The elliptic case

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Outline

- The inverse problem in groundwater flow
- Electrical impedance tomography
- Neumann to Dirichlet mapping
- Weak solutions
- Parameter-to-solution map
- The inverse problem

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References

M. Burger
Parameter identification
Lecture Notes, University Münster, 2008

H.T. Banks and K. Kunisch
Estimation Techniques for Distributed Parameter Systems
Birkhäuser, 1989

B. Kaltenbacher
Parameter identification in partial differential equations
Lecture Notes, University Stuttgart, 2008

Inverse problem in groundwater flow

A twodimensional model

$$-\nabla(q\nabla u) \;=\; r \text{ in } \Omega \subset \mathbb{R}^2$$

Here: u piezometric head, r given function, q unknown (transmissivity) parameter.

Determine from the solution u the transmissivity q of a porous medium

This is considered in an outstanding paper:

G.R. Richter
An inverse problem for the steady state diffusion equation
SIAM J. Appl. Math. 41, 1981

Inverse problem in groundwater flow-1

See also

I. Knowles

Uniqueness for an elliptic inverse problem

SIAM Journal on Appl. Math. 59, 1999

K. Ito and K. Kunisch

On the injectivity and linearization of the coefficient-to-solution mapping for elliptic boundary value problems

J. of Math. Anal. and Appl. 188, 1994

M. Hanke

A regularizing Levenberg–Marquardt scheme, with applications to inverse groundwater filtration problems

Inverse Problems 13, 1997

Inverse problem in groundwater flow-2

Assuming sufficient regularity, the equation may be stated as follows:

Hyperbolic description

$$L(q,u) := -\nabla q.\nabla u - q\Delta u \quad = \quad r \textit{ in } \Omega \subset \mathbb{R}^2 \;\; (\Omega \textit{ open and bounded})$$

How to solve this hyperbolic equation for the unknown q ?

- Consider characteristics (curves of steepest descent of u)
- The domain Ω should be covered by a family of characteristics of the solution u .
- The part Γ of $\partial\Omega$ where $\partial_\nu u$ is negative is called the *inflow region*.

The characteristic curve $s \longmapsto \xi(s)$ through (ξ_1^0, ξ_2^0) is the solution of

$$\frac{d\xi}{ds} = \frac{\nabla u(\xi)}{|\nabla u(\xi)|}\,,\ \xi(0) = (\xi_1^0, \xi_2^0) \;\; (s \text{ arclength }) .$$

Idea: Then one can compute the value of q along a characteristic since

$$\frac{d}{ds} q(\xi(s))|\nabla u(\xi(s)))|^{-1} + q(\xi(s))\Delta u(\xi(s)) = r(\xi(s))$$

Inverse problem in groundwater flow-3

Assumptions:

- $u \in C^2(\overline{\Omega})$.
- $r \in L_\infty(\Omega)$.
- $q \in L_\infty(\Omega)$, q continuous.
- q differentiable along the characteristics.

The discussion of the characteristics can be done under three different assumptions:

(1) $\inf_{\xi \in \Omega} |\nabla u(\xi)| > 0$

(2) $\inf_{\xi \in \Omega} \Delta u(\xi) > 0$

(3) $\inf_{\xi \in \Omega} \max\{|\nabla u(\xi)|, \Delta u(\xi)\} > 0$

Inverse problem in groundwater flow-4

Case (1): There is no critical point of u. The characteristic curves start in a boundary point and end in a boundary point and the whole domain is covered by characteristis. Where the characteristics start (inflow region) the parameter q has to be known. Then q can be computed in Ω.
Case (2): There is at most one critical point since u is a strongly convex function. In a critical point the value of q is known y looking at the hyperbolic equation. In other points one analyses the characteristics as in (1).
Case (3): There is at most one critical point since in a critical point we have local strong convexity, and if there would exist two critical points there should be at least a saddle point. The key property which one can deduce is that the domain Ω can be decomposed in regions where (1) or (2) applies.

