Part A: An introduction with four exemplary problems and some discussion about ill-posedness concepts

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Outline

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Ill-posedness of inverse problems and regularization

The goal of inverse problems consists in the identification of unobservable physical quantities (causes) by means of data from dependent but observable quantities (effects).

Inverse problems tend to be **ill-posed**, which e.g. means that small changes in the data may lead to arbitrarily large errors in the quantity to be identified. In this lecture, we will **introduce ill-posedness concepts more precisely.**

Roughly speaking, ill-posedness is the complex impact of **'smoothing' forward operators** occurring in the model.

Let X, Y be infinite dimensional Hilbert or Banach spaces with norms $\|\cdot\|$ and inner products resp. dual pairings $\langle\cdot,\cdot\rangle$.

We distinguish **linear inverse problems** with a **forward operator** $A \in \mathcal{L}(X,Y)$ which is a **bounded linear operator** and **nonlinear inverse problems**, where $F:\mathcal{D}(F) \subseteq X \longrightarrow Y$ is a **nonlinear forward operator** with **domain** $\mathcal{D}(F)$.

Solving inverse problems means solving **operator equations**

$$\boxed{Ax = y \qquad (x \in X,\ y \in Y) \qquad (*)}$$

for linear inverse problems and

$$\boxed{F(x) = y \qquad (x \in \mathcal{D}(F) \subseteq X,\ y \in Y) \qquad (**)}$$

for nonlinear inverse problems.

Due to the ill-posedness of inverse problems

$\Longrightarrow$ **Regularization methods are required!**

Regularization means that
objective and subjective a priori information
is used in addition to the data.

Varieties of regularization:

- **Descriptive regularization**
- **Variational regularization (Tikhonov-type regularization)**
- **Singular perturbation (Lavrentiev-type regularization)**
- **Iterative regularization**

Our focus is on Tikhonov and Lavrentiev regularization.

In the sequel, we always denote by $x^\dagger$ the exact solution of $(*)$ and $(**)$ with $Ax^\dagger = y$ and $F(x^\dagger) = y$, respectively.

Nonlinear Tikhonov-type regularization at first glance

For the stable approximate solution of $(**)$ we consider with convex and stabilizing functional $\mathcal{R} : \mathcal{D}(\mathcal{R}) \subseteq X :\to \mathbb{R}$ and for noisy data y^δ assuming a deterministic noise model

$$\|y - y^\delta\| \le \delta$$

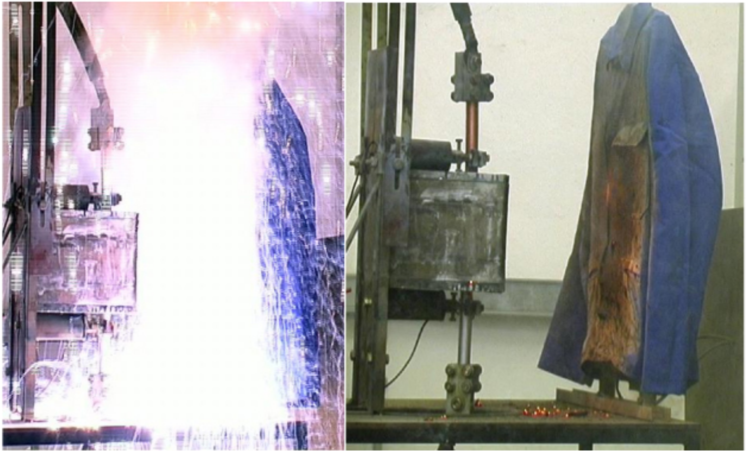
variational regularization of the form

$$\boxed{T_\alpha^\delta(x) := \frac{1}{p}\,\|F(x) - y^\delta\|^p + \alpha\,\mathcal{R}(x) \to \min,}$$

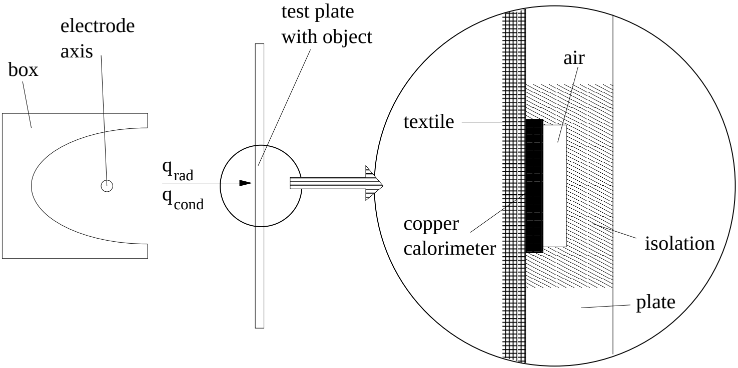
subject to $x \in \mathcal{D} := \mathcal{D}(F) \cap \mathcal{D}(\mathcal{R})$, with exponents $1 \le p < \infty$, regularization parameters $\alpha > 0$ and minimizers $x_\alpha^\delta \in \mathcal{D}(F)$.

Example 1: Find gas temperatures in fault arc tests!

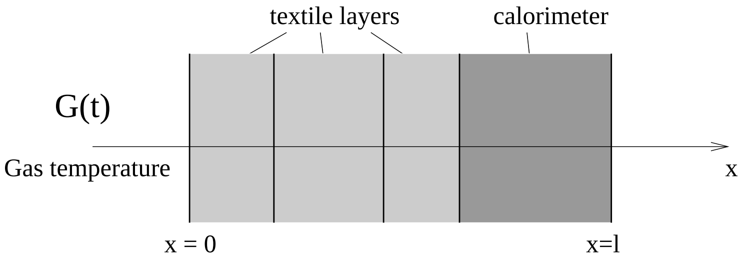
Fault arc tests are performed in textile research and in textile certification of protective clothes by the **Saxon Textile Research Institute Chemnitz**.
Protective clothes are used by people
working on electric installations
who are exposed to the risk of fault arc accidents,
potentially causing injury with heavy burns.



Electric fault arc tests for the certification of protective clothes



Schematic test arrangement



1D-model of the object

Notation:

x	1D local coordinate, $x \in (0, l)$,
t	time, $t \in [0, t^{end} = 30s]$,
$u = u(x,t)$	temperature in the object,
$G = G(t)$	temperature of the hot gas,
$C^A(x,t,u)$	apparent heat capacity,
$\kappa(x,u)$	thermal conductivity,
$f_{rad}(x,t,G(t),u(0,t))$	radiation heat source term,
h_0, h_s	heat transfer coefficients,
Q	space-time cylinder $(0,l) \times (0, t^{end})$

G and u relative temperatures w.r.t. ambient temperature T_0.

