Part C: Regularization and conditional stability: some case studies in a Hilbert space setting

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Let X and Y denote infinite dimensional **Hilbert spaces**, equipped with norms $\|\cdot\|$ and inner products $\langle\cdot,\cdot\rangle$.

We consider also here **linear inverse problems** in form of linear operator equations

$$\boxed{Ax = y \qquad (x \in X,\ y \in Y) \qquad (*)}$$

with a **bounded linear forward operator** $A \in \mathcal{L}(X,Y)$ and

nonlinear inverse problems in form of nonlinear equations

$$\boxed{F(x) = y \qquad (x \in \mathcal{D}(F) \subseteq X,\ y \in Y)\,. \qquad (**)}$$

We denote again by $x^\dagger$ solutions to equations $(*)$ and $(**)$.

We recall Nashed's ill-posedness concept for $(*)$

Definition (▷ Nashed 1987)

We call a linear operator equation $(*)$ **well-posed in the sense of Nashed** if the range $\mathcal{R}(A)$ of A is a closed subset of Y, consequently **ill-posed in the sense of Nashed** if the range is not closed, i.e. $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}^{Y}$. In the ill-posed case, the equation $(*)$ is called **ill-posed of type I** if the range $\mathcal{R}(A)$ contains an infinite dimensional closed subspace, and **ill-posed of type II** otherwise.

We recall local ill-posedness concept for $(**)$

Definition ($\triangleright$ HOFMANN/SCHERZER 1994)

The operator equation $(**)$ is called **locally well-posed** at the solution $x^\dagger \in \mathcal{D}(F)$ if there is a closed ball $\mathcal{B}_r(x^\dagger)$ with radius $r > 0$ and center $x^\dagger$ such that for every sequence $\{x_n\}_{n=1}^\infty \subset \mathcal{B}_r(x^\dagger) \cap \mathcal{D}(F)$ the convergence of images $\lim_{n\to\infty} \|F(x_n) - F(x^\dagger)\| = 0$ implies the convergence of the preimages $\lim_{n\to\infty} \|x_n - x^\dagger\| = 0$.
Otherwise $(**)$ is called **locally ill-posed** at $x^\dagger$.

Stable approximate solution of ill-posed linear problems $(*)$ and locally ill-posed nonlinear problems $(**)$ requires some kind of **stabilization.** We consider **conditional stability, Tikhonov regularization and Lavrentiev regularization** as different such approaches.

Outline

Tikhonov regularization and conditional stability

For ill-posed equations $(**)$ and a general situation of F, classical **Tikhonov regularization** is a very popular method, see, e.g., ▷ ENGL/HANKE/NEUBAUER 1996 (Chapter 10), with regularized solutions x_α^δ which are minimizers of

$$\|F(x)-y^\delta\|^2+\alpha\,\|x-\bar{x}\|^2 \to \min, \qquad \text{subject to} \quad x\in\mathcal{D}(F)\,. \qquad (Tik)$$

For a priori parameter choices $\alpha_*=\alpha_*(\delta)$ or a posteriori parameter choices $\alpha_*=\alpha_*(y^\delta,\delta)$ of the regularization parameter $\alpha>0$ satisfying

$$\alpha_*\to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha_*}\to 0 \qquad \text{as} \qquad \delta\to 0\,, \qquad (\searrow)$$

we have, under conditions on F and $\mathcal{D}(F)$, norm convergence in X of $x_{\alpha_*}^\delta$ to $\bar{x}$-minimum norm solutions $x^\dagger$ of $(**)$ as $\delta\to 0$.

