete iff every Cauchy sequence converges in $\mathcal{X}$.
- $X^* := \{\lambda : \mathcal{X} \longrightarrow \mathbb{R} : \lambda \text{ linear and continuous}\}$ is the dual space.
- $\|\lambda\|_{\mathcal{X}^*} := \sup\{|\langle\lambda, x\rangle| : \|x\| \leq 1\}$ defines a norm in $\mathcal{X}^*$.
- $(\mathcal{X}^*, \|\cdot\|_{\mathcal{X}^*})$ is a Banach space.
- $\mathcal{X} \neq \{\theta\}$ iff $\mathcal{X}^* \neq \{\theta\}$ (Theorem of Hahn-Banach).
- Examples: $\mathbb{R}, \mathbb{R}^d, C[a,b], C^m(\Omega)$ if Ω is a compact subset of $\mathbb{R}^d, \ldots$ (all endowed with the usual norms).
- The functionals $\lambda \in \mathcal{X}^* \backslash \{\theta\}$ define hyperplanes $H_{\lambda,a}$ as follows:

$$H_{\lambda,a} := \{x \in \mathcal{X} : \langle\lambda, x\rangle = a\}\,,\ a \in \mathbb{R}\,.$$

- The functionals $\lambda \in \mathcal{X}^* \backslash \{\theta\}$ define half spaces as follows:

$$H^+_{\lambda,a} := \{x \in \mathcal{X} : \langle\lambda, x\rangle \geq a\}\,,\ H^-_{\lambda,a} := \{x \in \mathcal{X} : \langle\lambda, x\rangle \leq a\}\,,\ a \in \mathbb{R}\,.$$

Hilbert spaces

Let $\mathcal{H}$ be a real vector space.

> *A mapping* $\langle\cdot|\cdot\rangle_{\mathcal{H}} = \langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ *is an inner product iff* $\langle\cdot|\cdot\rangle$ *is a definite, symmetric bilinear form.*

Let $\mathcal{H}$ be real vector space endowed with the inner product $\langle\cdot|\cdot\rangle$.

- The mapping $\|x\|_{\mathcal{H}} := \|x\| := \sqrt{\langle x|x\rangle}, x \in \mathcal{H}$, defines a norm in $\mathcal{H}$ (associated to the inner product).
- If $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Banach space then $\mathcal{H}$ is called a Hilbert space.
- We have the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \, , \; x, y \in \mathcal{H} \, .$$

- Examples: $\mathbb{R} = \mathbb{R}^1, \mathbb{R}^d$ endowed with the l_2-inner product:

$$\langle x, y\rangle := \sum_{i=1}^{d} x_i y_i \, .$$

- Example: $C^m(\Omega)$ where Ω is a compact subset of $\mathbb{R}^d$ endowed with the inner product

$$\langle x, y\rangle := \int_\Omega x(\xi) y(\xi) \, d\xi \, .$$

  But this space is no Hilbert spaces since it is not complete.

Geometrical properties of Banach and Hilbert spaces

There are serious differences between Banach and Hilbert spaces, mainly from the geometrical point of view.

Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|_{\mathcal{X}}$ and let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot|\cdot\rangle$ and associated norm $\|\cdot\|_{\mathcal{H}}$.

- $\mathcal{X}$ is a Hilbert space iff the parallelogram identity holds.
- In $\mathcal{H}$ we have an orthogonality relation:
  $$x, y \text{ are called orthogonal iff } \langle x|y\rangle = 0\,.$$
- As a consequence of orthogonality: In $\mathcal{H}$ we have the pythagorean law:
  $$\|x\|^2 + \|y\|^2 = \|x+y\|^2 \text{ if } x, y \text{ are orthogonal}\,.$$
- In $\mathcal{H}$ we have the Cauchy-Schwarz inequality:
  $$|\langle x|y\rangle| \le \|x\|_{\mathcal{H}}\|y\|_{\mathcal{H}}\,,\ x, y \in \mathcal{H}\,.$$

  As a consequence: We have an angle between vectors and subspaces.
- In the Hilbert space $\mathcal{H}$ the balls are „round“, i.e. each hyperplane touches a ball $\overline{B}_r(x)$ just in one point.
- One has the Theorem of Riesz: There is an isometry $R_{\mathcal{H}}$ from $\mathcal{H}^*$ onto $\mathcal{H}$ which has the property: for all $\lambda \in \mathcal{H}^*$ there exists a uniquely determined $y = R_{\mathcal{H}}(\lambda) \in \mathcal{H}$ with
  $$\langle \lambda, x\rangle = \langle y|x\rangle_{\mathcal{H}}\,,\ x \in \mathcal{H}\,.$$