Structure of the radiation source term:

$$f_{rad}(x,t,G(t),u(0,t))$$
$$= \gamma e^{-\gamma x} \left(q_a(t) + \beta_{Gas}(G(t) + T_0)^4 - \beta_{\mathrm{Obj}} \left((u(0,t) + T_0)^4 - T_0^4 \right) \right)$$

Goal: Identification of the time course of the unobservable hot gas temperature $G(t)$ near fault arc by lower temperature measurements $u(t,l)$ of lower temperatures behind the test plate.

Nonlinear forward operator

$$F: \; x := \{G(t),\; 0 < t \leq t^{end}\} \; \mapsto \; y = \{u(t,l),\; 0 < t \leq t^{end}\}$$

for operator equation $(**)$ is **implicitly given** by an initial boundary value problem for the heat equation:

Initial-boundary value problem of forward computations:

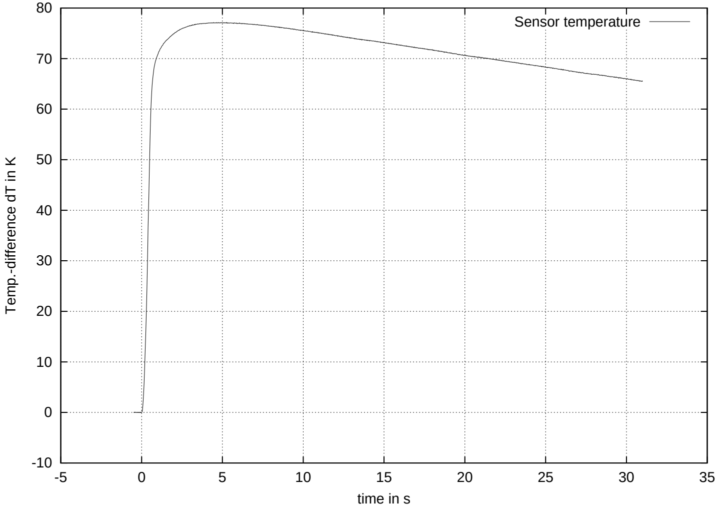
$$C^A(x,t,u)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\kappa(x,u)\frac{\partial u}{\partial x}\right) = f_{rad}(x,t,G(t),u(0,t)),$$

$$(x,t) \in Q,$$

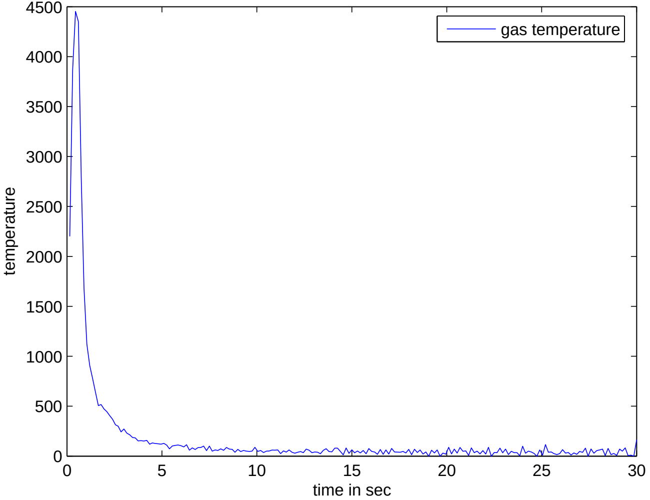
$$-\kappa(0,u(0,t))\frac{\partial u(0,t)}{\partial x} = h_0(G(t) - u(0,t)), \qquad t \in (0, t^{end}],$$

$$\kappa(l,u(l,t))\frac{\partial u(l,t)}{\partial x} = -h_s u(l,t), \qquad t \in (0, t^{end}],$$

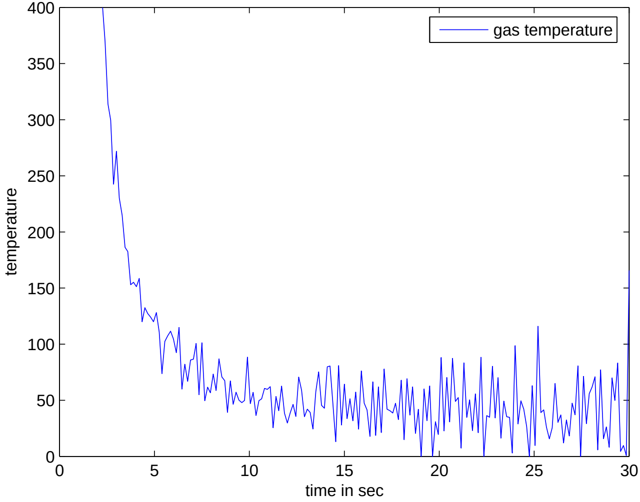
$$u(x,0) = 0, \qquad x \in [0,l].$$



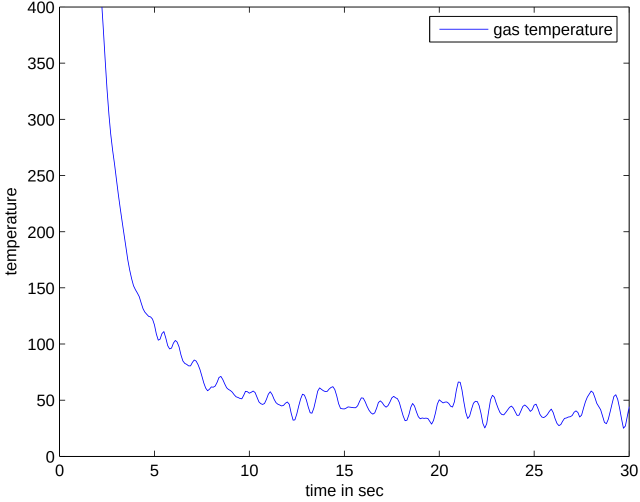
Calibration measurements of calorimeter temperature



