Experimental identification of transfer functions for diffusive and/or advective heat transfer for linear time invariant dynamical systems

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New Trends in Parameter Identification for Mathematical Models
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Contribution: Waseem Al Hadad
Experimental inverse problems in heat transfer and engineering  
METTI Group, SFT (French Heat Transfer Society)

Recently: interest in **convolutive models** and associated **inverse problems**

* Pollutant source identification in a ventilated domain (turbulence, transient concentration measurements)

* Transient thermal behaviour of heat exchanger (PhD W. Hadad, Fives Cryo postDoc)

* Virtual sensor construction in a furnace under vacuum conditions (PhD T. Loussouar, Safran Group)
Scope

1. Forced thermal response of Linear advective/diffusive systems with Time Independent (LTI) coefficients

2. The calibration problem
   2.1 Case of a heat exchanger
   2.2 Experimental Impedance/transmittance estimation for a half heat exchanger

3. Analysis of deconvolution deadlocks
   3.1 Reference case: 1D transient conduction
   3.2 Noisy matrix and Total Least Squares
   3.3 Comparison of calibration methods

4. Rectangular deconvolution
   4.1 Point versus averaged values for input and unknown
   4.2 Rectangular deconvolution (using « stairs » parameterizing)
   4.3 Rectangular estimation with \( n < m \) non uniform (NU) time steps

5. Conclusions/perspectives
1. Forced thermal response of an advective/diffusive system with time constant coefficients

Material *multicomponent* system = $K$ solid or fluid domains

Assumptions: *time constant* thermophysical properties and *velocity field*
Initial uniform state or steady state temperature field
+ one single separable thermal excitation

\[ Q_{\text{in}}(t) = m_{\text{in}} c_{\text{in}} T_{b}^{\text{in}}(t) \]

\[ Q_{S}(t) \text{ or } T_{S}(t) \neq T_{\text{init}} \]

\[ T_{\infty}(t) \neq T_{\text{init}} \]

**Time part of thermal excitation** \( u(t) \) (starts at time \( t = 0 \)):

- volumetrical heat source \( Q_{V}(t) \)
- surface heat or temperature source \( Q_{S}(t) \) or \( T_{S}(t) \)
- change of external fluid temperature \( T_{\infty}(t) \neq T_{\text{init}} \)
- change of temperature at one fluid inlet \( T_{b}^{\text{in}}(t) \)

**Fixed geometrical support**:

- point
- line
- surface
- volume
Change of perspective: one single heterogeneous fluid in one single domain
(if solid part: zero velocity)

\[ Q_{in}(t) = \dot{m}_{in} \dot{c}_{in} T_{in}^{b}(t) \]

\[ Q_s(t) \text{ or } T_s(t) \neq T_{init} \]

\[ T_{\infty}(t) \neq T_{init} \]

Transient separable thermal excitation:

Point response at any point \( P \):

\[ y(t) \equiv T(P, t) \]
Recap:

Physical system:
Set of solids AND fluid(s):
3D forced convection with constant velocities (in time but not in space)
P = ANY point in the system

One single thermal excitation defined by its support

Assumptions: Transient heat equation + boundary conditions with time-invariant coefficients + uniform initial temperature or steady state (the system is Linear and also Time-Invariant LITI)

\[
\begin{align*}
\rho c(P) \frac{\partial T}{\partial t}(P,t) + \rho c(P) \bar{u}(P) \cdot \bar{V} T(P,t) &= \bar{V} \cdot \left( \lambda(P) \bar{V} T(P,t) \right) + \frac{Q_v(t)}{V_{source}} f(P) \\
\text{Transient} & \quad \text{Advection} & \quad \text{Conduction} & \quad \text{Internal source}
\end{align*}
\]
Temperature rise at any point P:

\[ \theta(P, t) = T(P, t) - T_{\text{init}}(P) \]

Its Laplace transform:

\[ \tilde{\theta}(P, p) = \int_0^\infty \exp(-pt) \theta(P, t) \, dt \]