Reflexivity

Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|_{\mathcal{X}}$. Then $\mathcal{X}^*$ is Banach space too and we may consider the dual space $\mathcal{X}^{**} := (\mathcal{X}^*)^*$ of $\mathcal{X}^*$. We know already elements of $\mathcal{X}^{**}$ namely the functionals $\mu_x, x \in \mathcal{X}$:

$$\langle \mu_x, \lambda \rangle = \langle \lambda, x \rangle \,,\; \lambda \in \mathcal{X}^* \,.$$

This defines a mapping $\qquad J_{\mathcal{X}} : \mathcal{X} \ni x \;\longmapsto\; \mu_x \in \mathcal{X}^{**} \,.$

Definition

Let $\mathcal{X}$ be a Banach space. $\mathcal{X}$ is called reflexive iff the mapping $J_{\mathcal{X}}$ is bijective.

Actually, if $\mathcal{X}$ is reflexive then $J_{\mathcal{X}}$ is an isometry.

Fact

Every Hilbert space $\mathcal{H}$ is reflexive.

This follows from the existence of the Riesz-mapping and the fact that $\mathcal{H},^*, \mathcal{H}^{**}$ are again Hilbert spaces. Since a necessary condition for reflexivity is the completenes of a normed space (why?) we know already non-reflexive spaces.

Examples

$l_p, L_p(\Omega), 1 < p < \infty$, are reflexive Banach spaces. $l_1, l_p, L_1(\Omega), L_\infty(\Omega)$ are not reflexive.

Reflexivity-1

Definition

Let $\mathcal{X}$ be a Banach space. $\mathcal{X}$ is called separable if there exists a countable dense subset M of $\mathcal{X}$, i.e. its closure $\overline{M}$ satisfies $\overline{M} = \mathcal{X}$.

Definition

Let $\mathcal{X}$ be a Banach space. A sequence $(x^n)_{n\in\mathbb{N}}$ converges weakly to x iff $\lim_n\langle\lambda, x^n - x\rangle = 0$ for all $\lambda \in \mathcal{X}^*$.

Fact (Alaoglu-Banach)

The ball $\overline{B}_1(\theta)$ in a reflexive separable Banach space $\mathcal{X}$ is weakly sequentially compact, i.e. each sequence $(x^n)_{n\in\mathbb{N}}$ in the ball $\overline{B}_1(\theta)$ has a weakly convergent subsequence.

The fact above is a very powerful tool. In combination with the facts

- Every closed convex subset of a Banach space is weakly closed
- The norm in a Banach space is sequentially weakly lower semicontinuous

one gets very general existence theorems for optimization problems.

Lax Milgram Lemma

Theorem (Zarantonello)

Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{H} \longrightarrow \mathcal{H}^*$ be a mapping which satisfies

- *A is Lipschitz-continuous with Lipschitz-constant L, i.e.* $\|A(u) - A(v)\|_{\mathcal{H}^*} \leq L\|u - v\|_{\mathcal{H}}\,,\ u, v \in \mathcal{H}\,,$
- *A is strong monotone, i.e.* $\langle A(u) - A(v)|u - v\rangle \geq c\|u - v\|^2_{\mathcal{H}}, u, v \in \mathcal{H}$, *with* $c > 0\,.$

Then A is bijective.

Proof:

A is bijective iff $Au = \lambda$ is uniquely solvable for all $\lambda \in \mathcal{H}^*$.
Let $\lambda \in \mathcal{H}^*$. Then $Au = \lambda$ is equivalent to

$$R_{\mathcal{H}} \circ A(u) = R_{\mathcal{H}} \circ \lambda\,.$$

$R_{\mathcal{H}}$ Riesz-isometry from $\mathcal{H}^*$ onto $\mathcal{H}$.

Lax Milgram Lemma-1

$A' := R_{\mathcal{H}} \circ A, y := R_{\mathcal{H}} \circ \lambda$. Then $A' : \mathcal{H} \longrightarrow \mathcal{H}$ and $A'(u) = y$ is equivalent to the fixed point equation

$$F_k(u) = u \text{ where } F_k(u) := u + k(y - A'(u)), u \in \mathcal{H} .$$

with $k \in (0, 2cL^{-1})$. Then F_k is a contraction since for $u, v \in \mathcal{H}$:

$$\begin{aligned} \|F_k(u) - F_k(v)\|_{\mathcal{H}}^2 &= \|u - v\|_{\mathcal{H}}^2 - 2k\langle A'(u) - A'(v)|u - v\rangle \\ &\quad + k^2\|A'(u) - A'(v)\|_{\mathcal{H}^*}^2 \\ &\leq \|u - v\|_{\mathcal{H}}^2 - 2kc\|u - v\|_{\mathcal{H}}^2 + k^2L\|u - v\|_{\mathcal{H}}^2 \\ &= (1 + k(kL^2 - 2c))\|u - v\|^2 . \end{aligned}$$

Hence the equation $A'(u) = y$ has a uniquely determined solution for each $y \in \mathcal{H}$.