Domain Ω (Just for demonstration)

$$-\nabla(q\nabla u) = -\nabla q \cdot \nabla u + q\Delta u = r$$

Characteristic $\frac{d\xi}{ds} = \frac{\nabla u(\xi)}{|\nabla u(\xi)|}$ (s arclength)

Inverse problem in groundwater flow/References

There are various papers which discuss the elliptic identication from the numerical point of view (mostly by using finite elements). See for instance

- G.R. Richter
  Numerical identification of a spatially varying diffusion coefficient
  Mathematics of Computation 36, 1981
- R.S. Falk
  Error estimates for the numerical identification of a variable coefficient
  Math. Comp. 40, 1983
- R. Rannacher and B. Vexler
  A priori estimates for the finite element discretization of elliptic parameter identification problems with pointwise measurements
  SIAM J. Control Optim. 44, 2005

Calderon problem/Electrical impedance tomography

In 1980 A. P. Calderón published a short paper:

A.P. Calderón
On an inverse boundary value problem
Seminar on Numerical Analysis and its Applications to Quantum Physics, Rio de Janeiro, 1980

This pioneer contribution motivated many developments in inverse problems, in particular in the construction of solutions of partial differential equations to solve several inverse problems. The problem that Calderón considered was whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is known as *Electrical Impedance Tomography (EIT)*. Calderón was motivated by oil prospection. In the 40's of the last century he worked as an engineer for Yacimientos Petroliferos Fiscales, the state oil company of Argentina and he thought about this problem then although he did not publish his results until many years later.

Calderon problem/Electrical impedance tomography-1

Electrical impedance tomography

This is the technique to recover spatially distributed properties in the inaccessible interior of a body from electrical measurements. Applications: medical imaging, nondestructive testing, ...

Modelling

$$\begin{aligned} \nabla(q\nabla z) &= \theta \text{ in } \Omega \\ q\partial_\nu z &= g \text{ on } \partial\Omega \\ z &= f \text{ on } \partial\Omega \end{aligned}$$

- The problem is overdetermined.
- $q = q(\xi)$ electric conductivity inside a body $\Omega \subset \mathbb{R}^2/\mathbb{R}^3$
- z electric potential
- g (applied) electrical current at the boundary $\partial\Omega$
- f (measured) voltage at the boundary $\partial\Omega$

Neumann to Dirichlet mapping/Smooth conductivity

Admissible parameter:

$$Q_{\mathrm{ad}} := \{q \in C^2(\Omega) : \gamma \leq q(\xi), \xi \in \Omega \text{ for some } \gamma > 0\}$$

Forward problem: Given $q \in Q_{\mathrm{ad}}$ and $g \in C^2(\partial\Omega)$ solve

$$\begin{aligned} \nabla(q\nabla z) &= \theta \text{ in } \Omega \\ q\partial_\nu z &= g \text{ on } \partial\Omega \end{aligned}$$

This solution z exists in $C^2(\Omega)$ (uniquely determined!).

Consequence: Given $q \in Q_{\mathrm{ad}}$ we may consider

$$\Lambda_q : \mathrm{dom}(\Lambda_q) \ni g \longmapsto z_{|\partial\Omega} \in C^2(\partial\Omega)$$

where $z = \Lambda_q(g)$ is the solution of the forward problem above. Here the domain of definition $\mathrm{dom}(\Lambda_q)$ takes into account a normalization condition:

$$\mathrm{dom}(\Lambda_q) := \{g \in C^2(\partial\Omega) : \int_{\partial\Omega} g\, ds = 0\}$$

Neumann to Dirichlet mapping-1

Inverse problem
Determine q from the knowledge of the **Neumann to Dirichlet mapping (NtD)** Λ_q

The classical analysis in $\mathbb{R}^3$ is based on a transformation to a Schrödinger-type equation. The pair (q, z) is transformed in a pair (c, w) by the transformation

$$w := \sqrt{q}z \text{ and } c := \frac{1}{4}\frac{|\nabla q|^2}{q^2} - \frac{1}{2}\frac{\Delta q}{q}\,.$$

Then (c, w) solves

$$\Delta w + cw = f \text{ on } \partial\Omega$$

Identifiability:
c is identifiable due to the denseness of the products of harmonic functions in the space of $L_2(\Omega)$-functions with values in $\mathbb{C}$.