**Assumptions**: Transient heat equation + boundary conditions with time-invariant coefficient + uniform initial temperature (the system is **Linear** and also **Time-Invariant** LITI)

\[
\rho c(P) \frac{\partial T}{\partial t}(P, t) + \rho c(P) \vec{u}(P) \cdot \nabla T(P, t) = \nabla \cdot \left( \lambda(P) \nabla T(P, t) \right) + \frac{Q_v}{V_{\text{source}}} f(P)
\]

**Consequences**: Laplace transformed heat equation (no time derivative)

\[
\rho c(P) p \tilde{\theta}(P, p) + \rho c(P) \vec{u}(P) \cdot \nabla \tilde{\theta}(P, p) = \nabla \cdot \left( \lambda(P) \nabla \tilde{\theta}(P, p) \right) + \frac{Q_v(p)}{V_{\text{source}}} f(P)
\]
**Linear system with a single excitation**

Temperature or flux

⇒ response at any point P in the system

= simple product (Laplace domain)

\[ \bar{y}(P, p) = \bar{H}(P, p) \bar{u}(p) \]

or convolution product (time domain)

\[ y(P, t) = H(P, t) * u(t) = \int_0^t H(P, t-t') u(t') \, dt' \]

**Excitation u(t):**

- \( u(t) = Q_v(t) - Q_v^{init} \) or \( Q_s(t) - Q_s^{init} \)
- or \( T_s(t) - T_{s}^{init}(P_s) \)
- or \( T_{\infty}(t) - T^{init}_{\infty} \)
- or \( T_{b}^{in}(t) - T_{b}^{in,init} \)

**Response y(t) in any specific point P:**

\( y(t) = \theta(P, t) = T(P, t) - T_{init}(P) \)

or local heat flux in any direction \( \phi_x(P, t) \)

**Transfer function**

\( H(P, t) \)

**Response y**

<table>
<thead>
<tr>
<th>Excitation u</th>
<th>Response y</th>
<th>Transfer function H</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power source ( Q ) (watts)</td>
<td>Temperature difference ( \theta ) (kelvins)</td>
<td>Impedance ( Z ) (K.J^{-1})</td>
</tr>
<tr>
<td>Temperature difference ( \theta ) (kelvins)</td>
<td>Temperature difference ( \theta ) (kelvins)</td>
<td>Transmittance ( W ) (s^{-1})</td>
</tr>
<tr>
<td>Power source ( Q ) (watts)</td>
<td>Rate of heat flow ( \Phi ) (watts)</td>
<td>Transmittance ( W ) (s^{-1})</td>
</tr>
<tr>
<td>Temperature difference ( \theta ) (kelvins)</td>
<td>Rate of heat flow ( \Phi ) (watts)</td>
<td>Admittance ( Y ) (W.K^{-1}.s^{-1})</td>
</tr>
</tbody>
</table>
\[ y(P,t) = H(P,t) \ast u(t) = \int_0^t H(P,t-t') u(t') \, dt' \]

Steady state (ss) version of a transfer function

\[ H^{ss} = \frac{y^{ss}}{u^{ss}} = \int_0^\infty H(t) \, dt \]

asymptotic values

Time distribution

\[ T_{in} - T_{out} = R \, \Phi \]

Thermal resistance, flux pipe between 2 isothermal surfaces

\[ T^{ss} - T_{\infty} = Z^{ss} \, Q \]

Generalized resistance, no flux pipe

\[ u \equiv Q \text{ or } Q - Q^{ss} \text{ (thermal power)} \]
\[ y \equiv \theta = T - T_{init} \text{ (temperature variation)} \]
\[ \Rightarrow H \equiv Z \text{ (thermal impedance)} \]

\[ \text{Denis Maillet, Waseem Al Hadad, New trends in parameter identification, IMPA, Rio de Janeiro, Brasil, oct.-nov. 2017} \]
2. The calibration problem

2.1 Case of a heat exchanger

Assumptions

- Constant thermo-physical properties (fluid and walls) and velocities (LTI heat equation):
  \[ \frac{\partial \beta}{\partial t} = 0 \]
  \[ \beta \equiv u_{\text{hot}}, u_{\text{cold}}, \lambda, \rho, \ldots \]

- Uniform initial conditions/initial steady state:
  \[ T(P, t=0) = T_{\text{init}} \]
  \[ T(P, t=0) = T_{\text{ss}}(P) \]