Actually, A, A^{-1} are continuous.

Gelfand triple

Let $\mathcal{X}$ be a real reflexive Banach space and let $\mathcal{H}$ be a real Hilbert space. The (usual) notation for a Gelfand triple *is:*

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$$

This notation stands for the following construction.

Assumptions:

(1) Let $\mathcal{V}$ be a real reflexive Banach space and let $\mathcal{H}$ be a real Hilbert space.

(2) $\mathcal{V} \subset \mathcal{H}$ and $\iota : \mathcal{V} \ni x \longmapsto x \in \mathcal{H}$ is continuous.

(3) $\mathrm{ran}(\iota)$ is dense in $\mathcal{H}$ in the $\mathcal{H}$-norm-topology

(4) $\mathcal{H}$ is identified with its dual space $\mathcal{H}^*$ (via the Riesz Theorem)

Gelfand triple-1

Fact

(1) *$\mathcal{H} \subset \mathcal{V}^*$ and $\iota^* : \mathcal{H} \ni x \longmapsto x \in \mathcal{V}^*$ is continuous.*

(2) *$ran(\iota^*)$ is dense in $\mathcal{V}^*$ in the $\mathcal{V}^*$-norm-topology*

Ad (1): $\langle i^*(h), v\rangle := \langle h|v\rangle_{\mathcal{H}}$
Ad (2): Follows from the denseness of $\mathcal{V}$ in $\mathcal{H}$.

Continuity of ι:

$$\|\iota(x)\|_{\mathcal{H}} \le c\|x\|_{\mathcal{V}}\,,\ x \in \mathcal{V}\,,\ c \ge 0\,.$$

Without loss of generality: $c = 1$. Then

$$\|x\|_{\mathcal{V}^*} \le \|x\|_{\mathcal{H}} \le \|x\|_{\mathcal{V}}\,,\ x \in \mathcal{V}\,.$$

Now the construction

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathcal{V}^* \text{ or } \mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$$

is complete.

Gelfand triple-2

Examples

- $\mathcal{V} := \mathcal{H}_0^1(\Omega) \hookrightarrow \mathcal{H} := L_2(\Omega) \hookrightarrow \mathcal{V}^* := H^{-1}(\Omega)$. *Here* $\mathcal{H}_0^1(\Omega)$ *is the closure of* $C_0^1(\Omega)$ *under the norm*

$$\|f\|_{1,2} := \left(\int_\Omega |f(\xi)|^2\,ds\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla f(\xi)|^2\,ds\right)^{\frac{1}{2}}$$

where

$$C_0^1(\Omega) := \{f : \Omega \longrightarrow \mathbb{R} : f \text{ continuously differentiable, } supp(f) \text{ compact}\}$$

- *Let* $A : \mathcal{X} \longrightarrow \mathcal{Y}$ *be a linear injective compact operator; see below. Then we have a singular value decomposition* $(\sigma_n, e^n, f^n)_{n\in\mathbb{N}}$ *and* $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ *where*

$$\begin{aligned} \mathcal{V} &:= \{x \in \mathcal{H} : \sum_{i=1}^\infty |\langle x|e^i\rangle_{\mathcal{H}}|^2 \sigma_i^{-2} < \infty\} \\ \mathcal{V}^* &:= \{x \in \mathcal{H} : \sum_{i=1}^\infty |\langle x|e^i\rangle_{\mathcal{H}}|^2 \sigma_i^{2} < \infty\} \end{aligned}$$

Notice that $\lim_i \sigma_i = 0$

Lax Milgram Lemma again

Definition

Let $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ be a Gelfand triple and let $a : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ be a bilinear mapping.

- *a is called $\mathcal{V}$-**continuous** iff $|a(u, v)| \leq \gamma_0 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, u, v \in \mathcal{V}$.*
- *a is called $\mathcal{V}$-**coervive** iff $a(u, u) \geq \gamma_1 \|u\|^2_{\mathcal{V}}, u \in \mathcal{V}$.*

Here $\gamma_0 \geq 0, \gamma_1 > 0$.