Neumann to Dirichlet mapping – additional references

For a self-contained proof see

B. Kaltenbacher

Parameter identification in partial differential equations

Lecture Notes, University Stuttgart,2008

For the solution of the problem in $\mathbb{R}^2$ see

A.I. Nachman

Global uniqueness for a two dimensional inverse boundary value problem

Ann. of Mathematics 143, 1996

K. Astala and Päivärinta

Calderóns inverse conductivity problem in the plane

Ann. of Mathematics 163, 2006

Conclusion

The identification problem by using the Neumann to Dirichlet map may be considered solved if the parameter is assumed to be smooth.

Neumann to Dirichlet mapping/Factorization analysis

Now: The body Ω is homogeneous except in cavities $C_1, \dots C_I$.

$$C := \cup_{i=1}^{I} C_i \,.$$

Assumption: $\overline{C} \subset \Omega$, $\Omega \backslash \overline{C}$ connected

How to decide in a nondestructive way whether a point $\zeta \in \Omega$ belongs to C ?

Compare the solutions u, u^0 of

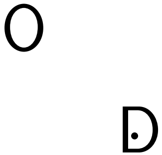
$$\begin{array}{rclcrcl} \Delta u & = & \theta \text{ in } \Omega\backslash\overline{C} & \qquad & \Delta u^0 & = & \theta \text{ in } \Omega \\ \partial_\nu u & = & g \text{ on } \partial\Omega & & \partial_\nu u^0 & = & g \text{ on } \partial\Omega \\ \partial_\nu u & = & \theta \text{ on } \partial C & & & & \end{array}$$

where

$$\begin{aligned} u \in H^1_\diamond(\Omega\backslash\overline{C}) &:= \left\{ v \in H^1(\Omega\backslash\overline{C}) : \int_{\partial\Omega} v\, ds = 0 \right\} \\ u^0 \in H^1_\diamond(\Omega) &:= \left\{ v \in H^1(\Omega) : \int_{\partial\Omega} v\, ds = 0 \right\} \end{aligned}$$

Cavities (Just for demonstration)

$$C = O \cup D$$



$$\begin{aligned} \Delta u &= \theta \text{ in } \Omega \setminus \overline{C} \\ \partial_\nu u &= g \text{ on } \partial\Omega \\ \partial_\nu u &= \theta \text{ on } \partial C \\ u &= f \text{ on } \partial\Omega \end{aligned}$$

$$\begin{aligned} \Delta u^0 &= \theta \text{ in } \Omega \\ \partial_\nu u^0 &= g \text{ on } \partial\Omega \\ u^0 &= f^0 \text{ on } \partial\Omega \end{aligned}$$

Result: Two (NtD)s:

$$\begin{aligned} \Lambda \quad &: \quad L^2_\diamond(\partial\Omega) \ni g \longmapsto f \in L^2_\diamond(\partial\Omega) \\ \Lambda_0 \quad &: \quad L^2_\diamond(\partial\Omega) \ni g \longmapsto f^0 \in L^2_\diamond(\partial\Omega)\,. \end{aligned}$$

Here: $L^2_\diamond(\partial\Omega) := \left\{ g \in L^2(\partial\Omega) : \int_{\partial\Omega} g\, ds = 0 \right\}$.

Observation The difference in the boundary potentials, $h := f - f^0$, is a function in the range of the operator $L := \Lambda - \Lambda_0$. Hence, ran(L) should be used to decide whether there are cavities.

Idea Find for each point $\zeta \in \Omega$ a function h_ζ with the property $h_\zeta \in$ ran(L) iff $\zeta \in C$. Actually, this works with a slightly change. Hence, we have to take our focus on properties of L .

Neumann to Dirichlet mapping/Factorization analysis-2

Fact

The operator $L := \Lambda - \Lambda_0 : L^2_\diamond(\partial\Omega) \longrightarrow L^2_\diamond(\partial\Omega)$ *posesses the following properties:*

(1) *L is linear and bounded.*

(2) *L is selfadjoint.*

(3) *L is positive definite.*

(4) *L is compact and ran(L) dense in $L^2_\diamond(\partial\Omega)$ but not closed.*