- Heat losses through convection/(linearized) radiation with environment through a uniform heat transfer coefficient \( h \) at temperature \( T_\infty = T_{\text{init}} \)

- One single heat source (inlet temperature increase) that starts at \( t = 0^+ \):
  \[ \theta_1(t) = T_1(t) - T_{\text{init}} \neq 0 \]

  Cause
  \[ \theta_4(t) = T_4(t) - T_{\text{init}} = 0 \]

  \[ \frac{\theta_4(t)}{\theta_4(t < 0)} \neq 0 \]

- Transient/unsteady thermal regime with observed responses at any point \( q \):
  \[ \theta_q(t) = T_q(t) - T_{\text{init}} \]
  \[ \theta_q(t \leq 0) = 0 \] and \( \theta_q(t > 0) \neq 0 \)

Consequences
Calculation of convolution products (transmittance case)
Parameterizing with piecewise constant functions, square case

$$\theta (P, t) = W (P, t) \ast \theta_1 (t)$$

$$= \int_0^t W (P, t-t') \theta_1 (t') \, dt'$$

$$= \int_0^t \theta_1 (P, t-t') W (P, t') \, dt'$$

$$\theta (P, t_i) \approx \Delta t \sum_{j=1}^m \theta_{j,i-j+1} W_j (P)$$

t_0 = 0 ; t_i = i \Delta t \text{ for } i = 1 \text{ to } m ; \Delta t = t_{\text{final}} / m$

$$z_i = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} z (t) \, dt \approx \frac{1}{2} \left( z(t_{i-1}) + z(t_i) \right) \text{ for } z(t) = \theta_1 \text{ or } W (P)$$

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First experiment:
- Calibration (inverse) problem

Next experiments:
- virtual sensor inverse problem (same as source estimation problem)
Model for $W$ identification calibration problem:

$$\theta_q = M(\theta_1) W^q$$

$\theta_q^{\exp} = \theta_q + \varepsilon_q$ and $\theta_1^{\exp} = \theta_1 + \varepsilon_1$

Ordinary (linear) least squares:

$$\hat{W}^q = \left(M(\theta_1^{\exp})\right)^{-1} \theta_q^{\exp}$$

Ill-posed problem: Inversion needs **regularization**

Here: Truncated SVD or 0 order Tikhonov

with discrepancy principle (Morozov)
Practical way of making the inlet temperature vary

\[ T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow T_4 \]

\[ Q(t) \rightarrow Z_1 \rightarrow \theta_1(t) \rightarrow W_1^q \rightarrow \theta_2(t) \rightarrow \theta_3(t) \]

\[ \theta_1 = M(Q) \cdot Z_1 \]

\[ \theta_q = M(\theta_1) \cdot W_1^q \]

2.2 – Experimental Impedance/transmittance estimation for a half heat exchanger

Identification of transmittance using experimental transient measurements (calibration)

<table>
<thead>
<tr>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_f$</th>
<th>$2l$</th>
<th>$2l_c$</th>
<th>$w$</th>
<th>$l_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>65</td>
<td>120</td>
<td>50</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Response: thermocouples

$d = 50.8 \, \mu m$
Identification of transfer function using **experimental** temperature recording:

Comparison of identified $Z$: step or periodical heating

**Step heating**

$T_{in}(t)$

$T_{out}(t)$

Upper and lower tranquilization chambers

identified impedances

$Z_{in}$ Step - Tikhonov

$Z_{out}$ Step - Tikhonov

$Z_{in}$ Periodical – Tikhonov

$Z_{out}$ Periodical – Tikhonov

$Z_{in\ or\ out} = (\mathbf{M}(Q))^{-1} \theta_{in\ or\ out}$

**Square periodical heating**

$Z$ (K J$^{-1}$)

Time (s)
Comparison of identified transmittance $W$ (outlet/inlet): step or periodical heating

**Step heating**

$T_\infty = 20.3^\circ C$

**Square periodical heating**

$T_\infty = 21.5^\circ C$, period = 129 s

Oscillations past first peak and for long times, zero initial level hard to recover:

Estimation of transmittance $W$ (noisy output and input) more difficult than $Z$ (noisy output only)

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3. Analysis of deconvolution deadlocks

