Experimental identification of transfer functions for diffusive and/or advective heat transfer for linear time invariant dynamical systems

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New Trends in Parameter Identification for Mathematical Models
IMPA, Rio de Janeiro, Brazil, October 30 – November 3, 2017

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Experimental inverse problems in heat transfer and engineering
METTI Group, SFT (French Heat Transfer Society)

Recently: interest in **convolutive models** and associated **inverse problems**

* Pollutant source identification in a ventilated domain (turbulence, transient concentration measurements)

* Transient thermal behaviour of heat exchanger (PhD W. Hadad, Fives Cryo postDoc)

* Virtual sensor construction in a furnace under vacuum conditions (PhD T. Loussouar, Safran Group)

Scope

1. **Forced thermal response of Linear advective/diffusive systems with Time Independent (LTI) coefficients**

2. **The calibration problem**

   2.1 Case of a heat exchanger
   2.2 Experimental Impedance/transmittance estimation for a half heat exchanger

3. **Analysis of deconvolution deadlocks**

   3.1 Reference case: 1D transient conduction
   3.2 Noisy matrix and Total Least Squares
   3.3 Comparison of calibration methods

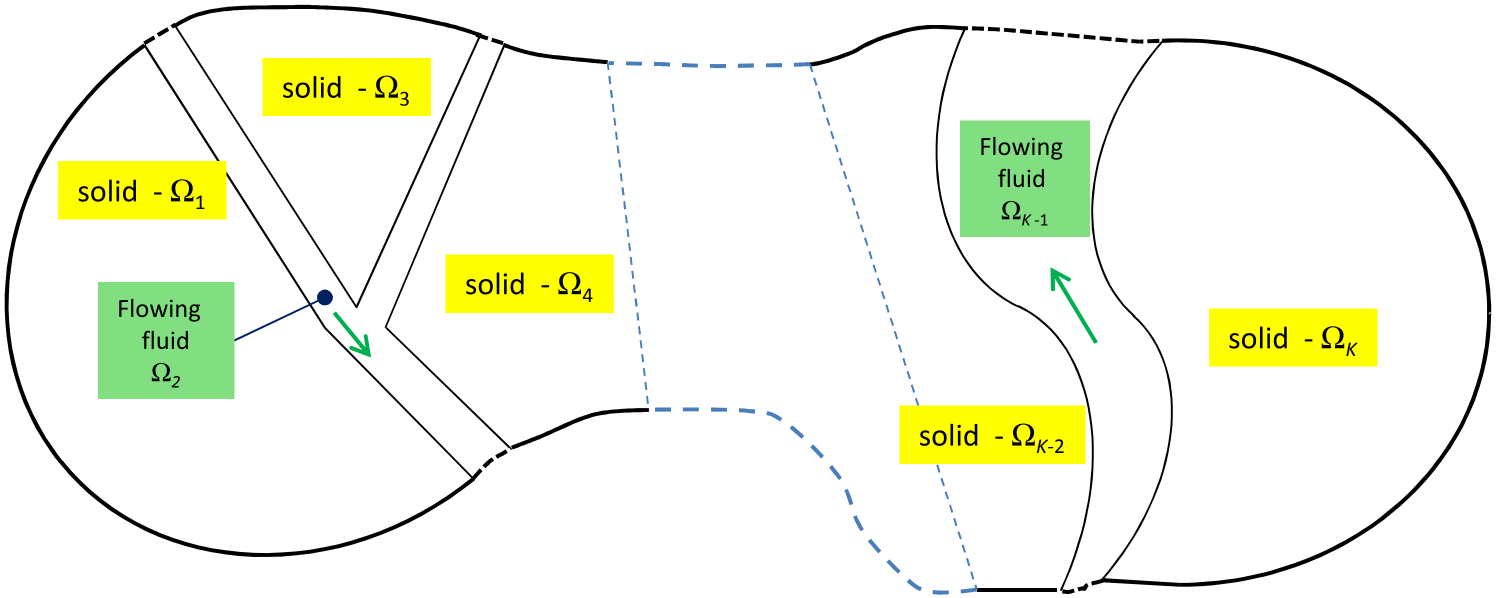
4. **Rectangular deconvolution**

   4.1. Point versus averaged values for input and unknown
   4.2 Rectangular deconvolution (using « stairs » parameterizing)
   4.3 Rectangular estimation with *n* < *m* non uniform (NU) time steps