A. Kirsch
An Introduction to the Mathematical Theory of Inverse Problems
Springer, Second Edition, 2011

Neumann to Dirichlet mapping/Factorization analysis-3

Since L is selfadjoint, positive definite and compact, L has an eigenvalue decomposition

$$L = \sum_{j=1}^{\infty} \lambda_j \langle e^j | \cdot \rangle e^j$$

with $\lambda_j > 0, j \in \mathbb{N}, \lim_j \lambda_j = 0, \langle e^j, e^i \rangle = \delta_{ij}, i, j = 1, \ldots, \infty$.
From

$$L = \sum_{j=1}^{\infty} \lambda_j \langle e^j | \cdot \rangle e^j$$

we obtain the family $(L^s)_{s>0}$ of operators generated by L where

$$L^s = \sum_{j=1}^{\infty} \lambda_j^s \langle e^j | \cdot \rangle e^j$$

with

$$\mathrm{ran}(L^s) = \{g \in L^2_\diamond(\partial\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2s} |\langle e^j | g \rangle|^2 < \infty\}.$$

Consequence:
Each L^s has the properties linearity, selfadjointness, compactness.

Let $\zeta \in \Omega$ and let

$$D_\zeta(\xi) := \frac{1}{\omega_d} \frac{(\zeta - \xi) \cdot a}{|\zeta - \xi|^d}\, , \xi \in \Omega \backslash \{\zeta\}$$

the dipole in ζ with axis $a \in \mathbb{R}^d, |a| = 1$. Here ω_d is the surface measure of the unit sphere in $\mathbb{R}^d$. Let u_ζ be the solution of

$$\begin{aligned} \Delta u_\zeta &= 0 \text{ in } \Omega \\ \partial_\nu u &= -\partial_\nu D_\zeta \text{ on } \partial\Omega \end{aligned}$$

Neumann to Dirichlet mapping/Factorization analysis-6

Fact

(1) $H_\zeta := D_\zeta + u_\zeta$ *is harmonic in* $\Omega\backslash\{\zeta\}$.

(2) $\partial_\nu H_\zeta = \theta$ *on* $\partial\Omega$.

(3) *If* $h_\zeta :=$ *trace* (H_ζ) *on* $\partial\Omega$ *belongs to* $ran(L)$ *then* ζ *in* C.

(4) *In general, there exist* $\zeta \in C$ *with* $h_\zeta \notin ran(L)$.

(5) $\zeta \in ran(L^{\frac{1}{2}})$ *iff* $\zeta \in C$.

Ad (3) This follows from the uniqueness of an extension of the harmonic function H_ζ onto $\Omega\backslash\{\zeta\}$.
Ad (5) We have to refer to the references.

- The property (5) above opens the door for a variety of algorithms to determine the inclusion domain.
- Generalization: Reconstruction from current-voltage measurements from a part Σ of $\partial\Omega$ only.
- Monotonicity: Testing of L by small anomalities.

Neumann to Dirichlet mapping/Additional references

B. Harrach
Recent progress on the factorization method for electrical impedance tomography
Comp. and Math. Methods in Medicine, Article ID 425184, 2013

M. Hanke and M. Brühl
Recent progress in electrical impedance tomography
Inverse Problems 19, 2003

B. Harrach
Inverse coefficient problems in elliptic partial differential equations
Proceedings, Technical University Munich, 2015

Lax Milgram Lemma (bounded form)

Theorem (Lax-Milgram)

Let $\mathcal{H}$ be a Hilbert space and let $a : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ be a bilinear mapping which satisfies

- $|a(u,v)| \leq \gamma_0 \|u\|_{\mathcal{H}} \|v\|v\|_{\mathcal{H}}\,,\ u, v \in \mathcal{H}\,,$
- $a(u,u) \geq \gamma \|u\|^2_{\mathcal{H}}\,,\ u \in \mathcal{H}\,,$ *with* $\gamma > 0\,.$

Then there is linear mapping $A : \mathcal{H} \longrightarrow \mathcal{H}^*$ defined by
$a(u,v) = \langle A(u), v\rangle, u, v \in \mathcal{H}\,.$
Moreover, A is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}^*$ with

$$\|Au\|_{\mathcal{H}^*} \leq \gamma_0 \|u\|_{\mathcal{H}}\,,\ \|A^{-1}\lambda\|_{\mathcal{H}} \leq \gamma^{-1}\|\lambda\|_{\mathcal{H}^*}\,,\ u \in \mathcal{H}, \lambda \in \mathcal{H}^*\,.$$

Proof:

This theorem follows by an application of the Zarantonello-Theorem (last lecture).