The convergence is based on the **minimizing property**

$$\|F(x_\alpha^\delta)-y^\delta\|^2+\alpha\|x_\alpha^\delta-\bar{x}\|^2 \le \|F(x^\dagger)-y^\delta\|^2+\alpha\|x^\dagger-\bar{x}\|^2 \quad (MP)$$

of the regularized solution $x_\alpha^\delta \in \mathcal{D}(F)$, which yields

$$\|x_\alpha^\delta - \bar{x}\| \le \sqrt{\frac{\delta^2}{\alpha} + \|x^\dagger - \bar{x}\|^2}$$

and

$$\|F(x_\alpha^\delta) - y^\delta\| \le \sqrt{\delta^2 + \alpha\|x^\dagger - \bar{x}\|^2}.$$

If F is **weakly sequentially closed**, then we have weak convergence (in subsequences) of $x_{\alpha_*}^\delta$ to $\bar{x}$-minimum norm solutions under ($\searrow$) as $\delta \to 0$. Since Hilbert spaces have the **Kadec-Klee** property, this yields even norm convergence.

Convergence rates for Tikhonov regularization

To obtain convergence rates, solution smoothness with respect to the forward operator is needed, which is expressed by some kind of **source condition.**

We recall a form expressed by **variational inequalities**:

Variational source condition (VSC)

We assume to have a constant $0 < \beta \leq 1$ and a **concave** index function φ such that

$$\beta \, \|x - x^\dagger\|^2 \leq \|x - \bar{x}\|^2 - \|x^\dagger - \bar{x}\|^2 + \varphi(\|F(x) - F(x^\dagger)\|) \quad \forall\, x \in \mathcal{M}\,.$$

The function $\varphi : [0, \infty) \to [0, \infty)$ is called an **index function** if it is continuous and strictly increasing with $\varphi(0) = 0$.

Under variational source conditions (*VSC*)

$$\alpha_* = \frac{\delta^2}{\varphi(\delta)}$$

is an appropriate a priori choice of the regularization parameter.

For practical use, however, discrepancy principles are more important. Consider for prescribed $0 < q < 1$ and large $\alpha_0 > 0$

$$\Delta_q := \left\{ \alpha_j : \quad \alpha_j := q^j \alpha_0, \quad j = 1, 2, \ldots \right\}.$$

Sequential discrepancy principle (SDP)

For prescribed $\tau > 1$ we say that the regularization parameter $\alpha_* \in \Delta_q$ is chosen according to the sequential discrepancy principle (SDP) if

$$\|F(x^\delta_{\alpha_*}) - y^\delta\| \le \tau\delta < \|F(x^\delta_{\alpha_*/q}) - y^\delta\|.$$

Theorem ▷ HOFMANN/MATHÉ 2012

Suppose that $x^\dagger$ obeys (VSC) for some $0 < \beta \le 1$ and some concave index function φ.

(i) Let $\alpha_* = \alpha_*(\delta) > 0$ be selected according to the a priori parameter choice $\alpha_* := \frac{\delta^2}{\varphi(\delta)}$.

(ii) For prescribed $\tau > 1$ let $\alpha_* = \alpha_*(\delta, y^\delta) > 0$ be chosen according to the sequential discrepancy principle (SDP).

Provided that $x^\delta_{\alpha_*} \in \mathcal{M}$ for all $0 < \delta \le \delta_{max}$ and some $\delta_{max} > 0$ we have for both parameter choices (i) and (ii) the convergence rate

$$\|x^\delta_{\alpha_*} - x^\dagger\|^2 = \mathcal{O}(\varphi(\delta)) \qquad \text{as} \;\; \delta \to 0.$$

We mention that both parameter choices in (i) and (ii) satisfy

$$\alpha_* \to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha_*} \to 0 \qquad \text{as} \qquad \delta \to 0\,. \qquad\qquad (\searrow)$$

Conditional stability

Stable approximate solutions can also be found by having **conditional stability estimates** of the form

$$\|x-x^\dagger\|^2 \le \varphi(\|F(x)-F(x^\dagger)\|) \quad \text{for all} \quad x \in \mathcal{D}(F)\cap Q \qquad (CSE)$$

for a concave index function φ and an appropriate set $Q \supset \{x^\dagger\}$.