Theorem

Let $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ be a Gelfand triple and let $a : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ be a $\mathcal{V}$-continuous and $\mathcal{V}$-coervive bilinear form. Then there exists a linear continuous mapping $A : \mathcal{V} \longrightarrow \mathcal{V}^*$ with

$$a(u, v) = \langle A(u), v \rangle \, , \; u, v \in \mathcal{V} \, .$$

Lax Milgram Lemma again-1

Theorem (Lax-Milgram)

Let $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ be a Gelfand triple and let $a : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ be a $\mathcal{V}$-continuous, $\mathcal{V}$-coercive bilinear form. Then there exists a linear continuous bijective mapping $A : \mathcal{V} \longrightarrow \mathcal{V}^*$ with

$$a(u, v) = \langle A(u), v \rangle \, , \; u, v \in \mathcal{V} \, .$$

Moreover,

$$A^{-1} : \mathcal{V}^* \longrightarrow \mathcal{V} \text{ is continuous with } \|A^{-1}\| \leq \gamma_1^{-1}$$

and

$$D_A := \{u \in \mathcal{V} : A(u) \in \mathcal{H}\}$$

is dense in $\mathcal{V}$ and $\mathcal{H}$.

Linear equations

$A : \mathcal{X} \longrightarrow \mathcal{Y}$: Forward operator/Linear continuous operator
$\mathcal{X}$: Solution space/Hilbert space with norm $\| \cdot \|_{\mathcal{X}}$.
$\mathcal{Y}$: Image space/Hilbert space with norm $\| \cdot \|_{\mathcal{Y}}$.

Problem statement

Solve the equation $\quad Ax = y$

(1) Infinte-dimensional setting is necessary for applications
(2) Usually, $\mathrm{ran}(A)$ is dense in $\mathcal{Y}$ but not closed.
(3) The unbounded operator A^{-1} describes the inverse problem

Fact

If ran(A) *is dense in* $\mathcal{Y}$ *but not closed then* A^{-1} *is not continuous.*

Compact operators

There is class of problems which can be considered as the generic case of an ill-posed problem, namely the solution of linear equations which are governed by compact operators.

Definition

Let $A : \mathcal{X} \longrightarrow \mathcal{Y}$ be a linear operator between infinite dimensional Hilbert spaces $\mathcal{X}, \mathcal{Y}$. Then A is called a **compact operator** *if A maps the unit ball $B_1(\theta)$ in $\mathcal{X}$ into the subset $A(B_1(\theta))$ of $\mathcal{Y}$ whose closure is compact.* □

As a rule, integral operators with a smooth kernel function and defined on functions of finite support are compact operators.

If $A : \mathcal{X} \longrightarrow \mathcal{Y}$ is the limit of a sequence of compact operators $A_n : \mathcal{X} \longrightarrow \mathcal{Y}$ in the operator-topology then A is compact.

Compact operators-1

For a compact operator one has a very powerful „normal form“, as we will see next; for the proof of this normal form we refer to the literature.

Theorem (Singular value decomposition/SVD)

Let $A : \mathcal{X} \to \mathcal{Y}$ be an injective compact operator and assume that $\mathcal{X}$ is infinite dimensional. Then there exist sequences $(e^j)_{j\in\mathbb{N}}, (f^j)_{j\in\mathbb{N}}, (\sigma_j)_{j\in\mathbb{N}}$, called a **singular system**, *such that the following assertions hold:*

(a) $e^j \in \mathcal{X}, f^j \in \mathcal{Y}$ *for all* $j \in \mathbb{N}$;

(b) $\sigma_j \in \mathbb{R}, 0 < \sigma_{j+1} < \sigma_j$ *for all* $j \in \mathbb{N}$, $\lim_j \sigma_j = 0$;

(c) $\langle e^j, e^k \rangle = 0, \langle f^j, f^k \rangle = 0$ *for all* $j, k \in \mathbb{N}, j \neq k$;

(d) $Ae^j = \sigma_j f^j, A^* f^j = \sigma_j e^j$ *for all* $j \in \mathbb{N}$;

(e) $Ax = \sum_{j=1}^{\infty} \sigma_j \langle x, e^j \rangle f^j$ *for all* $x \in X$,
$A^* y = \sum_{j=1}^{\infty} \sigma_j \langle y, f^j \rangle e^j$ *for all* $y \in \mathcal{Y}$.

Control system (pde)

$$y' = Ay + Bu(t),\ t \in (0, T)\,,\ y(0) = y^0.$$

$y \in \mathcal{Y}$ state, $u(t) \in \mathcal{U}$ control action at time t, y_0 initial state and $S(T, u)$ state at time T under the control action u

At least for parabolic systems we know:

ran$(S(T, \cdot))$ is dense in $\mathcal{Y}$ but not closed

J.L. Lions 1986: Duality method to construct the space ran$(S(T, \cdot))$ based on the injectivity of the adjoint $S(T, \cdot)^*$.