5. Conclusions/perspectives

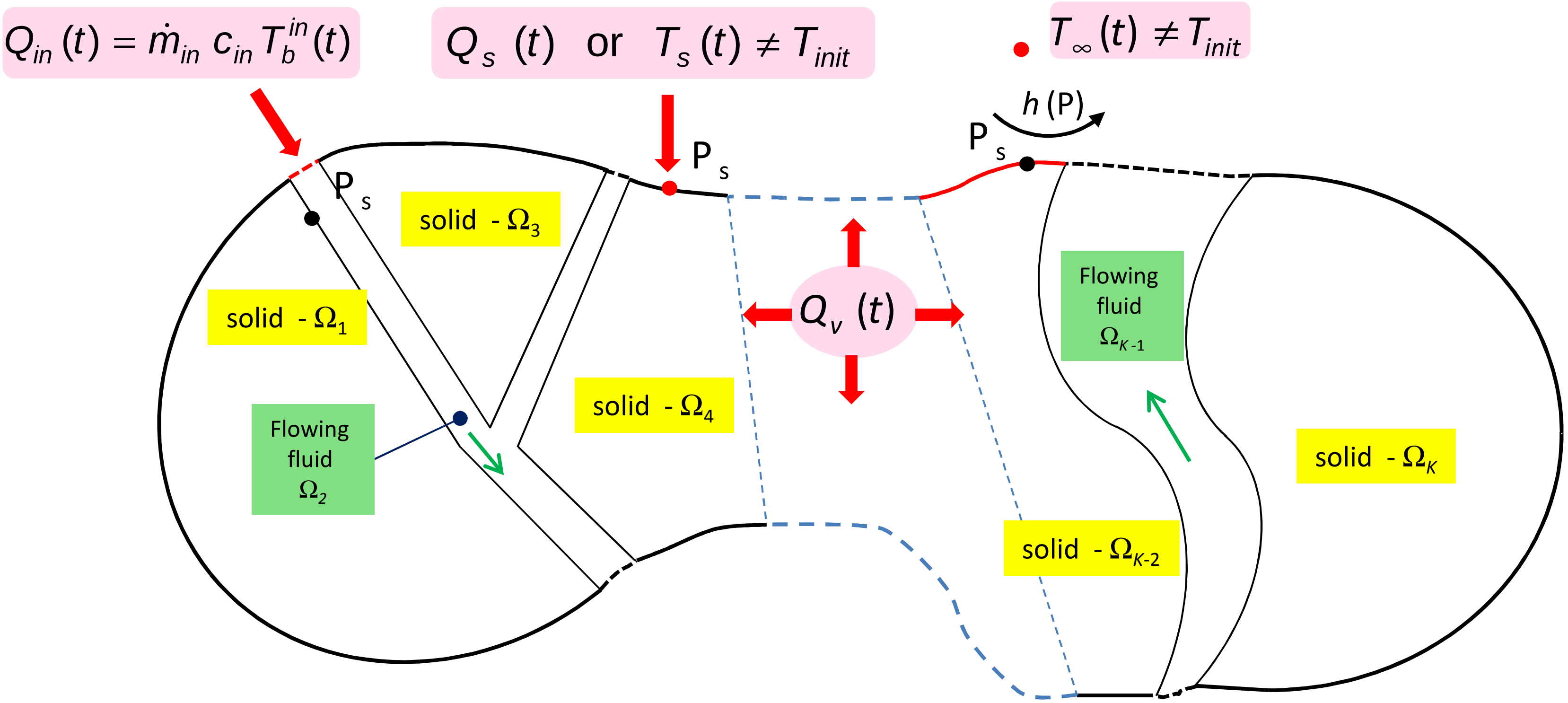
1. Forced thermal response of an advective/diffusive system with time constant coefficients

Material multicomponent system = K solid or fluid domains



Assumptions: **time constant thermophysical properties** and **velocity field**

Initial uniform state or steady state temperature field + one single separable thermal excitation



