Model reference adaptive systems-complete observation

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Outline

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- Modelling
- Weak solution and properties of the plant
- Derivation of the method
- Output–identifiability
- Parameter–identifiablity
- Stopping rule
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Introduction

Let us consider a very simple ordinary differential equation modelling a system:

$$z' + pz = f$$

Suppose we want to determine the parameter p. This determination should be done during the operation of the system in the time interval $[0, \infty)$ by observation of the state z.

Idea Consider a reference system which parallels the dynamic of the process model, feed this reference system by a parameter $t \longmapsto q(t)$ which is adapted by matching the state of the modelled system and the state of the reference system.

Implementation

$$\begin{array}{rcll} u' + u - z(t) + q(t)z(t) & = & f & , u(0) = u^0 \\ q' - x(t)(u - z(t)) & = & 0 & , q(0) = q^0 \end{array}$$

If we define the error quantities $e := u - z, r := q - p$ we obtain the error system

Introduction-1

$$
\begin{aligned}
e' + e + rz(t) &= 0 \quad , e(0) = u^0 - x(0) \\
r' - z(t)e &= 0 \quad , r(0) = q^0 - p
\end{aligned}
$$

Now we multiply the first equation by e, the second equation by r, and add the equations. This gives at the time t

$$e'(t)e(t) + |e(t)|^2 + r(t)z(t)e(t) + r'(t)r(t) - z(t)e(t)r(t) = 0$$

and we see that the error system becomes

$$\frac{d}{dt} L(e(t), r(t)) = -|e(t)|^2$$

where $L(e) := \frac{1}{2}|e|^2 + \frac{1}{2}|r|^2$ can be considered as a Ljapunov-function for the error system. Integrating this from t_0 to T gives

$$\frac{1}{2}|e(T)|^2 + \frac{1}{2}|r(T)|^2 + \int_{t_0}^{T} |e(t)|^2 dt = \frac{1}{2}|e(t_0)|^2 + \frac{1}{2}|r(t_0)|^2 \,, \; T \geq t_0 \,.$$

Introduction-2

Fact

(1) $T \longmapsto |e(T)|^2 + |r(T)|^2$ *is monotone decreasing (but not* $T \longmapsto |r(T)|^2$*, in general).*

(2) $\sup_{t\geq 0}\{|e(t)|^2 + |r(t)|^2\} + \int_0^\infty |e(t)|^2 dt < \infty$

Observation

- It should be „easy“ to prove $\lim_{t\to\infty} e(t) = 0$.
- The goal property $\lim_{t\to\infty} r(t) = \lim_{t\to\infty}(q(t) - p) = 0$ cannot be proved without an addidional assumptions; see the case $f = 0, z(0) = 0$.
- We could try compute p by the formula

$$p = \frac{1}{z(t)}(f(t) - z'(t))$$

(by using numerical differentiation if the data z are noisy).

References

Now, we make big step from this tutorial example to the adaptive identification of parameters in time dependent partial differential equations. We follow mainly

W. Scondo
Ein Modellabgleichungs Verfahren zur adaptiven ...
Dissertation, Goethe Universität Frankfurt/Main, 1987

J. Baumeister, W. Scondo, M.A. Demetriou and I.G. Rosen
On-line parameter estimation for infinitedimensional dynamical systems
SIAM J. Control Optim. 35 (1997), 678-713

See also

R. Boiger and B. Kaltenbacher
A online parameter identification method for time dependent partial differential equations
Inverse Problems, 32 (2016), 28 pp.