The HUM-construction in $\mathcal{X}$

$A : \mathcal{X} \longrightarrow \mathcal{Y}$ linear, injective, continuous , $\mathcal{X}, \mathcal{Y}$ Hilbert spaces

- $\| \cdot \|_d : \mathcal{X} \ni x \longmapsto \|Ax\|_{\mathcal{Y}} \in \mathbb{R}$
- $\mathcal{V}^*$ completion of $\mathcal{X}$ in this norm
- $\mathcal{V}$ subspace of $\mathcal{X}$ such hat $\mathcal{V}^*$ is the space of continuous functionals on $\mathcal{V}$
- $\mathcal{V} = \mathrm{ran}(A^*)$
- $\mathcal{V} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}^*$ (Gelfand triple)

- $\| \cdot \|_{dd} : \mathcal{X} \ni x \longmapsto \|A^*Ax\|_{\mathcal{X}} \in \mathbb{R}$
- $\mathcal{U}^*$ completion of $\mathcal{X}$ in this norm
- $\mathcal{U}$ subspace of $\mathcal{X}$ such hat $\mathcal{U}^*$ is the space of continuous functionals on $\mathcal{U}$
- $\mathcal{U} = \mathrm{ran}(A^*A)$
- $\mathcal{U} \hookrightarrow \mathcal{V} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}^* \hookrightarrow \mathcal{U}^*$ (Gelfand triple)

The HUM-construction in $\mathcal{X}$–a consequence

Fact

Let $\hat{x} \in \mathcal{X}$. The following conditions are equivalent:

(1) $\hat{x} \in \mathcal{V} = ran(A^*)$

(2) $|\langle \hat{x}|x\rangle_{\mathcal{X}}| \leq \kappa \|Ax\|_{\mathcal{Y}}, x \in \mathcal{X},$ *for some* $\kappa \geq 0$.

Proof:

By construction, the elements of $\mathcal{V}$ are the continuous functionals on $\mathcal{V}^*$, endowed with the norm $x \longmapsto \|Ax\|_{\mathcal{Y}}$.

The fact above is used in exploiting the the Neumann-to-Dirichlet mapping which we consider later on. See for instance:

B. Harrach
Recent progress on the factorization method for electrical impedance tomography.
Comp. and Math. Methods in Medicine, 2013

F. Frühauf and B. Gebauer and O. Scherzer
Deteckting interfaces in parabolic-elliptic problems from surface measurements
SIAM J. Numer. Anal. 45, 2007.

The HUM-construction in $\mathcal{Y}$

- $\|\cdot\|_r : \mathcal{Y} \ni y \longmapsto \|A^* y\|_{\mathcal{X}} \in \mathbb{R}$
- $\mathcal{W}^*$ completion of $\mathcal{Y}$ in this norm
- $\mathcal{W}$ subspace of $\mathcal{Y}$ such hat $\mathcal{W}^*$ is the space of continuous functionals on $\mathcal{W}$
- $\mathcal{W} = \mathrm{ran}(A)$
- $\mathcal{W} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{W}^*$ (Gelfand triple)

Similar

$$\mathcal{Z} \hookrightarrow \mathcal{W} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{W}^* \hookrightarrow \mathcal{Z}^*$$

The Gelfand triple $\mathcal{W} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{W}^*$ is the construction of J.L. Lions transferred into an abstract setting

The HUM-diagram

Theorem

All the operators in the following diagram are isomorphisms when the subspaces $\mathcal{U}, \mathcal{V}, \mathcal{V}^*, \mathcal{U}^*, \mathcal{Z}, \mathcal{W}, \mathcal{W}^*, \mathcal{Z}^*$ are endowed with the topologies introduced above.

$$
\begin{array}{ccccccccc}
\mathcal{U} & \hookrightarrow & \mathcal{V} & \hookrightarrow & \mathcal{X} & \hookrightarrow & \mathcal{V}^* & \hookrightarrow & \mathcal{U}^* \\
 & \swarrow A & & \swarrow A & & \swarrow A & & \swarrow A & \\
\mathcal{Z} & \hookrightarrow & \mathcal{W} & \hookrightarrow & \mathcal{Y} & \hookrightarrow & \mathcal{W}^* & \hookrightarrow & \mathcal{Z}^* \\
 & \swarrow A^* & & \swarrow A^* & & \swarrow A^* & & \swarrow A^* & \\
\mathcal{U} & \hookrightarrow & \mathcal{V} & \hookrightarrow & \mathcal{X} & \hookrightarrow & \mathcal{V}^* & \hookrightarrow & \mathcal{U}^*
\end{array}
$$

Notice that A is an extension or a restriction of the given A on several places.