Lax Milgram Lemma (unbounded form)

Definition

Let $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ *be a Gelfand triple and let* $a : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ *be a bilinear mapping.*

- *a is called* $\mathcal{V}$-**continuous** *iff* $|a(u,v)| \leq \gamma_0 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, u, v \in \mathcal{V}$.
- *a is called* $\mathcal{V}$-**coervive** *iff* $a(u,u) \geq \gamma \|u\|_{\mathcal{V}}^2, u \in \mathcal{V}$.

Here $\gamma_0 \geq 0, \gamma > 0$.

Theorem (Lax-Milgram)

Let $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ *be a Gelfand triple and let* $a : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ *be a* $\mathcal{V}$*-continuous,* $\mathcal{V}$*-coercive bilinear form. Then there exists a linear continuous bijective mapping* $A : \mathcal{V} \longrightarrow \mathcal{V}^*$ *with*

$$a(u,v) = \langle A(u), v \rangle \, , \; u, v \in \mathcal{V} \, .$$

Moreover,

$$A^{-1} : \mathcal{V}^* \longrightarrow \mathcal{V} \text{ is continuous with } \|A^{-1}\lambda\|_{\mathcal{V}} \leq \gamma^{-1} \|\lambda\|_{\mathcal{V}^*} \, , \; \lambda \in \mathcal{V}^* \, .$$

Lax Milgram Lemma (unbounded form)-1

Proof:
Let $u \in \mathcal{V}$. Since $v \longmapsto a(u, v)$ is linear and continuous there exists (a uniquely determined) $z_u \in \mathcal{V}$ such that $a(u, v) = \langle z_u | v \rangle_{\mathcal{V}}$ for all $v \in \mathcal{V}$. This defines a linear continuous mapping $A' : \mathcal{V} \ni u \longmapsto z_u \in \mathcal{V}$ with

$$a(u, v) = \langle A'u | v \rangle_{\mathcal{V}} \, , \; u \in \mathcal{V} \, .$$

Moreover, we read off

$$\|A'u\|_{\mathcal{V}} \leq \gamma_0 \|u\|_{\mathcal{V}} \, , \; \|A'u\|_{\mathcal{V}} \geq \gamma \|u\|_{\mathcal{V}} \, , \; u, v \in \mathcal{V} \, .$$

This shows that $A' : \mathcal{V} \longrightarrow \mathcal{V}$ is an isomorphism. Using the Riesz mapping $R_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}^*$ we obtain with $A := R_{\mathcal{V}} \circ A'$ an isomorphism $A : \mathcal{V} \longrightarrow \mathcal{V}^*$ with

$$\langle Au, v \rangle = \langle A'u | v \rangle_{\mathcal{V}} = a(u, v) \, , \; u, v \in \mathcal{V} \, .$$

Now, the proof is complete.

Existence of the parameter to solution

Consider the (BVP)

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= \theta \text{ in } \partial\Omega \end{aligned}$$

where L is given as

$$Lu = -\sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u) - \sum_{j=1}^{d} b^j \partial_j u + cu$$

This boundary value problem (BVP) should be considered in the Gelfand triple

$$\mathcal{V} := H_0^1(\Omega) \hookrightarrow \mathcal{H} := L_2(\Omega) \hookrightarrow \mathcal{V}^* := H^{-1}(\Omega)$$

Existence of the parameter to solution map-1

Assumptions:

(1) $\Omega \subset \mathbb{R}^d$ open and bounded.

(2) All coefficients a^{ij}, b^j, c are L_∞-functions.

(3) $a^{ij}(\xi) = a^{ji}(\xi)$ for all $i, j = 1, \dots, d, \xi \in \Omega$.

(4) L is strongly elliptic, i.e. there exists a constant $\delta > 0$ such that

$$\sum_{i,j=1}^{d} a^{ij}(\xi)\zeta_i\zeta_j \geq \delta \sum_{i,j=1}^{d} \zeta_i^2\,,\ \xi \in \Omega, \zeta = (\zeta_1, \dots, \zeta_d) \in \mathbb{R}^d\,.$$

(5) $f \in H^{-1}(\Omega)$.