3.1 Reference case: 1D transient conduction

Heat eq. \[
\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{a} \frac{\partial \theta}{\partial t}
\]
boundary \[
\varphi = -\lambda \frac{\partial \theta}{\partial t} = q(t) \quad \text{at} \quad x = 0 \quad \text{for} \quad t > 0; \\
\varphi = -\lambda \frac{\partial \theta}{\partial t} = h \theta \quad \text{at} \quad x = \ell \quad \text{for} \quad t > 0
\]
initial \[
\theta = 0 \quad \text{at} \quad t = 0 \quad \text{for} \quad 0 \leq x \leq \ell
\]
Laplace transform \[
\bar{\psi} (x, p) = \int_0^t \psi(x, t) \exp(-pt) \, dt \quad \text{for} \quad \psi = \theta \quad \text{or} \quad \varphi
\]
Transmittance \[
W(t) = \mathcal{L}^{-1} \left[ \frac{1}{\cosh(\beta \ell) + h \sinh(\beta \ell) / (\lambda \beta)} \right]
\]
with \( \beta^2 = p/a \)

\[\theta(t) = T(t) - T_\infty \]

change of variables

\[H \equiv W, \quad u \equiv \theta_1, \quad y \equiv \theta_2\]
Comparison: analytical $W$ and identified $W$ from synthetic profiles (COMSOL)

Analytical Laplace $W$ + numerical inversion of Laplace

\[ W(t) = L^{-1} \left[ \frac{1}{\cosh (\beta \ell) + h \sinh (\beta \ell) / (\lambda \beta)} \right] \]

Identified $W$ by OLS and TLS

\[ W \text{ from } \theta_1 \text{ and } \theta_2 \]

Numerical Inversion of Laplace Transforms by Hoog’s algorithm

Input: \[ \theta_1(t) = \left( 1 - e^{-\frac{t}{\tau}} \right) \theta_1^{ss} \text{ with } \tau = 30 \text{ s} \quad ; \quad \theta_1^{ss} = 30 \, ^\circ\text{C} \quad \text{and} \quad \Delta t = 0.5 \text{ s} \]

<table>
<thead>
<tr>
<th>$t_f$ (s)</th>
<th>$\ell$ (mm)</th>
<th>$h$ (W.m$^{-2}$.K$^{-1}$)</th>
<th>$\lambda$ (W.m$^{-1}$.K$^{-1}$)</th>
<th>$\rho c$ (kJ.m$^{-3}$.K$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>700</td>
<td>50</td>
<td>10</td>
<td>43</td>
<td>3666</td>
</tr>
</tbody>
</table>
Comparison: analytical W and identified W from synthetic profiles (COMSOL)

Comparison without noise:

Temperature profiles (COMSOL)

Validation without committing an INVERSE CRIME

Analytical and identified W (without regularization)
3.2 Noisy matrix and Total Least Squares

**Calibration problem:** how to get the best transfer function $H(t)$ at a point $P$?

**Model**

$$y(P, t) = u(t) * H(P, t) = \int_{0}^{t} u(t-t') H(t-t') \, dt'$$

$$y = M(u) \, H$$  

or  

$$y = A \, H$$  

with  

$$A = M(u)$$

**Measurements**

Available information: discrete noisy values of $y(P, t)$ and $u(t)$

$$y^{exp}(t_i) = y(t_i) + \varepsilon_i$$

$$u^{exp}(t_i) = u(t_i) + \tau_i$$

$$\Rightarrow \quad A^{exp} = A + \varepsilon_u$$  

with  

$$\varepsilon_u = M(\tau)$$

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Identification from \( u \) and \( y \) measurements:

\[ y = M(u)H \]

Total Least Squares (TLS) solution:

Augmented matrix \( G = [M(u) | y] \)

\( \hat{H} \) such as minimum residuals \( r_y \) and \( r_u \)

Frobenius norm:

\[ J_{TLS}(H) = \| r_G(H) \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{m+1} g_{i,j}^2 \]

with \( r_G(H) = G^{exp} - G \)

SVD Form of \( G \) → ill-posed → Regularized form: Truncated TLS
3.3 Comparison of calibration methods

Comparison: analytical $W$ and identified $W$ from synthetic profiles (COMSOL)