P. Kügler
Online parameter identification without Ricatti-type equations in a class of time-dependent partial differential equations: an extended state approach with potential to partial observations
Inverse Problems, 26 (2010)

Modelling

Model equation

$$D_t w + A_0(w) + A(q,w) \; = \; f(t) \; in \; \mathcal{V}^*, t \in (0,\infty); \; w(0) \; = \; \zeta$$

Compare for example with

$$D_t w + cw - \nabla(q\nabla w) = f(t), t \in (0,\infty); \; w(0) \; = \; \zeta$$

c known, q unknown.

$$\text{(Model data)} \quad \left\{ \begin{array}{l} \mathcal{V}, \mathcal{H}, \mathcal{Q}, \mathcal{P} \text{ Hilbert spaces} \\ \mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^* \,, \, \mathcal{Q} \hookrightarrow \mathcal{P} \hookrightarrow \mathcal{Q}^* \\ A_0(\cdot) : \mathcal{V} \; \longrightarrow \; \mathcal{V}^* \text{ linear} \\ w(t) \in \mathcal{V} \text{ state at time} \\ Q_{\text{ad}} \subset \mathcal{Q} \text{ set of admissible parameters} \\ q \in \mathcal{Q}_{\text{ad}} \text{ parameter (to be identified)} \\ A(\cdot,\cdot) : \mathcal{Q} \times \mathcal{V} \; \longrightarrow \; \mathcal{V}^* \text{ bilinear} \\ \zeta \in \mathcal{H} \text{ initial state} \\ f : (0,\infty) \; \longrightarrow \; \mathcal{V}^* \end{array} \right.$$

Weak solution and properties of the plant

Weak solution $w : [0, \infty) \longrightarrow \mathcal{H}$ with

- $w(t) \in \mathcal{V}, t > 0,$
- $\langle D_t w(t), v\rangle + \langle A_0(w(t)), v\rangle + \langle A(q, w(t)), v\rangle = \langle f(t), v\rangle$ for all $t > 0, v \in \mathcal{V}$
- $w(0) = \zeta$

Assumptions

(1) $A_0 : \mathcal{V} \longrightarrow \mathcal{V}^*$ linear and continuous, i.e. $|\langle A_0(u), v\rangle| \leq c_0 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, u, v \in \mathcal{V},$ with a constant $c_0 \geq 0$.

(2) $A(\cdot, \cdot) : \mathcal{Q} \times \mathcal{V} \longrightarrow \mathcal{V}^*$ bilinear and with a constant $c_q \geq 0$

$$|\langle A(q, u), v\rangle| \leq c_q \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, u, v \in \mathcal{V}, q \in \mathcal{Q}_{\text{ad}},$$

(3) $\langle A_0(v), v\rangle + \langle A(q, v), v\rangle + \beta_0 \|v\|^2_{\mathcal{H}} \geq c_1 \|v\|^2_{\mathcal{V}}$ for all $(q, v) \in \mathcal{Q}_{\text{ad}} \times \mathcal{V}$ where $\beta_0, c_1 \in \mathbb{R}$ with $c_1 > 0$.

(4) $f \in L_2(0, T; \mathcal{V}^*)$ for all $T > 0$

Weak solutions

Fact

If assumptions above hold then there exists for each $q \in \mathcal{Q}_{ad}$ a uniquely determined weak solution w of the model equation which satisfies:

$$z \in W((0,T);\mathcal{V}) \text{ for all } T > 0\,,\ z \in C([0,\infty);\mathcal{H})\,.$$

The space $W((0,T);\mathcal{V})$ is defined as follows:

$$W((0,T);\mathcal{V}) := \{w : (0,T] \longrightarrow \mathcal{V} : D_t w(t) \in \mathcal{V}^* \text{ for all } t \in (0,T]\}\,.$$

$W((0,T);\mathcal{V})$ is a Hilbert space and each function in $w \in W((0,T);\mathcal{V})$ satisfies $w \in C([0,T];\mathcal{H})$.

Plant

Let $p \in Q_{\text{ad}}$.

Model equation for the process

$$D_t z + A_0(z) + A(p,z) \ = \ f(t) \ in \ \mathcal{V}^*, t \in (0,\infty); \ z(0) \ = \ \zeta$$

We know by the theorem above that the model equation for the process has a uniquely determined solution z .