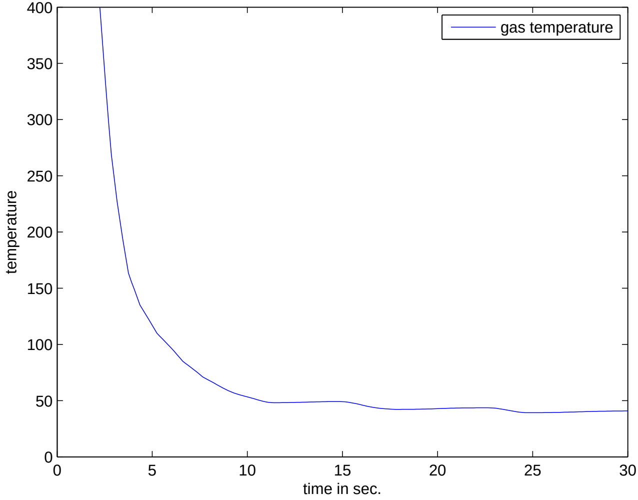
Gas temperature least-squares reconstruction without regularization



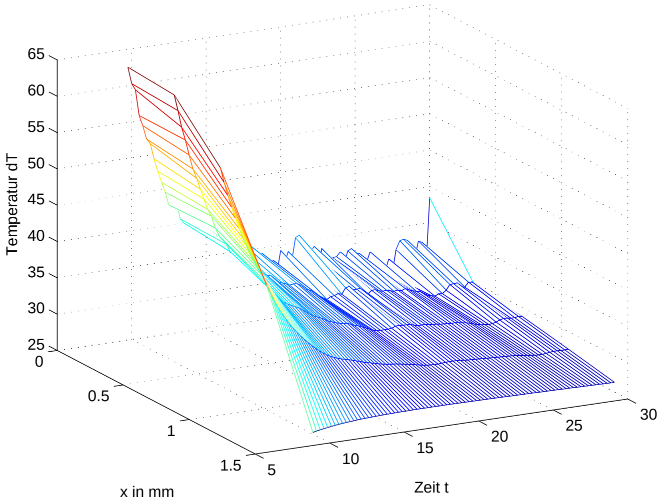
Gas temperature reconstruction without regularization (zoom)



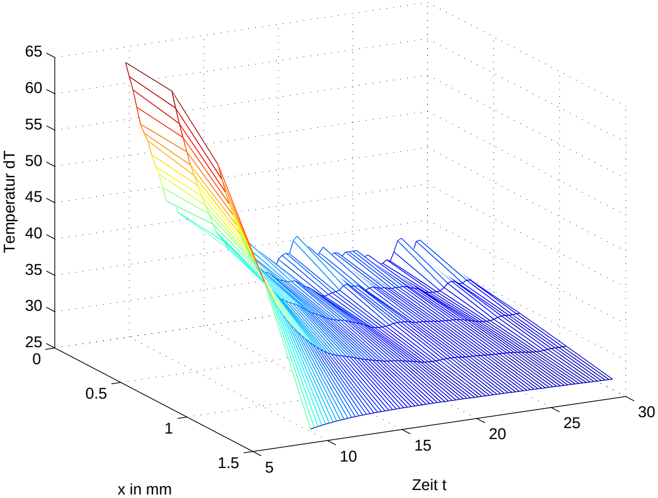
Gas temperature second order Tikhonov regularization (zoom)



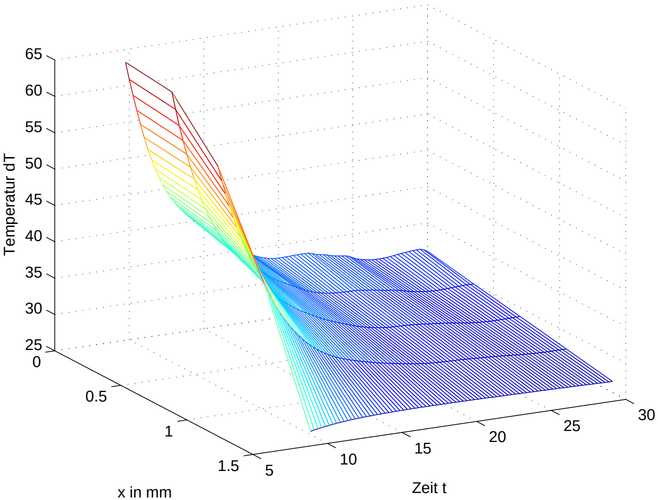
Gas temperature obtained by descriptive regularization (zoom)



Forward computations with textile layers based on unregularized gas temperature



Forward computations with textile layers based on Tikhonov regularized gas temperature

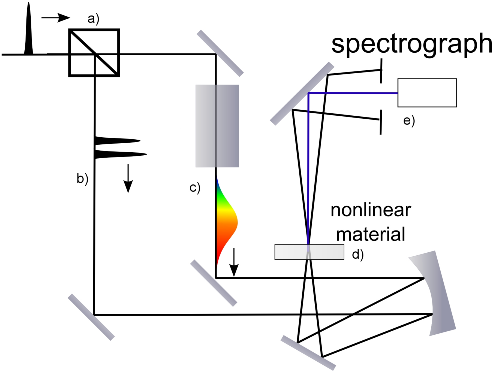


Forward computations with textile layers based on descriptively regularized gas temperature

Example 2: A problem in short-term laser optics

SPIDER = Spectral Phase Interferometry for Direct Electric Field Reconstruction

Special version **Self-Diffraction (SD) SPIDER** was developed by a research group of **Max Born Institute for Nonlinear Optics, Berlin**



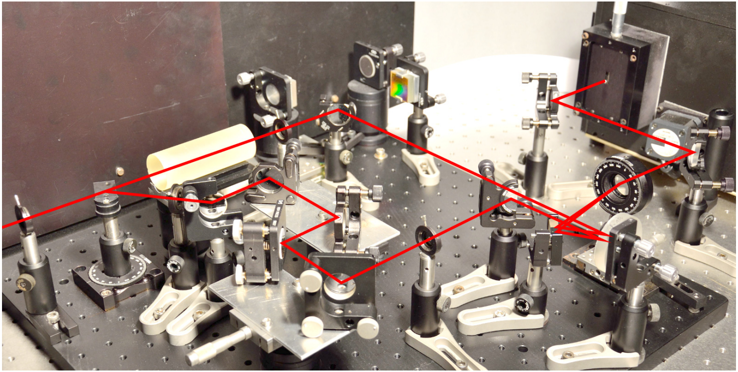


Figure: Measurement setup in self-diffraction spectral interferometry.