Comparison with noise: $\theta_1$ noised

$\sigma_1 = 1K$, $\sigma_2 = 0K$

Temperature profiles (COMSOL)

Analytical and identified $W$ (with regularization)

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Comparison: analytical W and identified W from synthetic profiles (COMSOL)

Comparison with noise: $\theta_2$ noised

Temperature profiles (COMSOL)

Analytical and identified W (with regularization)

Noise over signal ratio: $\frac{\|\theta_2^{\text{noised}} - \theta_2^{\text{exact}}\|}{\|\theta_2^{\text{noised}}\|} = 3.90\%$
Comparison: analytical $W$ and identified $W$ from synthetic profiles (COMSOL)

Comparison with noise: $\theta_1, \theta_2$ noised

$\sigma_1 = \sigma_2 = 1K$

Temperature profiles (COMSOL)

Analytical and identified $W$ (with regularization)
Conclusion of this comparison of deconvolution techniques

- Noise on the response $\theta_2$ more penalizing than noise on the source $\theta_1$.

- The truncated total least squares do not take into account the convolutive structure of the matrix $M(\theta_1)$
  $\Rightarrow$ no improvement of the estimate

- Important to improve short times values of the identified TF in a calibration experiment:
  $\Rightarrow$ largest impact on posterior inverse input estimation experiments.

![Graphs showing comparison of deconvolution techniques](image)
4. Rectangular deconvolution

Calibration problem:

\[ \theta_2 = A \mathbf{W} \quad \text{with} \quad A = M(\theta_1) \]

Can the ill-posed problem be more parsimonious?

\[ \rightarrow \] Less many unknowns \( n \) than output data: \( n \ll m \)

Idea: tailoring the definition of the estimate

Calibration = model identification for a given structure

2 paths deserve to be investigated **BEFORE** regularization:

- Rectangular parametrization using piecewise constant parametrization for \( W \)
- ARX model construction (perspective)
4.1. Point versus averaged values for input and unknown

Singular values of $\mathbf{M}(\theta_1)$ depending on type of parameterization of $\theta_1$ and on its noise

$$\mathbf{M}(\theta_1) = \mathbf{U} \mathbf{S} \mathbf{V}^T$$

$$s_1 \left( \mathbf{M}(\theta_{1, \alpha, \sigma_1}) \right) \approx s_1 \left( \mathbf{M}(\theta_{1, \alpha, \sigma_1}) \right) \quad \text{cond} \left( \mathbf{M}(\theta_{1, \alpha}) \right) = s_1 / s_{\alpha} \quad \text{for} \quad 1 \leq \alpha \leq m$$

$\alpha$: truncation order - $\sigma_1$ = noise level

$$\text{cond} \left( \mathbf{M}(\theta_{1, \sigma_1}) \right) \geq \text{cond} \left( \mathbf{M}(\theta_{1, \alpha, \sigma_1}) \right)$$
**Effect of the type of parameterization: noise on \( \theta_2 \) only**

Sampled = parameters = instantaneous value (parameterization over a basis of « hat » functions): \( z_i = z(t_i) \)

Parameterized = averaged value (parameterization over a base of « doors » functions): \( z_i = 0.5 \left( z(t_i) + z(t_{i-1}) \right) \)

for \( z = \theta_1 \) or \( W \)

\[
\text{Estimated transmittance } W \ (1/s)
\]

Exact \( W \): sampled (continuous) and parameterized (stairs)

Estimated \( W \): sampled (continuous) and parameterized (stairs)

i.i.d. noise on the output: \( \sigma_2 = 1 \) K; \( \varepsilon_1 = 0 \)

Truncated SVD: \( \alpha = 19 \) for \( m = 1400 \) times

no noise: \( \varepsilon_1 = \varepsilon_2 = 0 \)

\[
\text{bias} (\beta) = E \left( \hat{\beta} \right) - \beta^{\text{exact}}
\]

\[
\left\| \text{bias} \left( \hat{W}_{\alpha_{\text{opt}}, \sigma_2}^{\text{stairs}} \right) \right\| \leq \left\| \text{bias} \left( \hat{W}_{\alpha_{\text{opt}}, \sigma_2}^{\text{sampled}} \right) \right\|
\]
4.2 Rectangular deconvolution (using « stairs » parameterizing)

\[ z = W \quad \text{or} \quad \theta_1 \]

\[ c = 3 \]