Often Q depends on properties of $x^\dagger$ and is not known a priori. Consequently, (CSE) is not directly applicable for finding stable approximate solutions to $(**)$. Additional tools are needed.

▷ CHENG/YAMAMOTO IP 2000 ▷ HOFMANN/YAMAMOTO IP 2010

Combine (CSE) and Tikhonov regularization: $Q = B_r(x^\dagger)$

Under $\|x - x^\dagger\|^2 \le \varphi(\|F(x) - F(x^\dagger)\|)$ for all $x \in \mathcal{D}(F) \cap B_r(x^\dagger)$ the sol. $x^\dagger$ is **unique** in $B_r(x^\dagger)$, $(**)$ is **locally well-posed** at $x^\dagger$.

An appropriate a priori choice $\alpha_* = \alpha_*(\delta)$ is here

$$\underline{c}\,\delta^2 \le \alpha_*(\delta) \le \overline{c}\,\delta^2, \quad 0 < \underline{c} \le \overline{c} < \infty, \qquad \text{satisfying}$$

$$\alpha_* \to 0 \quad \text{and} \quad 0 < 1/\overline{c} \le \delta^2/\alpha_* \le 1/\underline{c} < \infty \to 0 \quad \text{as} \quad \delta \to 0\,. \; (\longrightarrow)$$

From (MP) we derive $\|x_\alpha^\delta - \bar{x}\|^2 \le \frac{\delta^2}{\alpha} + \|x^\dagger - \bar{x}\|^2$, consequently

$$\|x_\alpha^\delta - x^\dagger\|^2 \le 2(\|x_\alpha^\delta - \bar{x}\|^2 + \|x^\dagger - \bar{x}\|^2) \le \frac{2\delta^2}{\alpha} + 4\|x^\dagger - \bar{x}\|^2$$

and hence

$$\|x_{\alpha_*}^\delta - x^\dagger\|^2 \le \frac{2}{\underline{c}} + 4\|x^\dagger - \bar{x}\|^2.$$

For $r > \sqrt{\frac{2}{\underline{c}} + 4\|x^\dagger - \bar{x}\|^2}$ we thus have $x_{\alpha_*}^\delta \in \mathcal{D}(F) \cap B_r(x^\dagger)$.

(CSE) applies for $x_{\alpha_*}^\delta \in \mathcal{D}(F) \cap B_r(x^\dagger) = \mathcal{D}(F) \cap Q$ and yields

$$\|x_{\alpha_*}^\delta - x^\dagger\|^2 \le \varphi(\|F(x_{\alpha_*}^\delta) - y^\delta\| + \|F(x^\dagger) - y^\delta\|),$$

moreover with

$$\|F(x_{\alpha_*}^\delta) - y^\delta\| \le \sqrt{\delta^2 + \alpha_*\|x^\dagger - \bar{x}\|^2} \le \sqrt{\delta^2 + \overline{c}\,\delta^2\|x^\dagger - \bar{x}\|^2}.$$

For concave index functions φ we find constant $K > 0$ such that

$$\|x_{\alpha_*}^\delta - x^\dagger\|^2 \le K\,\varphi(\delta) = \mathcal{O}(\varphi(\delta)) \qquad \text{as } \delta \to 0\,.$$

Using regularization under conditional stability is like putting into the hole Q while playing golf!

Outline

Oversmoothing regularization in Hilbert scales

If in generalized Tikhonov regularization

$$\|F(x)-y^\delta\|^2+\alpha\,\mathcal{R}(x) \to \min, \quad \text{subject to} \quad x \in \mathcal{D} := \mathcal{D}(F)\cap\mathcal{D}(\mathcal{R}),$$

with convex penalty functionals $\mathcal{R}$, solutions $x^\dagger \in \mathcal{D}(F)$ to $(**)$ are **not smooth enough** w.r.t. $\mathcal{R}$ such that $\underline{\mathcal{R}(x^\dagger) = \infty}$, then for regularized solutions x_α^δ the minimizing property