Definition

The pair (p,z) is a **plant** *iff*

- *z is a weak solution of the model equation*
- $|\langle A(q,z(t)),v\rangle| \leq c_2\|q\|_{\mathcal{Q}}\|v\|_{\mathcal{V}}$, $t \geq 0, q \in \mathcal{Q}, v \in \mathcal{V}$, *with a constant* $c_2 \geq 0$

The second condition is essential the assumption that a plant is in $L_\infty([0,\infty);\mathcal{V})$.

Reference system and adaptive rule

Reference model

$$D_t u + A_0(z(t)) + A(q(t), z(t)) + C(u - z(t)) \ = \ f(t) \text{ in } \mathcal{V}^*\,,\ t > 0$$

(a) How to choose C?

(b) How to create $t \longmapsto q(t)$?

(c) $\lim_{t\to\infty}(q(t) - p) = \theta$?

(d) Necessity of $\lim_{t\to\infty}(u(t) - z(t)) = \theta$

Adaptation rule/Ansatz:

$$D_t q + F(z(t), u - z(t)) = \theta\,,\ t > 0$$

Error quantities

$$e(t) := u(t) - z(t), r(t) := q(t) - p\,,\ t \geq 0\,.$$

Argumentation with a Ljapunov function

Error system

$$\begin{aligned} D_t e + C(e) + A(r, z(t)) &= \theta\,,\ t > 0\,,\ e(0) = e_0 := \theta \\ D_t r + F(z(t), e) &= \theta\,,\ t > 0\,,\ r(0) = r_0 := q_0 - p\,. \end{aligned}$$

Goal:
$(\theta, \theta) \in \mathcal{P} \times \mathcal{H}$ should be a critical point for the „most simplest" Ljapunov function of the error system.

The most simplest Ljapunov function:

$$L(r, e) := \frac{1}{2}\|r\|_{\mathcal{P}}^2 + \frac{1}{2}\|e\|_{\mathcal{H}}^2$$

Gradient of the Ljapunov function along a solution of the error system is (in an informal computation)

$$\langle D_t e, e\rangle + \langle D_t r, r\rangle = -\langle C(e), e\rangle - \langle A(r, z(t), e\rangle + \langle F(z(t), e), r\rangle$$

Adaptation rule

Conclusion

$$\langle A(r, z(t)), e\rangle = \langle F(z(t), e), r\rangle \, , \; t \geq 0$$

or more general

$$\langle A(q, z(t)), v\rangle = \langle F(z(t), e), q\rangle \text{ for all } t \geq 0, q \in \mathcal{Q}, v \in \mathcal{V}$$

If (p, z) is a plant then

$$|\langle A(q, z(t)), v\rangle| \leq c_2 \|q\|_{\mathcal{Q}} \|v\|_{\mathcal{V}} \, , \; t \geq 0, q \in \mathcal{Q}, v \in \mathcal{V}$$

and hence there exists a continuous bilinear mapping
$b : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{Q}^*$ with

$$\langle A(q, z(t)), v\rangle = \langle b(z(t), v), q) \text{ for all } t \geq 0, q \in \mathcal{Q}, v \in \mathcal{V} \, .$$

Let (p, z) be a plant, then

$$D_t z + A_0(z) + A(p, z) \; = \; f(t) \text{ in } \mathcal{V}^*, t \geq 0, z(0) \; = \; \zeta$$

and the model reference adaptive system is given by

(MRAS)

$$\begin{cases} D_t u + C(u - z(t)) + A_0(z(t)) + A(q, z(t)) & = & f(t) & \text{in } \mathcal{V}^*, t > 0, \\ u(0) & = & \zeta & \\ D_t q - b(z(t), u - z(t)) & = & \theta & \text{in } \mathcal{Q}^*, t > 0, \\ q(0) & = & q_0 & \end{cases}$$

Resulting error system:

$$\begin{aligned} D_t e + Ce + A(r, z(t)) \; &= \; \theta\,, \; t > 0, e(0) \; = \; \theta\,, \\ D_t r - b(z(t), e) \; &= \; \theta\,, \; t > 0, r(0) \; = \; r_0 \; := \; q_0 - p\,. \end{aligned}$$

Parabolic example

Consider the model system

$$\begin{aligned} D_t z - (pz')' &= f(s,t)\,,\ (s,t) \in (0,1)\times(0,\infty) \\ z(0,t) = z(1,t) &= 0\,,\ t>0 \\ z(s,0) &= \zeta(s) \end{aligned}$$

Here, p is the parameter which should be identified.