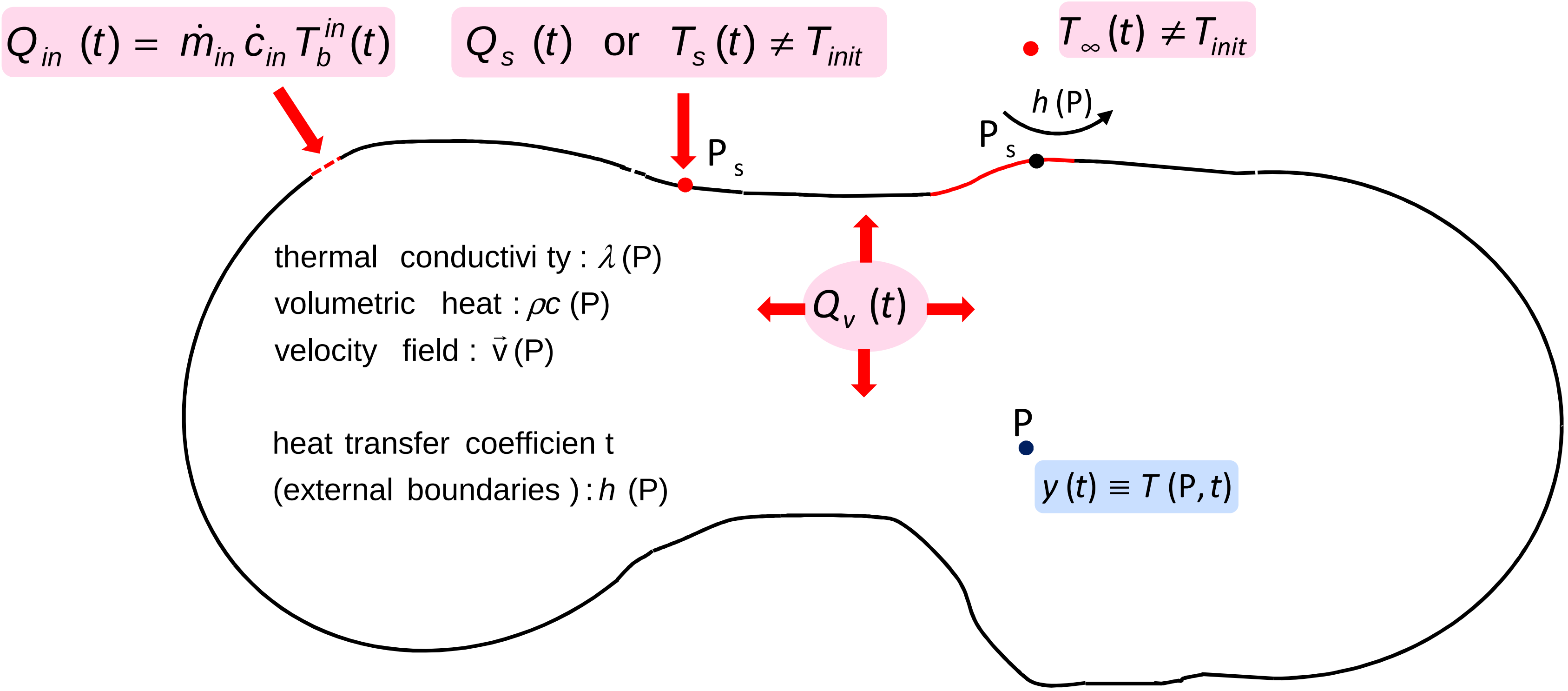
Time part of thermal excitation $u(t)$ (starts at time t = 0) :

- volumetretric heat source $Q_v(t)$
- surface heat or temperature source $Q_s(t)$ or $T_s(t)$
- change of external fluid temperature $T_\infty(t) \neq T_{init}$
- change of temperature at one fluid inlet $T_b^{in}(t)$

Fixed geometrical support:

- point
- line
- surface
- volume

Change of perspective: one single **heterogeneous fluid** in **one single domain**
(if solid part : zero velocity)