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\,\mathcal{R}(x_\alpha^\delta) \le \|F(x^\dagger) - y^\delta\|^2 + \alpha\,\mathcal{R}(x^\dagger) \quad (MP)$$

does not help anyway. In $\triangleright$ HOFMANN/MATHÉ 2017 we found situations, where for $\alpha_* = \alpha_*(y^\delta, \delta)$ satisfying

$$\|F(x_{\alpha_*})^\delta - y^\delta\| = \tau\delta, \quad \text{for some} \quad \tau > 1\,, \qquad (DP)$$

$$\alpha_* \to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha_*} \to \infty \qquad \text{as} \qquad \delta \to 0\,. \qquad (\nearrow)$$

Precisely, we have extended the rates results of the seminal paper by ▷ NATTERER 1984 from the linear case $(*)$ to the nonlinear case $(**)$ formulated in a Hilbert scale setting generated by an operator B as follows:

For an unbounded linear operator $B : \mathcal{D}(B) \subset X \to X$, which is self-adjoint and satisfies with some $m > 0$

$$\|Bx\| \geq m\,\|x\| \qquad \forall\, x \in \mathcal{D}(B)\,.$$

The Hilbert scale $\{X_\tau\}_{\tau\in\mathbb{R}}$ is based on norms $\|x\|_\tau = \|B^\tau x\|$.

We consider with $\mathcal{D} := \mathcal{D}(F) \cap \mathcal{D}(B)$ Tikhonov regularization as

$$\|F(x) - y^\delta\|^2 + \alpha\,\|B(x - \bar{x})\|^2 \to \min, \quad \text{subject to} \quad x \in \mathcal{D}.$$

We assume that

$$\bar{x} \in \mathcal{D} = \mathcal{D}(F) \cap \mathcal{D}(B) \text{ and } x^\dagger \in \mathcal{D}(F)\,, \text{ but } \underline{x^\dagger \notin \mathcal{D}(B)}$$

and that there is some $\underline{0 < p < 1 \text{ with } \|B^p x^\dagger\| < \infty}$.

Moreover, let for some $a > 0$ the nonlinearity condition

$$c_a \, \|B^{-a}(x - x^\dagger)\| \le \|F(x) - F(x^\dagger)\| \le C_a \, \|B^{-a}(x - x^\dagger)\| \quad \forall \, x \in \mathcal{D}$$

hold.

Theorem ▷ HOFMANN/MATHÉ 2017

Under the assumptions stated we have for all $0 < p < 1$ the convergence rate

$$\|x_{\alpha_*}^\delta - x^\dagger\| = \mathcal{O}(\delta^{\frac{p}{a+p}}) \qquad \text{as } \delta \to 0$$

whenever the regularization parameter α_* is chosen according to the discrepancy principe (DP).

Overview

Classical Tikhonov regularization in Hilbert spaces requires for **finite penalty values** at $x^\dagger$ to choose α_* according to

$$\alpha_* \to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha_*} \to 0 \qquad \text{as} \qquad \delta \to 0\,. \qquad (\searrow)$$

Under **conditional stability** we should choose α_* according to

$$\alpha_* \to 0 \quad \text{and} \quad 0 < \underline{C} \le \frac{\delta^2}{\alpha_*} \le \overline{C} < \infty \to 0 \quad \text{as} \quad \delta \to 0\,, \quad (\longrightarrow)$$

but in the context of **oversmoothing regularization** as

$$\alpha_* \to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha_*} \to \infty \qquad \text{as} \qquad \delta \to 0. \qquad (\nearrow)$$

Outline

On Lavrentiev regularization in Hilbert space

For a **monotone operator** $F: \mathcal{D}(F) \subseteq X \to X$, i.e.,

$$\langle F(x) - F(\tilde{x}), x - \tilde{x} \rangle \geq 0 \qquad \forall\, x, \tilde{x} \in \mathcal{D}(F), \qquad (Mon)$$

and if $(**)$ is a model of an inverse problem, then due to local ill-posedness it makes sense to solve a **singularly perturbed well-posed auxiliary equation**

$$F(x_\alpha^\delta) + \alpha(x_\alpha^\delta - \bar{x}) = y^\delta\,. \qquad (Lav).$$