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Appl. Mathematics and Optimization 31, 1995

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Randsteuerung linearer Volterra-Integrodifferentialgleichungen am Beispiel visko-elastischer Festkörper
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Hilbert and Banach uniqueness method for linear ill-posed problems.
Preeprint, 2009

Linear equations/Stability

Equation

$$Ax = y$$

Assumptions

A0) $\mathcal{X}, \mathcal{Y}$ *Hilbert spaces*
A1) $A : \mathcal{X} \longrightarrow \mathcal{Y}$ *linear, continuous and injective*
A2) $ran(A)$ *is dense in* $\mathcal{Y}$ *(but not closed in general)*

Consequences

A3) A^* *is injective*
A4) $ran(A^*)$ *is dense in* $\mathcal{X}$

Linear equations/Stability estimate

Theorem

$$\sup\{\|x - x^\dagger\|_{\mathcal{X}} \;:\; \|Ax - Ax^\dagger\|_{\mathcal{Y}} \leq \varepsilon, x, x^\dagger \in \mathcal{V}, \|x\|_{\mathcal{V}}, \|x^\dagger\|_{\mathcal{V}} \leq E\}$$

$$\leq \sqrt{2}\sqrt{E\,\varepsilon}$$

Proof:

$$\|A(x - x^\dagger)\|_{\mathcal{Y}} = \|x - x^\dagger\|_{\mathcal{V}^*}\,,\; \|x - x^\dagger\|_{\mathcal{X}} \leq \|x - x^\dagger\|_{\mathcal{V}}^{\frac{1}{2}} \|x - x^\dagger\|_{\mathcal{V}^*}^{\frac{1}{2}}$$

- Worst case error estimate: ε noise level
- Need of a-priori bounds $\|x\|_{\mathcal{V}} \leq E, \|x^\dagger\|_{\mathcal{V}} \leq E$
- Importance of $\mathcal{V}$: space of a-priori knowledge
- Difference to the well-posed case: $\sqrt{\varepsilon}$

Interpolation inequality: If $v = A^* w, w \in \mathcal{Y}$, then $\|v\|_{\mathcal{V}} = \|w\|_{\mathcal{Y}}$. Hence

$$\|v\|_{\mathcal{X}}^2 = \langle v|v\rangle_{\mathcal{X}} = \langle Av|w\rangle_{\mathcal{Y}} \leq \|Av\|_{\mathcal{Y}}\|w\|_{\mathcal{Y}} = \|v\|_{\mathcal{V}^*}\|v\|_{\mathcal{V}}\,.$$

Method of Tikhonov

- $x^\dagger$ is the exact solution of $Ax = y^\dagger$.
- Noisy data: $y^\varepsilon \in \mathcal{Y}$ with $\|y^\dagger - y^\varepsilon\|_{\mathcal{Y}} \le \varepsilon$.

Classical Method of Tikhonov

$$\textit{Minimize} \quad \tfrac{1}{2}\|Ax - y^\varepsilon\|^2_{\mathcal{Y}} + \tfrac{\alpha}{2}\|x\|^2_{\mathcal{X}}$$

- The existence of a minimizer is obvious.
- $\alpha > 0$ is a regularization parameter.
- Find a strategy $\alpha = \alpha(\varepsilon)$ such that $\lim_{\varepsilon\to 0} x^{\varepsilon,\alpha(\varepsilon)} = x^\dagger$.
- Optimality equation: $A^*Ax^{\varepsilon,\alpha} + \alpha x^{\varepsilon,\alpha} = A^*y^\varepsilon$

We present three convergence theorems, i.e. results for the fact

$$\lim_{\varepsilon\to 0} x^{\varepsilon,\alpha(\varepsilon)} = x^\dagger$$

where the parameter choice strategy $\alpha = \alpha(\varepsilon)$ is specified in dependence on the source condition.

Method of Tikhonov: Convergence without source condition

Theorem

Suppose that the assumptions A0),A1),A2) hold and let the regularization parameter $\alpha = \alpha(\varepsilon)$ be chosen as follows:

$$\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0\,, \ \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{\alpha(\varepsilon)} = 0\,.$$

Then

$$\lim_{\varepsilon \to 0} x^{\varepsilon,\alpha(\varepsilon)} = x^\dagger$$

Proof:

The necessary condition of optimality for the solution $x^{\varepsilon,\alpha}$ is given by

$$A^* A x^{\varepsilon,\alpha} + \alpha x^{\varepsilon,\alpha} = A^* y^\varepsilon\,,$$

and we see that $x^{\varepsilon,\alpha}$ belongs to $\mathcal{V}$.