$Q_{in}\ (t) = \dot{m}_{in}\,\dot{c}_{in}\,T_b^{in}(t)$

$Q_s\ (t)$ or $T_s\,(t) \neq T_{init}$

$T_\infty\,(t) \neq T_{init}$

$h\,(\mathrm{P})$

P_s

P_s

thermal conductivity : $\lambda\,(\mathrm{P})$

volumetric heat : $\rho c\ (\mathrm{P})$

velocity field : $\vec{\mathrm{v}}\,(\mathrm{P})$

heat transfer coefficient

(external boundaries) : $h\ (\mathrm{P})$

$Q_v\ (t)$

P

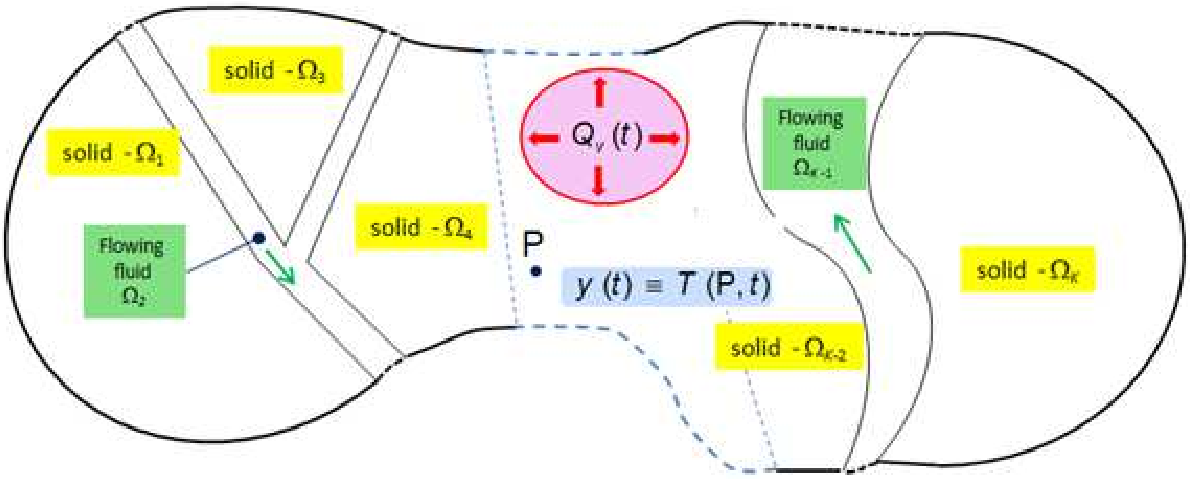
$y\,(t) \equiv T\,(\mathrm{P},t)$

Transient separable thermal excitation :

Point response at any point P :

$$u\,(t) \quad \Longrightarrow \quad y\ (t)\ \equiv\ T\ (\mathrm{P},t)$$

Recap:



Physical system:

Set of solids AND fluid(s):

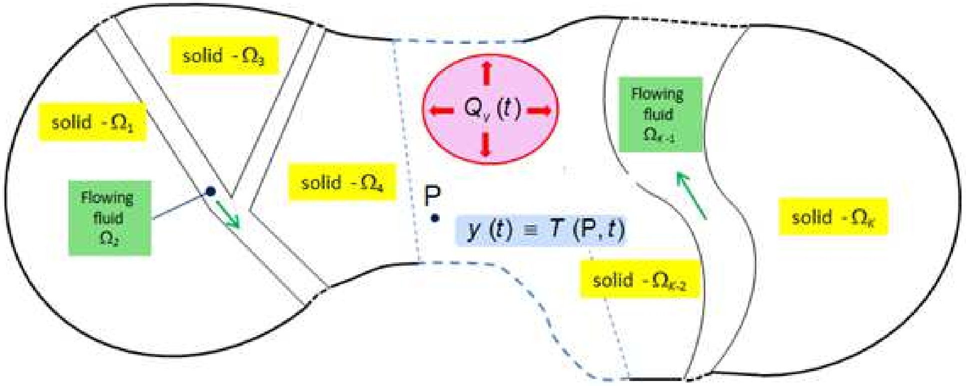
3D forced convection
with constant velocities
(in time but not in space)

P = ANY point in the system

One single thermal excitation defined by its support

Assumptions : Transient heat equation + boundary conditions with time-invariant coefficients + uniform initial temperature or steady state (the system is Linear and also Time-Invariant LITI)

$$\underset{\text{Transient}}{\boxed{\rho c(\mathrm{P})\frac{\partial T}{\partial t}(\mathrm{P},t)}} + \underset{\text{Advection}}{\boxed{\rho c(\mathrm{P})\vec{u}\,(\mathrm{P}).\,\vec{\nabla}T\,(\mathrm{P},t)}} = \underset{\text{Conduction}}{\boxed{\vec{\nabla}.\left(\lambda(\mathrm{P})\,\vec{\nabla}T\,(\mathrm{P},t)\right)}} + \underset{\text{Internal source}}{\boxed{\frac{Q_v\,(t)}{V_{\text{source}}}f(\mathrm{P})}}$$



Temperature rise at any point P:

$$\theta\,(\mathrm{P},t) = T\,(\mathrm{P},t) - T_{init}\,(\mathrm{P})$$

Its Laplace transform :

$$\bar{\theta}\,(\mathrm{P},p) = \int_0^{\infty} \exp(-p\,t)\,\theta\,(\mathrm{P},t)\;\mathrm{d}t$$