We choose

$$\mathcal{H} := L_2(0,1), \mathcal{V} = H_0^1(0,1), \mathcal{P} = L_2[0,1], \mathcal{Q} = H^1[0,1]$$

endowed with the usual inner products. Then we have

$$\langle A(q,u), v\rangle = \int_0^1 q(s)u'(s)v'(s)\,ds =$$

and we obtain the adaptation rule $D_t q - z'(s,t)(u' - z'(s,t)) = 0\,.$

Parabolic example-1

The model reference adaptive system can be designed in a classical formulation as follows:

$$\begin{array}{rcl} D_t u - u'' + z''(s,t) - (qz'(s,t))' & = & f(s,t)\,,\ (s,t) \in (0,1) \times (0,\infty) \\ u(s,t) \ = \ u(1,t) & = & 0\,,\ t > 0 \\ u(s,0) & = & \zeta(s)\,,\ s \in (0,1) \\ D_t q - z'(s,t)(u' - z'(s,t)) & = & \theta\,,\ (s,t) \in (0,1) \times (0,\infty) \\ q(s,0) & = & q^0(s) \end{array}$$

The assumptions above can be verified. A plant z is a function satisfying the model system in the weak sense and satisfying

$$\Big| \int_0^1 q(s) z'(s,t) v'(s) ds \Big| \le c_2 \|q\|_{\mathcal{P}} \|v\|_{\mathcal{V}}\,,\ t \ge 0, q \in \mathcal{Q}, v \in \mathcal{V},$$

with a constant $c_2 \ge 0$. This holds if $z' \in Ł_\infty([0,\infty); \mathcal{V})$.

Properties of (MRAS)

Let us come back to (MRAS). We have

$$\langle D_t e(t), e(t)\rangle + \langle D_t r(t), r(t)\rangle + \langle C(e(t), e(t)\rangle = 0$$

Since the gradient of the Ljapunov function L is given by

$$\langle D_t e(t), e(t)\rangle + \langle D_t r(t), r(t)\rangle$$

we should have:

$$\langle C(e(t)), e(t)\rangle > 0\,.$$

Assumption:

(1) $C : \mathcal{V} \longrightarrow \mathcal{V}^*$ is coercive, i.e.
$\langle C(u), u\rangle \geq c_5 \|u\|_{\mathcal{V}}^2, u \in \mathcal{V},$ with a constant $c_5 > 0\,.$

(2) $C : \mathcal{V} \longrightarrow \mathcal{V}^*$ is a linear mapping which is continuous, i.e.
$|\langle C(u), v\rangle| \leq c_6 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, u, v \in \mathcal{V},$ with a constant $c_6 \geq 0\,.$

Properties of (MRAS)-1

Fact

Let (p, z) be a plant and let the additional assumptions be satisfied. Then there exists a uniquely determined solution (e, r) of the error system which satisfies

- $e \in W((0,T);\mathcal{V})$ *for all* $T > 0\,.$
- $e \in C([0,\infty);\mathcal{H}) \cap L_\infty((0,\infty);\mathcal{H}) \cap L_2((0,\infty);\mathcal{V}),\,.$
- $r \in W((0,T),\mathcal{Q})$ *for all* $T > 0\,.$
- $r \in C([0,\infty);\mathcal{Q}) \cap L_\infty((0,\infty);\mathcal{P})\,.$
- $D_t r \in L_\infty((0,\infty);\mathcal{Q}^*)\,.$
- $E : (0,\infty) \ni t \longmapsto E(t) := L(e(t), r(t)) := \frac{1}{2}\|e(t)\|_{\mathcal{H}}^2 + \frac{1}{2}\|r(t)\|_{\mathcal{P}}^2 \in \mathbb{R}$ *is nonincreasing.*
- *For* $0 \le t_0 \le t$