- Less many parameters than measurement times: \( n \ll m \)
- Simplest method = use of a basis of \( n \) piecewise constant functions:
  \[ m/n = c \text{ (integer)} \]

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Square model:
\[ \theta_2^{\text{square}} = M (\theta_1) W^{\text{square}} = A W^{\text{square}} \]

Rectangular model:
\[ \theta_2^{\text{rectangular}} = M (\theta_1) W^{\text{square}} = X W \]

The \( n \) rectangular sensitivity vectors are simply the averaged values of the \( m \) square ones.

The sensitivity matrix and sensitivity vectors are repeated for some lines.
Effect of rectangular inversion: 1 unknown every $c=16$ time steps

Square & rectangular estimation, no noise: $\varepsilon_1 = \varepsilon_2 = 0$

i.i.d. noise on the output: $\sigma_2 = 1\,K ; \varepsilon_1 = 0$

Square Truncated SVD: $\alpha = 19$ for $n = m = 1400$ times

Rectangular TSVD, for $n = 87$

Rectangular case with $n = 87$ unknowns $\quad \text{cond}(A) = 2530$

Same relative RMS residual

$$\left\| \theta_2^{\sigma_2} - \theta_2^{\text{square, recalculated}}(\hat{W}_{\alpha_{opt}}) \right\| / \left\| \theta_2^{\sigma_1} \right\| = 3.88\%$$

$$\left\| \theta_2^{\sigma_2} - \theta_2^{\text{rectangular, recalculated}}(\hat{W}_{OLS}) \right\| / \left\| \theta_2^{\sigma_1} \right\| = 3.88\%$$

No gain in term of estimation bias
Norm of each sensitivity vector ($1/s$)

Study of the norm (length) of the sensitivity vectors

(rectangular case, $n = 87$ sensitivity vectors)

$s_{k}$

(square case, $m = 1400$ sensitivity vectors)

$A_{j}$

low sensitivity to long times estimates

time relative to each sensitivity vector (s)
4.3 Rectangular estimation with \( n < m \) non uniform (NU) time steps

\[
X^\text{NU}_k = \frac{1}{c_k} \sum_{j = a_k-1+1}^{a_k} A_j \quad \text{where} \quad a_k = \sum_{k' = 1}^{k-1} c_{k'}
\]

Each of the \( n \) rectangular sensitivity vectors are simply the averaged values of \( c_k \) square ones

\[
\theta_{\text{rectangular, NU}}^2 = X^\text{NU} W \quad \text{where} \quad X^\text{NU} \equiv [X_1 \ X_2 \ \cdots \ X_n]
\]

with \( X^\text{NU} = G^\text{NU} A = G^\text{NU} \ M (\theta_1) \)

**Question:** how to chose the \( n \) limits of the \( n \) different time steps?
First try: constant level past time $t = 400$ s (steady state reached)

Study of the norm (length) of the sensitivity vectors

- Norms of the sensitivity vectors (1/s)
  - $\| X_{k}^{NU} \|$ (NU rectangular case, $n = 87$ sensitivity vectors)
  - $\| X_{k} \|$ (rectangular case, $n = 51$ sensitivity vectors)
  - $\| A_{j} \|$ (square case, $m = 1400$ sensitivity vectors)

Time interval relative to each sensitivity vector (s)
Rectangular TSVD, NU case, \( n = 51, \) 18 singular values kept

Exact \( W, m = n = 1400 \)

Rectangular TSVD, for \( n = 87, \) 19 singular values kept

Improvement of zero level and of the transmittance for short times
5. Conclusions/perspectives

- Importance and applicability of **transfer functions** (impedances, transmittances, ...) in (exact) **reduced** convolutive model structures for **Linear Time Invariant** physical systems (detailed model = PDE, integro-differential equations, ...)