Laplace parameter

Assumptions : Transient heat equation + boundary conditions with time-invariant coefficient + uniform initial temperature (the system is Linear and also Time-Invariant LITI)

$$\underbrace{\boxed{\rho c(\mathrm{P})\frac{\partial T}{\partial t}(\mathrm{P},t)}}_{\text{Transient}} + \underbrace{\boxed{\rho c(\mathrm{P})\vec{u}\,(\mathrm{P}).\,\vec{\nabla}T\,(\mathrm{P},t)}}_{\text{Advection}} = \underbrace{\boxed{\vec{\nabla}.\left(\lambda(\mathrm{P})\,\vec{\nabla}T\,(\mathrm{P},t)\right)}}_{\text{Conduction}} + \underbrace{\boxed{\frac{Q_v\,(t)}{V_{\text{source}}}f\,(\mathrm{P})}}_{\text{Internal source}}$$

Consequences : Laplace transformed heat equation (no time derivative)

$$\underbrace{\boxed{\rho c(\mathrm{P})\,p\,\bar{\theta}(\mathrm{P},p)}}_{\text{Transient}} + \underbrace{\boxed{\rho c(\mathrm{P})\vec{u}\,(\mathrm{P}).\,\vec{\nabla}\bar{\theta}\,(\mathrm{P},p)}}_{\text{Advection}} = \underbrace{\boxed{\vec{\nabla}.\left(\lambda(\mathrm{P})\,\vec{\nabla}\bar{\theta}\,(\mathrm{P},p)\right)}}_{\text{Conduction}} + \underbrace{\boxed{\frac{\bar{Q}_v\,(p)}{V_{\text{source}}}f\,(\mathrm{P})}}_{\text{Internal source}}$$

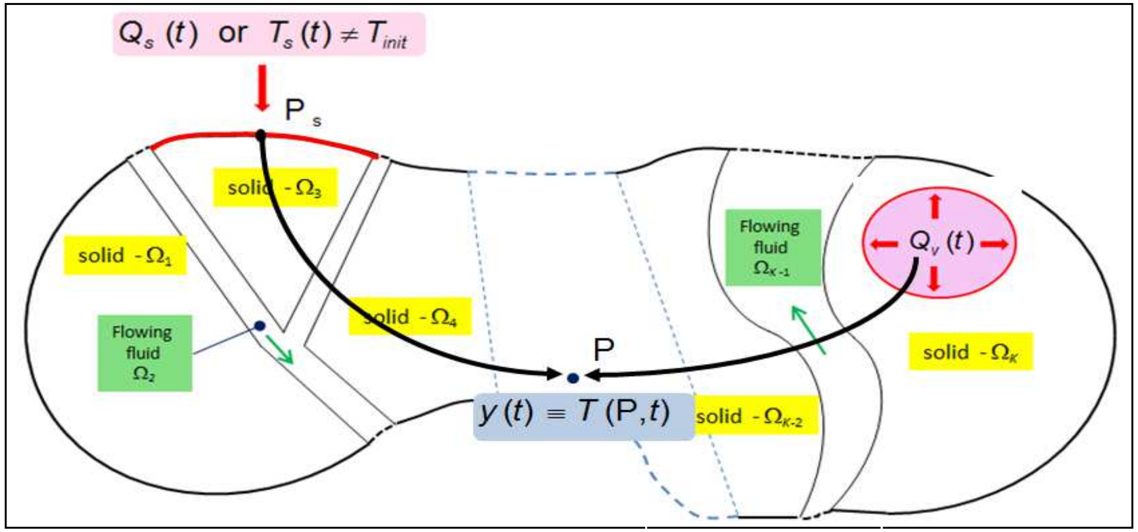
Linear system with a single excitation

Temperature or flux

$\Rightarrow$ response at any point P in the system

= simple product (Laplace domain)

$$\overline{y}\,(\mathrm{P}, p) = \overline{H}\;(\mathrm{P}, p)\;\;\overline{u}\;(p)$$



or convolution product (time domain)

excitation

response

$$y\,(\mathrm{P},\,t) = H\;(\mathrm{P},\,t)\;*\;\;u\;(t) = \int_0^t\;H\;(\mathrm{P},\;t - t'\,)\;u\;(t'\,)\;\mathrm{d}t'$$