Now, we define the bilinear mapping a on $H_0^1(\Omega) \times H_0^1(\Omega)$ as follows:

$$\begin{aligned} a(u,v) \ &:= \ \sum_{i,j=1}^{d} \int_\Omega a^{ij}(\xi)\partial u(\xi)\partial v(\xi)d\xi \\ &\quad + \sum_{j=1}^{d} \int_\Omega b^j(\xi)\partial u(\xi)v(\xi)d\xi + \int_\Omega c(\xi)u(\xi)v(\xi)d\xi \end{aligned}$$

Existence of the parameter to solution map-2

Definition

$u \in H_0^1(\Omega)$ is called a **weak solution** *of the (BVP) iff*

$$a(u,v) = \langle f, v\rangle \quad \textit{for all } v \in H_0^1(\Omega)\,.$$

Due to the assumptions concerning the coefficients we obtain the following estimates:

$$\begin{aligned} |a(u,v)| &\leq \gamma_0 \|u\|_{H_0^1} \|v\|_{H_0^1} \text{ for all } u,v \in H_0^1(\Omega) \\ a(u,u) &\geq \gamma \|u\|^2_{H_1^0(\Omega)} - \beta \|u\|^2_{L_2(\Omega)} \text{ for all } u \in H_0^1(\Omega) \end{aligned}$$

This enables us to use the Lax Milgram Lemma to get a weak solution in the case that β vanishes. This is especially the case if the coefficients b^j, c vanish.

Here is reference to an outstandig book on partial differential equations:

L.C. Evans
Partial differential equations
American Mathematical Society, 2010

Existence of the parameter to solution map-3

The elliptic parameter identification should be considered in the following framework.

(1) Given a Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ of Hilbert spaces which is used to describe the state.

(2) Given a Gelfand triple $\mathcal{Q} \hookrightarrow \mathcal{P} \hookrightarrow \mathcal{Q}^*$ of Hilbert spaces which is used to describe the parameter.

(3) Given for each $q \in \mathcal{Q}$ a bilinear form $a(q;\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$.

(4) Q_{ad} is a subset of $\mathcal{Q}$ which describes the admissible parameters.

(5) For each $q \in Q_{\mathrm{ad}}$ there exist constants $\gamma_0 \geq 0, \gamma(q) > 0$ such that

$$|a(q;u,v)| \leq \gamma_0 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}\,,\ a(q;u,u) \geq \gamma(q)\|u\|^2_{\mathcal{V}} \text{ for all } u,v \in \mathcal{V}\,.$$

(6) Given $f \in \mathcal{V}^*$.

Existence of the parameter to solution map-4

Usually, the bilinear map a is decomposed in parameter dependent part and a parameter independent part:

$$a(q; u, v) = a_0(u, v) + a_1(q; u, v)$$

Then the assumption in (5) above must be replaced by

(5) For each $q \in Q_{\text{ad}}$ there exist constants $\gamma_0 \geq 0, \gamma_1 \geq, \gamma(q) > 0$ such that

$$|a_0(u, v)| \leq \gamma_0 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}\,,\ |a_1(q; u, v)| \leq \gamma_1 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}\,,\ u, v \in \mathcal{V}$$

and

$$a_0(u, u) + a_1(q; u, u) \geq \gamma(q) \|u\|_{\mathcal{V}}^2 \text{ for all } u \in \mathcal{V}\,,$$

for some $\gamma(q) > 0$.

Existence of the parameter to solution map-5

Parameter to solution mapping (PtS)

$$F : Q_{ad} \ni q \longmapsto u \in \mathcal{V}$$

where $u := F(q)$ *solves the variational equation*

$$a_0(u,v) + a_1(q;u,v) = \langle f, v\rangle \quad \text{for all } v \in \mathcal{V}\,.$$

The **inverse problem of parameter identification** in this framework consists in solving the equation $F(q) = y$ where $q \in \mathcal{Q}, y \in \mathcal{V}$. Unfortunately, in practice we have the following situation:

- *Given the pair* $(q^\dagger, y^\dagger) \in Q_{ad} \times \mathcal{V}$ *with* $F(q^\dagger) = y^\dagger$.
- *Given an approximation* $y^\varepsilon \in \mathcal{H}$ *with* $\|y^\varepsilon - y^\dagger\|_{\mathcal{H}} \leq \varepsilon, \varepsilon > 0$.
- *Find a reasonable approximation* q^ε *for* $q^\dagger$ *using the data* y^ε *only.*

Notice that the point to solution mapping should adapted: the image space of F should be $\mathcal{H}$.