- Convolution products can be given a **commutative** vector/matrix form in discrete time

  ⇒ ill-posed inverse problems: identification problem (**calibration** first, inverse input problem (**source estimation**) or inverse (or direct) **virtual sensor**, or use in a Non Destructive Testing procedure **next**

- Path to improve the quality of estimation of transmittance: **rectangular deconvolution** and pertinent way of **tayloring** its unknown parameters

- Perspective: use of **ARX structures** (AutoRegressive models with eXternal inputs) for better estimation of transfer functions
Muito obrigado!
Model for $W$ identification
calibration problem:

\[ \theta_q = M(\theta_1) W^q \]

\[ \theta_{q\exp} = \theta_q + \varepsilon_q \text{ and } \theta_{1\exp} = \theta_1 + \varepsilon_1 \]

- Ordinary least squares:
  \[ \hat{W}^q = (M(\theta_{1\exp})^{-1}{\theta}_{q\exp} \]

- SVD decomposition of square sensitivity matrix:
  \[ M(\theta_{1\exp}) = U S V^T \text{ with } S = \text{diag}(s_1, s_2, \ldots, s_m) \]

- TSVD:
  \[ \hat{W}_{\alpha}^q = V S_{\alpha}^{-1} U^T \theta_{q\exp} \text{ with } S_{\alpha}^{-1} = \text{diag}(1/s_1, 1/s_2, \ldots, 1/s_\alpha, 0, \ldots, 0) \]

- Zero order Tikhonov:
  \[ \hat{W}_{\mu}^q = \text{Arg}(\min_{W} \left( \| r(W) \|_2^2 + \mu \| W \|_2^2 \right)) \text{ where } r(W) = \theta_{q\exp} - M(\theta_{1\exp}) W \]

or:
  \[ \hat{W}_{\mu}^q = V S_{\mu}^{-1} U^T \theta_{q\exp} \text{ with } S_{\mu}^{-1} = F_{\mu} S \text{ where } F_{\mu} = \text{diag}\left( \frac{s_1^2}{\mu^2 + s_1^2}, \frac{s_2^2}{\mu^2 + s_2^2}, \ldots, \frac{s_m^2}{\mu^2 + s_m^2} \right) \]

- Choice of the hyperparameters $\gamma = \alpha$ or $\mu$ by discrepancy principle (Morozov):
  \[ \| r(\hat{W}_{\gamma}) \|_2^2 \approx m \sigma^2 \]

Ill-posed problem:
Inversion needs **regularization**
Here: Truncated SVD or 0 order Tikhonov
Total Least Squares (TLS) solution: augmented matrix $G = \begin{bmatrix} M(u) \mid y \end{bmatrix}$

$\hat{H}$ such as minimum residuals $r_y$ and $r_u$

$$J_{TLS}(H) = \left\| r_G(H) \right\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{m+1} g_{ij}^2$$

with $r_G(H) = G^{exp} - G$

$$\hat{H}_{TLS} = -V_{12}v_{22}^{-1}$$

with $G^{exp} = U \sum V^T$

and $V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$

Regularized form: T-TSVD

$$\hat{H}_{T-TLS} = -V_{12}^{\alpha_2} \left( V_{22}^{\alpha_2} \right)^T / \left\| V_{22}^{\alpha_2} \right\|^2$$

with $V = \begin{bmatrix} V_{11}^{\alpha_2} & V_{12}^{\alpha_2} \\ V_{21}^{\alpha_2} & V_{22}^{\alpha_2} \end{bmatrix}$
4.1. Point versus averaged values for input and unknown

Back to the parameterization problem

\[ \theta_2 = \mathbf{M} (\theta_1) \mathbf{W} \]

- \( z(t) = W(t) \) or \( \theta_1(t) \): defined on a basis of \( m \) piecewise constant functions

- \( \theta_2 \): vector of \( m \) sampled values

\[ z_i = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} z(t) \, dt \approx \frac{1}{2} \left( z(t_{i-1}) + z(t_i) \right) \Rightarrow \text{cond} \left( \mathbf{M}(\theta_i^{\text{stairs}}) \right) = 3.64 \times 10^8 \]

for \( z = \theta_1 \) or \( W \)

Other choice (sampled values)

\[ z_i = z(t_i) \quad \text{for} \quad z = \theta_1 \) or \( W \] \Rightarrow \text{cond} \left( \mathbf{M}(\theta_i^{\text{sampled}}) \right) = 2.05 \times 10^5