$$\|e(t)\|_{\mathcal{H}}^2 + \|r(t)\|_{\mathcal{P}}^2 + 2c_5 \int_{t_0}^{t} \|e(s)\|_{\mathcal{V}}^2\, ds \le \|e(t_0)\|_{\mathcal{H}}^2 + \|r(t_0)\|_{\mathcal{P}}^2\,.$$

Properties of (MRAS)-2

Moreover

Fact

(1)

$$\sup_{t\in(0,\infty)} (\|e(t)\|^2_{\mathcal{H}} + \|r(t)\|^2_{\mathcal{P}}) + \int_0^\infty \|e(s)\|^2_{\mathcal{V}}\, ds < \infty$$

(2) *Given $l > 0$ we have*

$$\lim_{t_0\to\infty} \left(\sup_{t\in[t_0,t_0+l]} \|e(t)\|^2_{\mathcal{H}} + \int_{t_0}^{t_0+l} \|e(s)\|^2_{\mathcal{V}}\, ds \right) = 0\,.$$

The property (2) above is a key ingredient in proving output-identifiabilty.

Output-identifiability

Fact

Let (p, z) be a plant and let the additional assumptions be satisfied. We have

$$\lim_{t\to\infty} \|u(t) - z(t)\|_{\mathcal{H}} = 0\,,\ r^* := \lim_{t\to\infty} \|q(t) - p\|_{\mathcal{P}} \text{ exists}.$$

Proof:

- $L^* := \lim_{t\to\infty}(\|u(t) - z(t)\|^2_{\mathcal{H}} + \|q(t) - p\|^2_{\mathcal{P}})$ exists due to monotonicity of L. If $\lim_{t\to\infty} \|u(t) - z(t)\|_{\mathcal{H}} = 0$ then r^* exists.
- Assume by contradiction: $\lim_{t\to\infty} \|e(s)\|_{\mathcal{H}}$ does not hold.
- If $l > 0$ we have $\lim_{t_0\to\infty} \|e\|_{[t_0, t_0+l]} = 0$ uniformly in l where

$$\|e\|_{[t_0,t_0+l]} := \sup_{t\in[t_0,t_0+l]} \|e(t)\|_{\mathcal{H}} + \left(\int_{t_0}^{t_0+l} \|e(t)\|^2_{\mathcal{V}}\, dt \right)^{\frac{1}{2}}$$

Output-identifiability-2

- $\forall \tau > 0 \forall \kappa > 0 \exists t_0 > 0 \, \forall \, t_1, t_2 \geq t_0$

$$\left(|t_2 - t_1| \leq \kappa \text{ implies } \big| \, \|e(t_2)\|_{\mathcal{H}} - \|e(t_1)\|_{\mathcal{H}} \big| < \tau \right)$$

Then there exists $\delta > 0$ and a sequence $(t_n)_{n \in \mathbb{N}}$ with

$$t_n > 0, t_{n+1} - t_n \geq 2, n \in \mathbb{N}, \lim_n \|e(s)\|^2_{\mathcal{H}} \geq \delta \, .$$

By using the fact $\|v\|_{\mathcal{H}} \geq \|v\|_{\mathcal{V}}$ we obtain

$$\int_0^\infty \|e(s)\|^2_{\mathcal{V}} \, ds = \infty$$

which is a contradiction.

Parameter identifiability: $lim_{t \to \infty} \|q(t) - p\|_{\mathcal{P}} = 0 \, .$ *???*

Additional assumption !!!