The physical model leads to an **autoconvolution problem**

$$[F(x)](s) := \int_0^s k(s,t)x(s-t)x(t)dt = y(s) \quad (0 \le s \le 2) \quad (**)$$

with **explicitly given nonlinear** forward operator

$$F : X = L^2_{\mathbb{C}}(0,1) \to Y = L^2_{\mathbb{C}}(0,2).$$

We have to determine a **complex-valued** function x (characteristics of a short-term - femtosecond - laser pulse) from complex-valued data of y provided that the complex-valued kernel k in the integral equation is available.

Observable quantities: $|\hat{x}|, |y^\delta|, \arg(y^\delta), |\hat{k}|, \arg(\hat{k})$
Completely unknown and preferably to determine: $\arg(x)$

For SD-SPIDER the first derivative (**group delay**) of the phase $\arg(x)$ is of particular interest.

Tikhonov regularized solutions x_α^δ minimizing

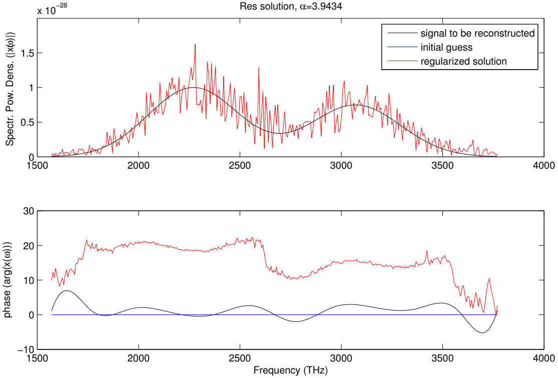
$$T_\alpha^\delta(x) := \frac{1}{2}\|F(x) - y^\delta\|^2_{L^2_{\mathbb{C}}(0,2)} + \alpha\,\mathcal{R}(x)$$

are used in a discretized version, where the penalty $\mathcal{R}(x)$ approximates the L^2-norm square of the 2nd derivative of x.

Adapted a posteriori choice of $\alpha > 0$: $\big||x_\alpha^\delta| - |\hat{x}|\big| \to \min$!

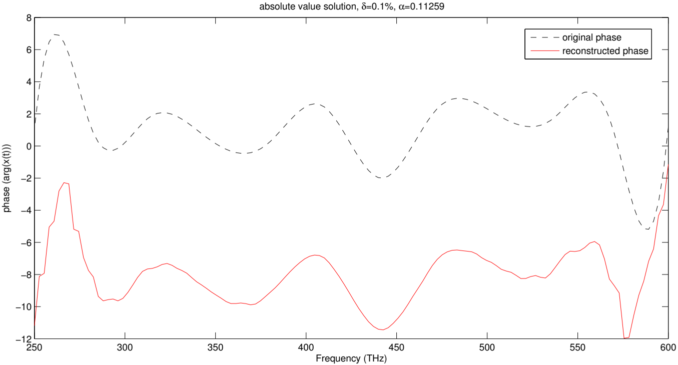
Varieties of Levenberg-Marquardt iterations help to find regularized solutions!

Ill-posedness of the deautoconvolution problem leads to oscillating and wrong reconstructions when noisy data occur (here: 1% noise) and no or insufficient regularization is used:



Reconstructed phase with Tikhonov-type regularization:

The group delay (first derivatives of the phase) is reconstructed reasonably well for appropriately chosen regularization parameter $\alpha > 0$. The phase has an offset of 2π. Only at the right boundary the curves do not match while the left boundary is reconstructed in an acceptable way.



Example 3: A problem in inverse option pricing

Calibrating local volatility surfaces from market data is an ill-posed nonlinear inverse problem in finance.

Consider the price process $P(t)$ for an asset

$$\frac{dP(t)}{P(t)} = \mu\, dt + \sigma(t)\, dW(t) \quad (t \geq 0,\ P(0) > 0).$$

A **benchmark problem** for studying phenomena is the calibration of time-dependent volatilities $\sigma(t),\ 0 \leq t \leq T$, from maturity-dependent option prices $u(t),\ 0 \leq t \leq T$, of European call options with a fixed strike $K > 0$.

For parameters $P > 0$, $K > 0$, $r \geq 0$, $t \geq 0$ and $s \geq 0$ we introduce the **Black-Scholes function** as

$$U_{BS}(P,K,r,t,s) := \begin{cases} P\Phi(d_1) - Ke^{-r\,t}\,\Phi(d_2) & (s > 0) \\ \max(P - Ke^{-r\,t}, 0) & (s = 0) \end{cases}$$

with

$$d_1 := \frac{\ln\left(\frac{P}{K}\right) + r\,t + \frac{s}{2}}{\sqrt{s}}, \quad d_2 := d_1 - \sqrt{s}$$

and the cumulative density function

$$\Phi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\frac{\eta^2}{2}}\,d\eta.$$

of the standard normal distribution.

For

$$a(t) := \sigma^2(t) \qquad \text{and} \qquad S(t) = \int_0^t a(\tau)\, d\tau$$

the associated forward operator in $(**)$ is here $\;F: \; a \mapsto u$ with

$$[F(a)](t) := U_{BS}(P, K, r, t, S(t)) \quad (0 \le t \le T).$$

Hence, we have a composition $F = N \circ J$ with the nonlinear

Nemytskii operator $\;[N(S)](t) := k(t, S(t))\; (0 \le t \le T)\;$ for

$$k(t, v) = U_{BS}(P, K, r, t, v) \quad ((t, v) \in [0, T] \times [0, \infty)),$$

and with the linear **integral operator**

$$[J\,a](t) := \int_0^t a(\tau)\, d\tau \quad (0 \le t \le T).$$

This calibration problem can be written as an equation $(**)$.