The analog w.r.t. $(*)$ (cf. Example 4 in Part A) attains the form

$$A\,x + \alpha(x_\alpha^\delta - \bar{x}) = y^\delta\,. \qquad (Lav).$$

Lavrentiev regularization is simpler than the Tikhonov regularization, but its use it restricted to the smaller class of monotone forward operators.

Useful assumptions

- $F : X \to X, \ \mathcal{D}(F) = X$ (X separable real Hilbert space).
- F is a monotone and hemicontinuous operator.

Then F is even **maximally monotone** and we have a weak-to-norm sequential closedness as

$$x_n \rightharpoonup \tilde{x} \quad \text{and} \quad F(x_n) \to z_0 \quad \Rightarrow \quad F(\tilde{x}) = z_0.$$

There occur well-posed and ill-posed situations.
The best situation of global well-posedness is characterized by **strong monotonicity**

$$\langle F(x) - F(\tilde{x}), x - \tilde{x} \rangle \geq C \, \|x - \tilde{x}\|^2 \qquad \text{for all} \quad x, \tilde{x} \in X,$$

with some constant $C > 0$, which implies the **coercivity** condition

$$\lim_{\|x\| \to \infty} \frac{\langle F(x), x \rangle}{\|x\|} = \infty.$$

Under the above assumptions $F : X \to X$ is **surjective** due to the **Browder-Minty theorem** if the coercivity condition holds. If, moreover, F is strongly monotone, then F is **bijective** and $F^{-1} : X \to X$ is Lipschitz continuous as

$$\|F^{-1}(y) - F^{-1}(\tilde{y})\| \le \frac{1}{C} \|y - \tilde{y}\| \qquad \text{for all} \quad y, \tilde{y} \in X. \qquad (Lip)$$

There are classes of ill-posed inverse problems with monotone F occurring in natural sciences and engineering, where (Lip) fails. Then we have operator equations $(**)$ of the **first kind**, but the associated equations of the **second kind**

$$G(x) = y \qquad \text{with} \qquad G(x) := F(x) + \alpha x$$

satisfy (Lip) with $C = \alpha$ for all $\alpha > 0$.

This motivates Lavrentiev regularization (Lav) for stabilizing $(**)$.

Example ▷ B. KALTENBACHER

As an ill-posed example we consider the identification of the source term q in the elliptic boundary value problem

$$\begin{aligned} -\Delta u + \xi(u) &= q \text{ in } \mathcal{G} \\ u &= 0 \text{ on } \partial\mathcal{G} \end{aligned}$$

from measurements of u in $\mathcal{G}$, where $\xi : \mathbb{R} \to \mathbb{R}$ is some Lipschitz continuously differentiable monotonically increasing function and $\mathcal{G} \subseteq \mathbb{R}^3$ a smooth domain.

Then the corresponding **nonlinear** forward operator
$F : X := L^2(\mathcal{G}) \to H^2(\mathcal{G}) \subseteq L^2(\mathcal{G})$,
mapping $q \mapsto u$, is **monotone and hemicontinuous.**

Lavrentiev regularization is always helpful if bijectivity of F and hence Lipschitz property of F^{-1} fails, for example because coercivity fails or well-posedness occurs only in a local sense.

This is the case if F is **locally strongly monotone**

$$\langle F(x) - F(x^\dagger), x - x^\dagger \rangle \geq C\,\|x - x^\dagger\|^2 \qquad \text{for all} \quad x \in B_r(x^\dagger),$$

with $C > 0$ and $r > 0$, or if F is **locally uniformly monotone**

$$\langle F(x) - F(x^\dagger), x - x^\dagger \rangle \geq \zeta(\|x - x^\dagger\|) \qquad \text{for all} \quad x \in B_r(x^\dagger)$$

with some index function ζ and $r > 0$.