Parameter identifiability

Definition

Let (p, z) be a plant. The state z is **asymptotically persistently excited** *if there exist numbers $l > 0, \mu > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $(0, \infty)$ with $\lim_n t_n = \infty$ such that the following condition holds:*

$$\forall\, h \in \mathcal{Q}\, \forall\, n \in \mathbb{N}\, \exists t_{n,1}, t_{n,2} \in [t_n, t_n + l]\, \exists v \in \mathcal{V} \backslash \{\theta\}$$

$$\left(\left| \int_{t_{n,1}}^{t_{n,2}} \langle A(h, z(t)), v \rangle \, ds \right| \geq \mu \|h\|_{\mathcal{P}} \|v\|_{\mathcal{V}} \right)$$

Fact

Let (p, z) be a plant and let the additional assumptions be satisfied. Suppose that the state z is asymptotically persistently excited. Then we have parameter-identifiability:

$$\lim_{t\to\infty} q(t) = p \text{ in the space } \mathcal{P}\,.$$

Uniformly persistently excited plant

Definition

The state z is **uniformly persistently excited** *if there exist numbers $l > 0, \mu > 0$ such that*

$$\forall\, h \in \mathcal{Q}\, \forall\, t_0 \in (0,\infty)\, \exists t_1, t_2 \in [t_0, t_0 + l]\, \exists v \in \mathcal{V} \backslash \{\theta\}$$

$$\left(\left| \int_{t_1}^{t_2} \langle A(h, z(t)), v \rangle \, ds \right| \geq \mu \|h\|_{\mathcal{P}} \|v\|_{\mathcal{V}} \right)$$

The property *uniformly persistently excited* may be used to show that the convergence in

$$\lim_{t\to\infty} \|u(t) - z(t)\|_{\mathcal{H}}\,,\ \lim_{t\to\infty} (\|q(t) - p\|_{\mathcal{P}}$$

is exponentially fast.

Clearly, the property *uniformly persistently excited* is stronger than the property *asymptotically persistently excited*.

Stopping rule/exact data

Fact

Let (p, z) be a plant and let the additional assumptions be satisfied. Suppose that the state z is uniformly asymptotically persistently excited and let l, μ be chosen as in Definition of this property. Then for all $t_0 > 0$ we have

$$\|r(t_0)\|_{\mathcal{P}} \leq \mu^{-1} c_7 \, \|e\|_{[t_0, t_0+l]} \, ,$$

Stoping rule Let $l > 0$:

Given an accuracy parameter $\sigma > 0$ choose the stopping time $\tau > 0$ as the smallest time t_0 such that

$$\|e\|_{[t_0, t_0+l]} \leq \sigma \, .$$

Remarks

- Convergence without excitation assumption to a set of parameters
- Small error model & uniformly persitently excitation: convergence to a small set of parameters
- General error model: (MRAS) has to regularized.
- Discretization of the scheme may be considered under an error model
- Applicability: ordinary differential equations, differential equations with delay
- Applicability: elliptic, parabolic, hyperbolic equations
- Adaptation rules can be realized with different smoothness requirements
- (MRAS) can be realized as an off-line method (Kaczmarz-type implementation)

An illustrating example concerning richness

We follow

M.A. Demetriou and I.G. Rosen
On the persistence of excitation in the adaptive estimation of distributed parameter systems
IEEE Trans. on Autom. Control 39, 1994

Model equation:

$$\begin{aligned} D_t z - (pz')' &= f(s,t)\,,\ (s,t) \in [0,1] \times (0,\infty) \\ z(0,t) = z(1,t) &= 0, t > 0, \\ z(s,0) &= 0, s \in (0,1) \end{aligned}$$

with p a positive constant, $f(s,t) = \alpha\sqrt{2}\sin(n\pi s)$. We choose

$$\mathcal{H} := L_2(0,1), \mathcal{V} = H_0^1(0,1), \mathcal{P} = \mathcal{Q} = \mathbb{R},$$

endowed with the usual inner products and have

$$\langle A(q,u), v \rangle = q \int_0^1 u'(s)v'(s)\,ds\,.$$

An illustrating example concerning richness-1

The model reference adaptive systems is

$$\begin{aligned}
D_t u - u'' + z''(t) - qz''(t) &= f(s,t)\,,\ (s,t) \in (0,1)\times(0,\infty) \\
u(0,t) \ = \ u(1,t) &= 0\,,\ t \in (0,\infty) \\
u(s,0) &= \zeta(s)\,,\ s \in (0,1) \\
D_t q - z'(t)(u' - z'(t)) &= \theta \text{ in } (0,1)\times(0,\infty) \\
q(s,0) &= q^0(s)\,,\ s \in (0,1)\,.
\end{aligned}$$