It is split into a **linear inner equation**

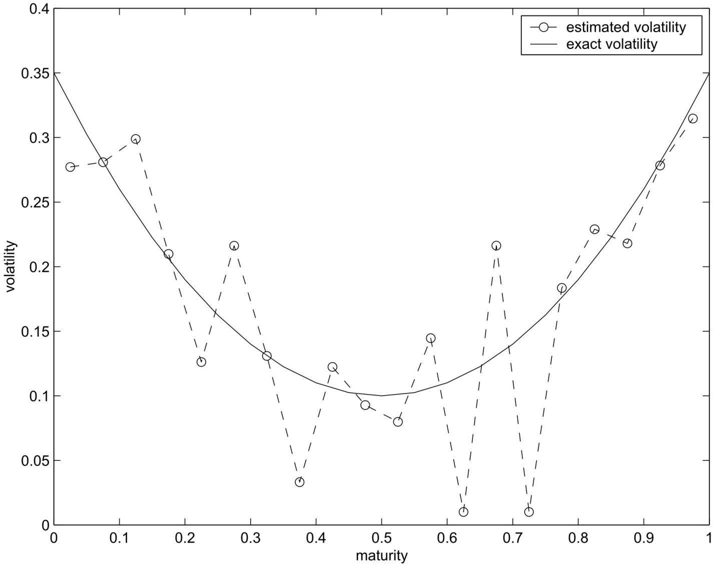
$$\boxed{J\,a = S \qquad (a \in D(F) \subset X,\; S \in Z) \qquad (in)}$$

and a **nonlinear outer equation**

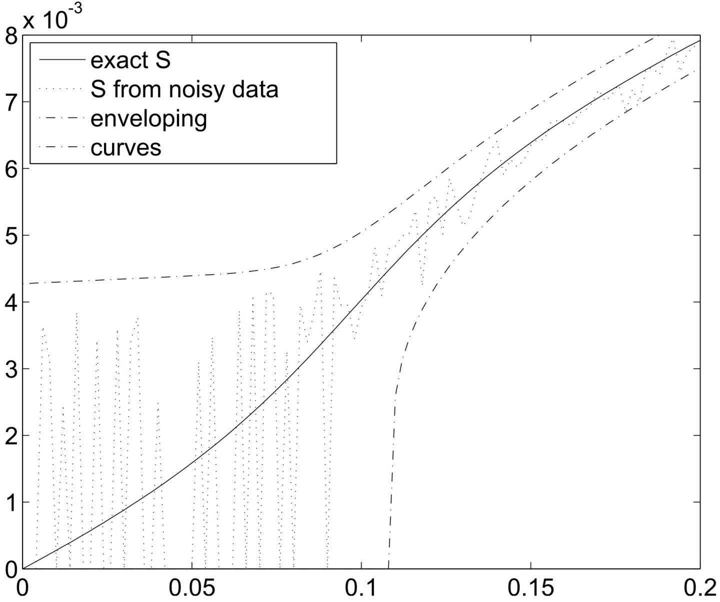
$$\boxed{N(S) = u \qquad (S \in Z,\; u \in Y), \quad (out)}$$

where X, Y, Z are Banach spaces of real functions over $[0, T]$.

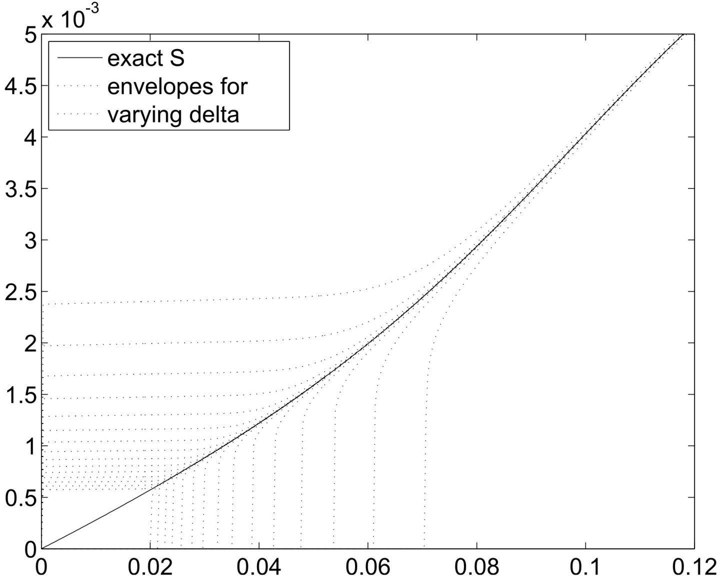
Least-squares solution of $(**)$ after discretization with 20 grid points



Oscillations near $t = 0$ in solving the outer equation



Reduction of oscillation areas for $\delta \to 0$:
Ill-conditioning but not ill-posedness of the outer equation in C-spaces



Example 4: A linear inverse source problem

Let Ω be an open bounded connected domain of $\mathbb{R}^d$ ($d = 2, 3$) with boundary $\partial\Omega$. We consider the elliptic system

$$\begin{aligned} -\nabla \cdot \big(Q\nabla\Phi\big) &= f \text{ in } \Omega, \\ Q\nabla\Phi \cdot \vec{n} &= j \text{ on } \partial\Omega \text{ and} \\ \Phi &= g \text{ on } \partial\Omega, \end{aligned}$$

where $\vec{n}$ is the unit outward normal on $\partial\Omega$ and the diffusion matrix Q is given. Furthermore, we assume that $Q := (q_{rs})_{1\le r,s\le d} \in L^\infty(\Omega)^{d\times d}$ is symmetric and satisfies the uniformly ellipticity condition

$$Q(x)\xi \cdot \xi = \sum_{1\le r,s\le d} q_{rs}(x)\xi_r\xi_s \ge \underline{q}|\xi|^2 \text{ a.e. in } \Omega$$

for all $\xi = (\xi_r)_{1\le r\le d} \in \mathbb{R}^d$ with some constant $\underline{q} > 0$.

The elliptic system is overdetermined, i.e., if the Neumann and Dirichlet boundary conditions $j \in H^{-1/2}(\partial\Omega)$, $g \in H^{1/2}(\partial\Omega)$, and the source term $f \in L^2(\Omega)$ are given, then there may be no Φ satisfying this system. Here we assume that the system is consistent and our **inverse problem** is to **reconstruct the source function** $f \in L^2(\Omega)$ in the elliptic system from noisy data

$$\left(j^\delta, g^\delta\right) \in H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

of the exact Neumann and Dirichlet data (j, g), such that

$$\|j^\delta - j\|_{H^{-1/2}(\partial\Omega)} + \|g^\delta - g\|_{H^{1/2}(\partial\Omega)} \le \delta,$$

with some noise level $\delta > 0$. Unfortunately, the mapping

$$f \in L^2(\Omega) \ \mapsto \ (j, g) \in H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

is **set-valued** and does not serve as a forward operator.