Proposition

Let local uniform monotonicity of F in $B_r(x^\dagger)$ hold with an index function ζ of the form $\zeta(t) = \theta(t)\,t,\ t > 0$, such that θ is also an index function. Then we have a conditional stability estimate

$$\|x - x^\dagger\| \le \theta^{-1}(\|F(x) - F(x^\dagger)\|) \quad \text{for all} \quad x \in B_r(x^\dagger), \qquad (CSE)$$

and the operator equation $(**)$ is locally well-posed at $x^\dagger$.
In the special case $\zeta(t) = C\,t^2$ of strong monotonicity we find

$$\|x - x^\dagger\| \le \frac{1}{C}\,\|F(x) - F(x^\dagger)\| \quad \text{for all} \quad x \in B_r(x^\dagger). \qquad (CSE)$$

One dimensional example

For $X := \mathbb{R}$ with $\|x\| := |x|$ we consider $(**)$ with the continuous monotone operator $F : \mathbb{R} \to \mathbb{R}$ defined for exponents $\kappa > 0$ as

$$F(x) := \begin{cases} -1 & \text{if} \quad -\infty < x < -1 \\ -(-x)^\kappa & \text{if} \quad -1 \le x \le 0 \\ x^\kappa & \text{if} \quad 0 < x \le 1 \\ 1 & \text{if} \quad 1 < x < \infty \end{cases},$$

which, however, is not bijective and not coercive.

Then we have local ill-posedness at $x^\dagger$ if $x^\dagger \le -1$ or $x^\dagger \ge 1$. At $x^\dagger = 0$ we have local well-posedness for all $\kappa > 0$ due to a local uniform monotonicity condition with $\zeta(t) = t^{\kappa+1}$ such that

$$\|x - x^\dagger\| \le \|F(x) - F(x^\dagger)\|^{1/\kappa} \quad \text{for all} \quad x \in B_1(0). \qquad (CSE)$$

Lavrentiev regularization allows for linear convergence if $\kappa = 1$, Hölder convergence rates for $\kappa > 1$, and there even occurs a superlinear convergence rate at $x^\dagger = 0$ if $0 < \kappa < 1$.

Convergence rates under variational source conditions

Let us consider a variational source condition adapted to the monotonicity structure:

Variational source condition (VSC-Lav)

We assume to have a constant $0 \le \beta < 1$ *and an index function* φ *such that for all* $x \in \mathcal{M}$

$$\langle x^\dagger - \bar{x}, x^\dagger - x \rangle \le \beta \|x^\dagger - x\|^2 + \varphi(\langle F(x) - F(x^\dagger), x - x^\dagger \rangle).$$

Here, φ must be an index function with $\lim_{t \to +0} \frac{\varphi(t)}{\sqrt{t}} \ge \underline{c} > 0$.

The next theorem is proved by the general Young inequality

$$ab \le \int_0^a f(t)\, dt + \int_0^b f^{-1}(t)\, dt$$

for $a, b \ge 0$ and an index function f.

Theorem ▷ H./KALTENBACHER/RESMERITA 2016

Let $(VSC - Lav)$ be satisfied for a solution to $(**)$ with $\mathcal{M}$ such that for the choice of $\alpha > 0$ all regularized solutions x_α^δ from (Lav) belong to $\mathcal{M}$ for sufficiently small $\delta > 0$. Then one has the estimate

$$\|x_\alpha^\delta - x^\dagger\|^2 \le \frac{1}{(1-\beta)^2}\frac{\delta^2}{\alpha^2} + \frac{2}{1-\beta}\Psi(\alpha),$$

for such $\delta > 0$, where $\Psi(\alpha)$ is introduced as follows:
Let f be an index function such that its antiderivative $\tilde{f}(s) := \int_0^s f(t)dt$ satisfies the condition $\tilde{f}(\varphi(s)) \le s$ for $s > 0$ and let $G(\alpha) \ge \int_0^\alpha f^{-1}(\tau)d\tau$. Then we set $\Psi(\alpha) := \frac{G(\alpha)}{\alpha}$.