Method of Tikhonov: Convergence without source condition-1

Since

$$A^*Ax^\dagger = A^*y^\dagger$$

holds we obtain for the error $e^{\varepsilon,\alpha} := x^{\varepsilon,\alpha} - x^\dagger$

$$A^*Ae^{\varepsilon,\alpha} + \alpha e^{\varepsilon,\alpha} = -\alpha x^\dagger + A^*(y^\varepsilon - y^\dagger)\,.$$

Let $v \in \mathcal{V}$ with $\|v\|_{\mathcal{V}} \le R$. Then $v = A^*w$ with $\|w\|_{\mathcal{Y}} \le R$.We We obtain

$$\begin{aligned}\|Ae^{\varepsilon,\alpha}\|_{\mathcal{Y}}^2 + \alpha\|e^{\varepsilon,\alpha}\|_{\mathcal{X}}^2 &= \langle -\alpha(x^\dagger - A^*w) - \alpha A^*w, e^{\varepsilon,\alpha}\rangle_X + \langle y^\varepsilon - y^\dagger, Ae^{\varepsilon,\alpha}\rangle_Y \\ &\le \alpha\|x^\dagger - A^*w\|_{\mathcal{X}}\|e^{\varepsilon,\alpha}\|_{\mathcal{X}} + \alpha\|w\|_{\mathcal{Y}}\|Ae^{\varepsilon,\alpha}\|_{\mathcal{Y}} + \varepsilon\|Ae^{\varepsilon,\alpha}\|_{\mathcal{Y}}\,.\end{aligned}$$

We set

$$\Delta_R := \inf\{\|x^\dagger - A^*w\|_{\mathcal{X}} \,|\, \|w\|_{\mathcal{Y}} \le R\}\,.$$

Method of Tikhonov: Convergence without source condition-2

Taking the infimum with respect to w with $\|w\|_{\mathcal{Y}} \le R$ we obtain

$$
\begin{aligned}
\|Ae^{\varepsilon,\alpha}\|^2_{\mathcal{Y}} + \alpha\|e^{\varepsilon,\alpha}\|^2_{\mathcal{X}} &\le \alpha\Delta_R\|e^{\varepsilon,\alpha}\|_{\mathcal{X}} + \alpha R\|Ae^{\varepsilon,\alpha}\|_{\mathcal{Y}} + \varepsilon\|Ae^{\varepsilon,\alpha}\|_{\mathcal{Y}} \\
&\le \alpha(\frac{1}{2}\Delta_R^2 + \frac{1}{2}\|e^{\varepsilon,\alpha}\|^2_{\mathcal{X}}) + \frac{1}{2}(\alpha R + \varepsilon)^2 + \frac{1}{2}\|Ae^{\varepsilon,\alpha}\|^2_{\mathcal{Y}} \\
&\le \alpha(\frac{1}{2}\Delta_R^2 + \frac{1}{2}\|e^{\varepsilon,\alpha}\|^2_{\mathcal{X}}) + (\alpha^2 R^2 + \varepsilon^2) + \frac{1}{2}\|Ae^{\varepsilon,\alpha}\|^2_{\mathcal{Y}}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|Ae^{\varepsilon,\alpha}\|^2_{\mathcal{Y}} &\le \alpha(\Delta_R^2 + 2\alpha R^2 + 2\frac{\varepsilon^2}{\alpha}) \\
\|e^{\varepsilon,\alpha}\|^2_{\mathcal{X}} &\le \Delta_R^2 + 2\alpha R^2 + 2\frac{\varepsilon^2}{\alpha}
\end{aligned}
$$

Now we apply the parameter choice strategy and the fact $\lim_{R\to\infty}\Delta_R = 0$ due to the denseness of $\mathcal{V} = \mathrm{ran}(A^*)$ in $\mathcal{X}$.

One can see that the solution $x^{\varepsilon,\alpha}$ belongs to $\mathcal{V}$. Therefore it is reasonable to consider a source condition in $\mathcal{V}$.

Method of Tikhonov: Convergence with source condition in $\mathcal{V}$

Theorem

Suppose that the assumptions A0),A1),A2) hold and that the source condition

$$x^\dagger \in \mathcal{V}\,,\ \|x^\dagger\|_{\mathcal{V}} \le E \ \textit{for some } E > 0$$

is satisfied. Let the regularization parameter $\alpha = \alpha(\varepsilon)$ be chosen as follows:

$$\alpha(\varepsilon) = \varepsilon\, E^{-1}\,,\ \varepsilon > 0\,.$$

Then

$$\|x^{\varepsilon,\alpha(\varepsilon)} - x^\dagger\|_{\mathcal{X}} \le 2\sqrt{2}E^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$$

Method of Tikhonov: Convergence with source condition in $\mathcal{V}$-1

Proof:
From the necessary condition we obtain

$$A^* A e^{\varepsilon,\alpha} + \alpha e^{\varepsilon,\alpha} = -\alpha A^* w^\dagger + A^*(y^\varepsilon - y^\dagger)\,.$$

and

$$\|Ae^{\varepsilon,\alpha}\|_{\mathcal{Y}}^2 + \alpha\|e^{\varepsilon,\alpha}\|_{\mathcal{X}}^2 \leq \frac{1}{2}(\alpha\|w^\dagger\|_{\mathcal{Y}} + \varepsilon)^2 + \frac{1}{2}\|Ae^{\varepsilon,\alpha}\|_{\mathcal{Y}}^2\,.$$