Now, for fixed (j, g) we consider the Neumann problem

$$-\nabla \cdot (Q\nabla u) = f \;\text{ in } \Omega \;\text{ and }\; Q\nabla u \cdot \vec{n} = j \;\text{ on } \partial\Omega.$$

By Riesz' representation theorem, we have for each $f \in L^2(\Omega)$ that there is a unique weak solution u of this problem and we can define the **Neumann operator** $f \mapsto \mathcal{N}_f j$, which maps in $L^2(\Omega)$ each f to the unique weak solution $\mathcal{N}_f j := u$ of the Neumann problem.

Similarly, for fixed (j, g) the Dirichlet problem

$$-\nabla \cdot (Q\nabla v) = f \;\text{ in } \Omega \;\text{ and }\; v = g \;\text{ on } \partial\Omega$$

yields the **Dirichlet operator** $f \mapsto \mathcal{D}_f g$, which maps in $L^2(\Omega)$ each f to the unique weak solution $\mathcal{D}_f g := v$ of the Dirichlet problem.

It can be shown that the non-empty solution set of our inverse problem, which can be characterized as

$$\mathcal{I}(j,g) := \left\{ f \in L^2(\Omega) : \mathcal{N}_f j = \mathcal{D}_f g \right\},$$

is **closed and convex, but not a singleton**.

On the contrary, the problem is even **highly underdetermined.**

Having an initial guess $\bar{f} \in L^2(\Omega)$, it makes sense to search for the **uniquely determined $\bar{f}$-minimum-norm solution** $f^\dagger$, which is defined as the minimizer of the extremal problem

$$\min_{f \in \mathcal{I}(j,g)} \|f - \bar{f}\|^2_{L^2(\Omega)}. \qquad (\mathcal{IP} - MN)$$

Tikhonov-type regularization with a Kohn/Vogelius misfit term

$$\mathcal{J}^\delta(f) := \int_\Omega Q\nabla \left(\mathcal{N}_f j^\delta - \mathcal{D}_f g^\delta\right) \cdot \nabla \left(\mathcal{N}_f j^\delta - \mathcal{D}_f g^\delta\right) dx$$

yields regularized solutions f_α^δ which are minimizers of

$$T_\alpha^\delta(f) := \mathcal{J}^\delta(f) + \alpha \|f - \bar{f}\|^2_{L^2(\Omega)} \to \min, \quad \text{subject to} \quad f \in L^2(\Omega).$$

Proposition ▷ M. Hinze, B. Hofmann, Q. Tran arXiv 2017

The minimizers $f = f_\alpha^\delta$ satisfy the equation

$$\left(\mathcal{N}_f j^\delta - \mathcal{D}_f g^\delta\right) + \alpha(f - \bar{f}) = 0. \tag{1}$$

This seems to be a form of **Lavrentiev-type regularization.**

For clearing this phenomenon we come back to the open question of formulating an appropriate forward operator. Really, for exact data the **implicit operator equation**

$$H_{j,g}(f) := \mathcal{N}_f j - \mathcal{D}_f g = 0$$

characterizes the inverse problem. For fixed noise-free data j and g we have that H is an **affine linear operator** applied to f.

Namely, since the elliptic PDE is linear, the superposition principle yields $\mathcal{N}_f j = \mathcal{N}_0 j + \mathcal{N}_f 0$ and $\mathcal{D}_f j = \mathcal{D}_0 j + \mathcal{D}_f 0$.

We set $A\,f := \mathcal{N}_f 0 - \mathcal{D}_f 0$ and $y := \mathcal{D}_0 g - \mathcal{N}_0 j.$ such that the **explicit operator equation** $A\,f = y$ fits the inverse problem.

Proposition

The operator A defined by the formula $Af := \mathcal{N}_f 0 - \mathcal{D}_f 0$ is a **linear, bounded and self-adjoint monotone linear operator** mapping in $L^2(\Omega)$. Moreover A has a non-trival nullspace $\mathcal{N}(A)$ and a non-closed range $\mathcal{R}(A)$.

Our inverse problem is modeled by the linear operator equation

$$\boxed{Af = y \qquad (x \in L^2(\Omega),\ y \in L^2(\Omega))\,. \qquad (*)}$$

Due to the proposition above $(*)$ is ill-posed.

From the monotonicity of the forward operator A it follows that the Lavrentiev regularization is successfully applicable.

A convergence rate result, which immediately follows from the classical theory of Lavrentiev's regularization method:

Theorem

Assume that there exists, for the $\overline{f}$-minimum-norm solution $f^\dagger$, a function $w \in L^2(\Omega)$ such that the **source condition**

$$f^\dagger - \overline{f} = A\,w := \mathcal{N}_w 0 - \mathcal{D}_w 0$$

holds true. Then we have the convergence rate

$$\|f_\alpha^\delta - f^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \qquad \text{as} \qquad \delta \to 0$$

whenever the regularization parameter α is chosen a priori as

$$\underline{c}\,\delta \le \alpha(\delta) \le \overline{c}\,\delta$$

with some constants $0 < \underline{c} \le \overline{c} < \infty$.

Outline

On ill-posedness concepts and stable solvability

Here, we restrict our consideration to **Hilbert spaces** X and Y.

For the understanding of well-posedness and ill-posedness concepts, Hadamard's classical definition plays a prominent role. This definition assumes that for the **well-posedness** of an operator equation **in the sense of Hadamard** all three of the following conditions are satisfied:

(i) For all $y \in Y$ there exists an admissible solution $x^\dagger$ of the operator equation (existence condition).

(ii) The solution of the operator equation is always uniquely determined (uniqueness condition).

(iii) The solutions depend stably on the data, i.e. small perturbations in the right-hand side y lead to only small errors in the solution x (stability condition).

Otherwise the corresponding operator equation is called **ill-posed in the sense of Hadamard**.

Hadamard's well-posedness concept in its entirety can only be of importance for linear equations $(*)$.

In the nonlinear case $(**)$, the range $F(\mathcal{D}(F))$ of F will rarely coincide with Y such that (i) is suspicious.