Theorem cont.

Moreover, for the a priori choice

$$\alpha_*(\delta) \sim \Theta^{-1}(\delta^2)$$

with $\Theta(\lambda) := \lambda^2 \Psi(\lambda)$, which satisfies

$$\alpha_*(\delta) \to 0 \quad \text{and} \quad \frac{\delta}{\alpha_*(\delta)} \to 0 \quad \text{as} \quad \delta \to 0,$$

this yields the convergence rate

$$\|x^\delta_{\alpha_*(\delta)} - x^\dagger\| = \mathcal{O}\left(\frac{\delta}{\Theta^{-1}(\delta^2)}\right) = \mathcal{O}\left(\sqrt{\Psi(\Theta^{-1}(\delta^2))}\right).$$

The best possible rate occurs for $\varphi(t) \sim \sqrt{t}$ and $\Psi(t) \sim t$ as

$$\|x^\delta_{\alpha_*(\delta)} - x^\dagger\| = \mathcal{O}(\delta^{\frac{1}{3}}) \qquad \text{if} \quad \alpha_*(\delta) \sim \delta^{\frac{2}{3}}.$$

Logarithmic type variational source conditions

We consider $(VSC - Lav)$ with

$$\varphi(t) = \frac{1}{-\ln t}, \quad f(t) = e^{-\frac{1}{t}} \le \frac{1}{t^2} e^{-\frac{1}{t}} = \varphi^{-1\prime}(t), \quad f^{-1}(t) = \frac{1}{-\ln t},$$

$$G(\alpha) = \alpha \frac{1}{-\ln \alpha} \ge \int_0^\alpha \frac{1}{-\ln t}\, dt, \quad \Theta(t) \sim t^2 \frac{1}{-\ln t}, \text{ for } \alpha, t \in (0,1).$$

This yields the logarithmic rate $\mathcal{O}\left(\sqrt{\Psi(\Theta^{-1}(\delta^2))}\right)$.

Since the a priori choice $\alpha_* = \alpha_*(\delta) \sim \Theta^{-1}(\delta^2)$ cannot be determined explicitly in this logarithmic case, a more convenient choice is $\alpha_*(\delta) \sim \sqrt{\delta}$ which implies

$$\|x_{\alpha_*}^\delta - x^\dagger\| = \mathcal{O}\left(\frac{1}{\sqrt{-\ln(\delta)}}\right) .$$

Hölder type variational source conditions

For exponents $\mu \in (0, \frac{1}{2}]$, we consider $(VSC - Lav)$ with

$$\varphi(t) = t^{\mu}, \quad f(t) = \frac{1}{\mu} t^{\frac{1-\mu}{\mu}}, \quad f^{-1}(t) = (\mu t)^{\frac{\mu}{1-\mu}},$$

$$\tilde{f}(s) = s^{\frac{1}{\mu}}, \quad G(\alpha) = (1-\mu)\mu^{\frac{\mu}{1-\mu}} \alpha^{\frac{1}{1-\mu}}, \quad \Phi(t) \sim t^{\frac{2-\mu}{1-\mu}}.$$

According to the theorem this yields the Hölder rate

$$\|x_{\alpha_*}^{\delta} - x^{\dagger}\| = \mathcal{O}(\delta^{\frac{\mu}{2-\mu}}) \qquad \text{if} \quad \alpha_*(\delta) \sim \delta^{\frac{2(1-\mu)}{2-\mu}}.$$

Note that $\mathcal{O}\left(\delta^{\frac{\mu}{2-\mu}}\right) = \mathcal{O}\left(\delta^{\frac{p}{p+1}}\right)$ for $\mu := \frac{2p}{2p+1},\ 0 < p \leq \frac{1}{2}$.

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