The state z of the plant (p,z) is given as follows:

$$z(s,t) = Z(t)\sqrt{2}\sin(n\pi s)\,,\ (s,t) \in (0,1)\times(0,\infty)\,,$$

where

$$Z(t) = \frac{\alpha}{pn^2\pi^2}(1 - \exp(-pn^2\pi^2))\,,\ t > 0\,.$$

An illustrating example concerning richness-2

Let $t_0 \geq 0, l > 0$. For $q \in \mathcal{Q}\backslash\{0\}, v \in \mathcal{V}, \|v\|_{\mathcal{V}} \leq 1$ we have for the defining inequality of *uniformly persistently excited*

$$\begin{aligned} \left| q \int_{t_0}^{t_0+l} \int_0^1 z'(s,t) v'(s)\, dsdt \right| &\geq \frac{|\alpha|}{p} \int_{t_0}^{t_0+l} (1 - \exp(-pn^2\pi^2 t)\, dt \\ &= \frac{|q||\alpha|}{p} (l - \frac{\exp(-pn^2\pi^2 t_0)}{pn^2\pi^2}) =: \kappa \end{aligned}$$

for $v(s) := \frac{1}{n\pi} \sin(n\pi s)$. For t_0 sufficiently large,

$$\kappa \geq \mu_0 |q| \|v\|_{\mathcal{V}} \text{ with } \mu_0 \approx \frac{|\alpha|}{p} l > 0 \,.$$

An illustrating example concerning richness-3

We would expect the following observations:

- The larger μ the better should be the convergence of $\lim_{t\to\infty} q(t) = p$.
- This convergence is influenced also by the operator C in the model reference equation. If we use the operator $C(u) := c^* u''$, we would expect that oszillation of the model reference state u is damped by a large c^* .

A hyperbolic example

Model equation:

$$
\begin{aligned}
D_t^2 z - (pz')' &= f(t) \text{ in } (0,1)\times(0,\infty) \\
z(0,t) = z(1,t) &= 0 \text{ in } (0,\infty) \\
z(s,0) = z^0(s), D_t z(s,0) &= z^1(s,0) \text{ in } [0,1]
\end{aligned}
$$

Numerical simulation:

$$p(s) = 1 + s, z(s,t) = \sin(\pi s + t),$$

$$C(u) := 2u'', q(s,0) = s, u(s,0) = \sin(\pi s)$$

Notice: $z'(1/2,0) = z'(3/2,0) = 0\,.$

An elliptic example

Model equation

$$\begin{aligned} -(pz')' &= f \text{ in } (0,1) \\ z(0) = z(1) &= 0 \end{aligned}$$

Numerical simulation:

$$p(s) = 1 + s, z(s) = \sin(\pi s), q(s,0) = s, u(s,0) = \sin(\pi s)$$

Notice: $z'(1/2, 0) = 0$.

- No pointwise convergence in $s = 1/2$ if we apply the adaptation rule in $\mathcal{Q} := L_2[0,1]$.
- Pointwise convergence in $[0,1]$ if we apply the adaptation rule in $\mathcal{Q} := H^1[0,1]$.

Outlook

Model equation

$$D_t z + A_0(z) + A(q, z) \ = \ f(t) \ \mathit{in} \ \mathcal{V}^*, t \in (0, \infty); \ z(0) \ = \ \zeta$$

Inspired by

R. Boiger and B. Kaltenbacher
A online parameter identification method for time dependent partial differential equations
Inverse Problems, 32 (2016), 28 pp.