With respect to (ii), F can be **injective** or **non-injective.** In the former case, the inverse operator $F^{-1} : F(\mathcal{D}(F)) \to \mathcal{D}(F)$ is single-valued and (ii) is fulfilled. In the latter case, F^{-1} is set-valued, moreover (ii) is violated, and $F^{-1}(y)$ characterizes the set of preimages to y.

In the following, we **have to distinguish** these two cases to define the stability condition (iii) in a more precise manner.

As we will recall, the **closedness of the range** $\mathcal{R}(A)$ is essential for the well-posedness of linear operator equations $(*)$. Hence, well-posedness and alternatively ill-posedness are **global properties** on X.

For operator equations $(**)$ with nonlinear operator F, the literature on inverse problems uses concepts of well-posedness and ill-posedness mostly in a rather rough manner, because in contrast to the linear case the closedness of the range $F(\mathcal{D}(F))$ of the forward operator F does not serve as an appropriate criterion.

Stable and unstable behavior of a nonlinear equation $(**)$ is not only a local property and can change from point to point, but one also has to **distinguish local properties in the image space and in the solution space**.

Let us first consider the local stability behavior in the **image space.** For forward operators F **injective** on $\mathcal{D}(F)$, stability at some point $y \in F(\mathcal{D}(F))$ means that the single-valued inverse operator $F^{-1} : F(\mathcal{D}(F)) \subseteq Y \to \mathcal{D}(F) \subseteq X$ is **continuous at** $y \in F(\mathcal{D}(F))$, i.e., for every sequence $\{y_n\}_{n=1}^{\infty} \subset F(\mathcal{D}(F))$ with $\lim\limits_{n\to\infty} \|y_n - y\| = 0$ we have that

$$\lim_{n\to\infty} \|F^{-1}(y_n) - F^{-1}(y)\| = 0\,.$$

In case of a **non-injective** operator F, the inverse F^{-1} is a set-valued mapping and stability or instability are based on continuity concepts of set-valued mappings. The **non-symmetric quasi-distance** $\mathrm{qdist}(\cdot,\cdot)$ seems to be an appropriate stability measure in the non-injective case.

Therefore, we suggest the following definition:

Definition

We call the operator equation $(**)$ **stably solvable** at the point $y \in F(\mathcal{D}(F))$ if we have for every sequence $\{y_n\}_{n=1}^{\infty} \subset F(\mathcal{D}(F))$ with $\lim_{n\to\infty} \|y_n - y\| = 0$ that

$$\lim_{n\to\infty} \mathrm{qdist}(F^{-1}(y_n), F^{-1}(y)) = 0\,,$$

where

$$\mathrm{qdist}(U, V) := \sup_{u\in U} \inf_{v\in V} \|u - v\|$$

denotes the quasi-distance between the sets U and V.

Nashed's ill-posedness concept for linear problems

Definition (▷ NASHED 1987)

We call a linear operator equation $(*)$ **well-posed in the sense of Nashed** if the range $\mathcal{R}(A)$ of A is a closed subset of Y, consequently **ill-posed in the sense of Nashed** if the range is not closed, i.e. $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}^{Y}$. In the ill-posed case, the equation $(*)$ is called **ill-posed of type I** if the range $\mathcal{R}(A)$ contains an infinite dimensional closed subspace, and **ill-posed of type II** otherwise.

Ill-posedness in the sense of Nashed requires $\dim \mathcal{R}(A) = \infty$. Then the equation $(*)$ is ill-posed of type II if and only if A is compact. Well-posedness, however, does not exclude the case of non-injective A possessing non-trivial null-spaces $\mathcal{N}(A)$.

Proposition

If $(*)$ is well-posed in the sense of Nashed, then the equation is **stably solvable everywhere** on $\mathcal{R}(A) = \overline{\mathcal{R}(A)}^Y$.
If $(*)$ is ill-posed in the sense of Nashed, the equation is **stably solvable nowhere.**

Proof: For $y_n, y \in \mathcal{R}(A)$ with $\lim_{n\to\infty} \|y_n - y\| = 0$ we have

$$F^{-1}(y) = \{x \in X :\ x = A^\dagger y + x_0,\ x_0 \in \mathcal{N}(A)\} \qquad \text{and}$$

$$F^{-1}(y_n) = \{x \in X :\ x = A^\dagger y_n + \tilde{x}_0,\ \tilde{x}_0 \in \mathcal{N}(A)\} \quad (n \in \mathbb{N}).$$

Since $A^\dagger y_n - A^\dagger y$ is orthogonal to $\mathcal{N}(A)$, the equality
$\min\limits_{x\in F^{-1}(y)} \|x_n - x\| = \|A^\dagger y_n - A^\dagger y\|$ is valid for all $x_n \in F^{-1}(y_n)$.
In particular, for $(*)$ well-posed in the sense of Nashed we have
$\mathrm{qdist}(F^{-1}(y_n), F^{-1}(y)) = \min\limits_{x\in F^{-1}(y)} \|x_n - x\| \le \|A^\dagger\|_{\mathcal{L}(Y,X)} \|y_n - y\| \to 0.$
For $(*)$ ill-posed in the sense of Nashed, $A^\dagger$ is unbounded and we have sequences $\{y_n\}_{n=1}^\infty$ in the range of A such that $\|A^\dagger y_n - A^\dagger y\| \not\to 0$ although $\|y_n - y\| \to 0$ as $n \to \infty$. $\square$

Local well-posedness and ill-posedness

In the nonlinear case $(**)$, there occurs in general a locally varying behavior of solutions, moreover often an overlap of instability of solutions with respect to small data perturbations and the existence of distinguished solution branches.

Definition ($\triangleright$ HOFMANN/SCHERZER 1994)

The operator equation $(**)$ is called **locally well-posed** at the solution $x^\dagger \in \mathcal{D}(F)$ if there is a closed ball $\mathcal{B}_r(x^\dagger)$ with radius $r > 0$ and center $x^\dagger$ such that for every sequence $\{x_n\}_{n=1}^\infty \subset \mathcal{B}_r(x^\dagger) \cap \mathcal{D}(F)$ the convergence of images $\lim_{n\to\infty} \|F(x_n) - F(x^\dagger)\| = 0$ implies the convergence of the preimages $\lim_{n\to\infty} \|x_n - x^\dagger\| = 0$.
Otherwise $(**)$ is called **locally ill-posed** at $x^\dagger$.