This implies

$$\begin{aligned} \|Ae^{\varepsilon,\alpha}\|_{\mathcal{Y}} &\leq \alpha\|w^\dagger\|_{\mathcal{X}} + \varepsilon \leq \alpha E + \varepsilon\,, \\ \|e^{\varepsilon,\alpha}\|_{\mathcal{V}^*} &\leq \alpha\|w^\dagger\|_{\mathcal{Y}} + \varepsilon \leq \alpha E + \varepsilon\,. \end{aligned}$$

since the norm in $\mathcal{V}^*$ is given as $\|A\cdot\|_{\mathcal{Y}}$.

Method of Tikhonov: Convergence with source condition in $\mathcal{V}$-2

From

$$e^{\varepsilon,\alpha} = A^*(-w^\dagger + \frac{1}{\alpha}(y^\varepsilon - y^\dagger) - \frac{1}{\alpha}Ae^{\varepsilon,\alpha})$$

we read off

$$\|e^{\varepsilon,\alpha}\|_V \leq \|w^\dagger\|_{\mathcal{Y}} + \frac{\varepsilon}{\alpha} + \frac{1}{\alpha}\|Ae^{\varepsilon,\alpha}\|_{\mathcal{Y}}$$

and we conclude

$$\|e^{\varepsilon,\alpha}\|_{\mathcal{V}} \leq 2(E + \frac{\varepsilon}{\alpha})$$

By the interpolation inequality we obtain

$$\|e^{\varepsilon,\alpha}\|_X \leq \sqrt{2\alpha}(E + \frac{\varepsilon}{\alpha})$$

If we choose

$$\alpha(\varepsilon) := \frac{\varepsilon}{E}\,,$$

then we obtain finally

$$\|e^{\varepsilon,\alpha(\delta)}\|_{\mathcal{X}} \leq 2\sqrt{2}\sqrt{E\,\varepsilon}$$

Method of Tikhonov: Convergence with source condition in $\mathcal{U}$

Theorem

Suppose that the assumptions A0),A1),A2) hold and that the source condition

$$x^{\dagger} \in \mathcal{U}\,,\ \|x^{\dagger}\|_{\mathcal{U}} \leq E \ \textit{for some}\ E > 0$$

is satisfied. Let the regularization parameter $\alpha = \alpha(\varepsilon)$ be chosen as follows:

$$\alpha(\varepsilon) = \varepsilon^{\frac{2}{3}}\, E^{-\frac{2}{3}}\,,\ \varepsilon > 0\,.$$

Then

$$\|x^{\varepsilon,\alpha(\varepsilon)} - x^{\dagger}\|_{\mathcal{X}} \leq 2E^{\frac{1}{3}}\,\varepsilon^{\frac{2}{3}}\,.$$

Proof:

The proof is a little bit more tricky.

Method of Tikhonov: Convergence with source condition in $\mathcal{U}$-1

Let $\alpha > 0$. We have

$$A^*Ae^{\varepsilon,\alpha} + \alpha e^{\varepsilon,\alpha} = -\alpha A^*Aw^\dagger + A^*(y^\varepsilon - y^\dagger)$$

with $\|w^\dagger\|_{\mathcal{X}} \le E$. Then $e^{\varepsilon,\alpha} = e^1 + e^2$ where e^1, e^2 solve

$$A^*Ae^1 + \alpha e^1 = -\alpha A^*Aw^\dagger \,,\; A^*Ae^1 + \alpha e^1 = A^*(y^\varepsilon - y^\dagger)\,,$$

respectively. It is easy to see that

$$\|Ae^1\|_{\mathcal{Y}} \le \varepsilon \,,\; \|e^1\|_{\mathcal{X}} \le \frac{\varepsilon}{\sqrt{\alpha}}\,.$$

We have $e^2 \in \mathcal{U}$ and since A^*A is an isomorphism from $\mathcal{X}$ onto $\mathcal{U}$ by construction we may consider e^2 as a solution of

$$e^2 + \alpha(A^*A)^{-1}e^2 = -\alpha w^\dagger\,.$$

Since $(A^*A)^{-1}$ is nonnegative we obtain the estimate

$$\|e^2\|_{\mathcal{X}} \le \alpha\|w^\dagger\|_{\mathcal{X}} \le \alpha E\,.$$

Therefore we have proved the estimate

$$\|e^{\varepsilon,\alpha}\|_{\mathcal{X}} \le \frac{\varepsilon}{\sqrt{\alpha}} + \alpha E$$

and the parameter choice strategy leads to the result.