Observation

- *Observation operator* $O : \mathcal{V} \ \longrightarrow \ \mathcal{Y}$
- *Observation space* $\mathcal{Y}$ *(Hilbert space)*
- *Observation* $y(t) := Oz(t), t > 0$

Outlook-1

Assumption: O is linear, continuous, surjective

Fact

Suppose that O is linear, continuous and surjective. Then:

- *The pseudo inverse $O^\dagger$ exists*
- $O^\dagger O : \mathcal{V} \longrightarrow \mathcal{V}$ *is linear and continuous*
- $ran(O^\dagger O) = ker(O)^\perp$
- $P := O^\dagger O$ *is an orthogonal projection onto* $ker(O)^\perp$
- $Q := I - O^\dagger O$ *is an orthogonal projection onto* $ker(O)$

Definition

$O^\dagger Oz$ *is called the* **observable part of** z *and* $I - O^\dagger Oz$ *is the* **unobservable part** *of* z .

Outlook-2

Notice The assumption O *is surjective* is crucial: In the most cases one does not have this property. Then one has to approximate $O^\dagger O$.

Now, we consider an linear orthogonal projection $P : \mathcal{V} \longrightarrow \mathcal{V}$. Let Q be the associated linear orthogonal projection $I - P$. Then we obtain linear continuous mappings

$$P^* : \mathcal{V}^* \longrightarrow \mathcal{V}^*, Q^* : \mathcal{V}^* \longrightarrow \mathcal{V}^*$$

defined as follows:

$$\langle P^*(\lambda), v\rangle = \langle \lambda, P(v)\rangle \, , \, \langle Q^*(\lambda), v\rangle = \langle \lambda, Q(v)\rangle \, , \, \lambda \in \mathcal{V}^*, v \in \mathcal{V} \, .$$

We set

$$\hat{\mathcal{V}} := \mathrm{ran}(P) \, , \, \check{\mathcal{V}} := \mathrm{ran}(Q) \, .$$

(MRAS)

$$\begin{cases} D_t u + C(u - z(t)) + A_0(z(t)) + A(q, z(t)) &= f(t) \quad \text{in } \mathcal{V}^*, t > 0, \\ u(0) &= \zeta \\ D_t q - b(z(t), u - z(t)) &= \theta \quad \text{in } \mathcal{Q}^*, t > 0, \\ q(0) &= q_0 \end{cases}$$

(MRASpro)

$$\begin{cases} D_t z + A_0(z) + A(p, z) &= f(t) \text{ in } \mathcal{V}^*, \\ z(0) &= \zeta \\ D_t u + C(Pu - Pz(t)) + P^* A_0(Qu + Pz(t)) & \\ +P^* A(q, Qu + Pz(t)) + Q^* M(Qu) &= f(t) \text{ in } \mathcal{V}^*, \\ u(0) &= \zeta \\ D_t q - b(Qu + Pz(t), Pu - Pz(t)) &= \theta \text{ in } \mathcal{Q}^*, \\ q(0) &= q_0 \end{cases}$$

Outlook-4

- Choose C and M in an appropriate way in order to find a decoupling for computing $\hat{u} := Pu, \breve{u} := Qu$
- Once $\breve{u}$ is computed find a solution of the resulting computational scheme for $\hat{u}$ and q .
- Analyze the assymptotic properties of $\hat{u}, q$.

Crucial point: Choice of C .
The property we want to exploit is the fact

$$\langle C\hat{w}, \breve{w}\rangle = 0 \text{ for } \hat{w} \in \hat{\mathcal{V}}, \breve{w} \in \breve{\mathcal{V}} .$$

This „leads us" to the choice

$$C : \mathcal{V} \longrightarrow \mathcal{V}^* , \langle C(v), v'\rangle := \gamma_0 \langle v|v'\rangle_{\mathcal{V}} , \; v, v' \in \mathcal{V} \text{ with } \gamma_0 > 0 .$$