Local well-posedness at $x^\dagger$ requires **local injectivity**, which means that $x^\dagger$ is the only solution in $\mathcal{B}_r(x^\dagger) \cap \mathcal{D}(F)$ and hence $x^\dagger$ is an **isolated solution** of the operator equation.

This often provokes criticism, but the underlying idea of this definition is that a really existing physical quantity $x^\dagger$ is the unique solution of $(**)$ in the ball $\mathcal{B}_r(x^\dagger) \cap \mathcal{D}(F)$ and can be recovered exactly when the measurement process may be taken arbitrarily precise, i.e. when $\delta \to 0$ can be implemented. The idea does not exclude the case that further branches of solutions to $(**)$ exist in $\mathcal{D}(F)$ outside of the ball.

Proposition

Let the operator F be locally injective at $x^\dagger \in \mathcal{D}(F)$. If the operator equation $(**)$ is stably solvable at $y = F(x^\dagger)$, then this equation is locally well-posed at $x^\dagger$.

The converse implication does not hold, in general.

The nonlinear operator equation $(**)$ can be stably solvable at some points $y = F(x^\dagger),\ x^\dagger \in \mathcal{D}(F)$, and not stably solvable at other points of the range.

Similarly, $(**)$ can be locally well-posed at some points $x^\dagger \in \mathcal{D}(F)$ and locally ill-posed at other points of the domain.

The following examples illustrate such behavior.

Four examples

Example 1 (one-dimensional example):

Let $X = Y := \mathbb{R}$ and consider the nonlinear mapping $F : \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) := \frac{x^2}{1 + x^4} \, .$$

Then the corresponding nonlinear equation $F(x) = y \; (**)$ is **locally well-posed** for all $x^\dagger \in \mathbb{R}$. However, the equation is evidently **not stably solvable** at $y = F(0) = 0$, because we have, for $y_n > 0$ tending to zero as $n \to \infty$, $\lim_{n\to\infty} \mathrm{qdist}(F^{-1}(y_n), F^{-1}(0)) = +\infty$. On the other hand, the equation is stably solvable for all other range points $y > 0$.

Example 2 (self-integration weighted identity operator):

Let $X = Y := L^2_{\mathbb{R}}(0,1)$ (Hilbert space of real-valued square integrable functions over the unit interval $(0,1)$), and let the quadratic operator $F : L^2_{\mathbb{R}}(0,1) \to L^2_{\mathbb{R}}(0,1)$ be given for $x \in X$ by

$$[F(x)](s) := \phi(x)\, x(s), \quad s \in (0,1), \quad \phi(x) := \int_0^1 x(t)dt.$$

This operator F is
locally injective on $X \backslash N$, where $N := F^{-1}(0)$, but
not locally injective at each point of N.

The corresponding operator equation $(**)$ is
locally well-posed everywhere on $X \backslash N$, but
locally ill-posed everywhere on N.

The equation is **stably solvable everywhere** on $F(X)$.

Example 3 (autoconvolution problem):

Let $X := L^2_{\mathbb{C}}(0,1)$ and $Y := L^2_{\mathbb{C}}(0,2)$ (complex-valued spaces). Then the **autoconvolution operator** F attains the form

$$[F(x)](s) := \begin{cases} \int\limits_0^s x(s-t)\,x(t)\,dt, & 0 \le s \le 1, \\ \int\limits_{s-1}^1 x(s-t)\,x(t)\,dt, & 1 < s \le 2. \end{cases}$$

As a consequence of Titchmarsh's convolution theorem, the corresponding operator equation $(**)$ possesses for arbitrary $y = F(x^\dagger),\ x^\dagger \in X$, the solution set $F^{-1}(y) = \{x^\dagger, -x^\dagger\}$.

For $\mathcal{D}(F) = X$ it was shown that this nonlinear operator equation $(**)$ is **locally ill-posed everywhere** and hence **stably solvable nowhere**. But **stable solvability can be achieved by an appropriate restriction of the domain**, for example to $\mathcal{D}(F) := \{x \in H^\alpha_{\mathbb{C}}(0,1) : \ \|x\|_{H^\alpha_{\mathbb{C}}} \le c\},\ \alpha > 0$.

Example 4 (another quadratic problem)

Let X be an infinite-dimensional, separable real Hilbert space with orthonormal basis $\{u_k\}_{k=1}^{\infty}$. Consider the operators in X:

$$Sx := \langle x, u_1 \rangle u_1 + \sum_{k=3}^{\infty} \sigma_k \langle x, u_k \rangle u_{k+1}, \qquad Tx := \sum_{k=2}^{\infty} \langle x, u_k \rangle u_k,$$

where $0 \neq \sigma_k \in \mathbb{R}$ satisfies $\sigma_k \to 0$ as $k \to \infty$. S and T are bounded linear operators, with S being compact and T having a closed range. Consider the quadratic operator $F : X \to X$:

$F(x) := B(x, x)$ from the bilin. form $B(x, y) := \langle x, u_1 \rangle Sy + \langle x, u_2 \rangle Ty$.

For the operator equation $(**)$ with $X = Y$ corresponding to this operator F, we have **stable solvability** at some points in the range of F, as well as **unstable solvability** at other points in the range.

Revisiting the linear case

For linear operator equation $(*)$, i.e. $F := A \in \mathcal{L}(X, Y)$ and $\mathcal{D}(F) = X$ in terms of $(**)$, local well-posedness at some points and local ill-posedness at other points **may not occur**:

Proposition

The linear operator equation $(*)$ is locally well-posed everywhere on X if the equation is well-posed the sense of Nashed, i.e. $\mathcal{R}(A) = \overline{\mathcal{R}(A)}^Y$, and if moreover the null-space of A is trivial, i.e. $\mathcal{N}(A) = \{0\}$. If at least one of both requirements fails, then $(*)$ is locally ill-posed everywhere on X.

Proof: Now, $(*)$ is locally ill-posed everywhere if $\mathcal{N}(A) \neq \{0\}$. For $\mathcal{N}(A) = \{0\}$, local well-posedness holds iff for $n \to \infty$ $\|A\,\Delta_n\| \to 0 \implies \|\Delta_n\| \to 0$ with $\|\Delta_n\| < r$ and $\Delta_n := x_n - x^\dagger$, which is valid iff A^{-1} is bounded, i.e. if $\mathcal{R}(A) = \overline{\mathcal{R}(A)}^Y$. $\square$

Outline

Relevant references:

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