Excitation $u\;(t)$ **:**

$u\;(t) = Q_v\,(t) - Q_v^{init}$ or $Q_s\,(t) - Q_s^{init}$

or $T_s\,(t)$ - $T_{init}\,(\mathrm{P}_s)$

or $T_\infty(t)$ - T_∞^{init}

or $T_b^{in}(t)$ - $T_b^{in,init}$

Transfer function

$H\,(\mathrm{P}\;,\,t)$

« init »= initial steady state or uniform temperature field

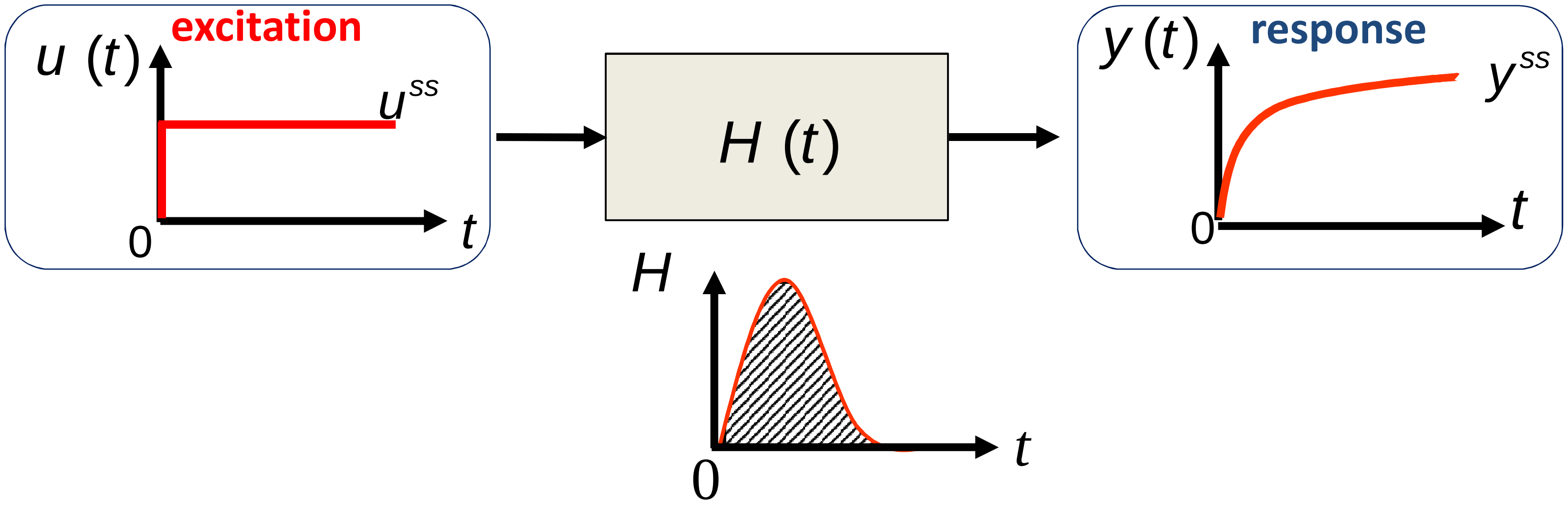
Response $y\;(t)$ **in any specific point P :**

$y\;(t) = \theta(\mathrm{P},\,t) = T\,(\mathrm{P},\,t)$ - $T_{init}\,(\mathrm{P})$

or local heat flux in any direction $\varphi_x\,(\mathrm{P},\,t)$

Excitation u	Response y	Transfer function H
Power source Q (watts)	Temperature difference θ (kelvins)	Impedance Z (K.J^{-1})
Temperature difference θ (kelvins)	Temperature difference θ (kelvins)	Transmittance W (s^{-1})
Power source Q (watts)	Rate of heat flow Φ (watts)	Transmittance W (s^{-1})
Temperature difference θ (kelvins)	Rate of heat flow Φ (watts)	Admittance Y (W.K^{-1}.s^{-1})

$$\boxed{y\,(\mathrm{P},t) = H\,(\mathrm{P},t) * \; u\,(t) = \int_0^t \; H\,(\mathrm{P},\, t - t'\,)\, u\,(t'\,)\, \mathrm{d}t'}$$



Steady state (ss) version
of a transfer function

$$H^{ss} = \boxed{\frac{y^{ss}}{u^{ss}}} = \boxed{\int_0^\infty H(t)\; dt}$$

asymptotic values

Time distribution

$$T_{in} - T_{out} = R \;\; \Phi$$

Thermal resistance, flux pipe
between 2 isothermal surfaces

case:

$u \equiv Q \;\; or \;\; Q - Q^{SS}$ (thermal power)

$y \;\equiv\; \theta = T - T_{init}$ (P) (temperature variation)

$\Rightarrow H \equiv Z$ (thermal impedance)

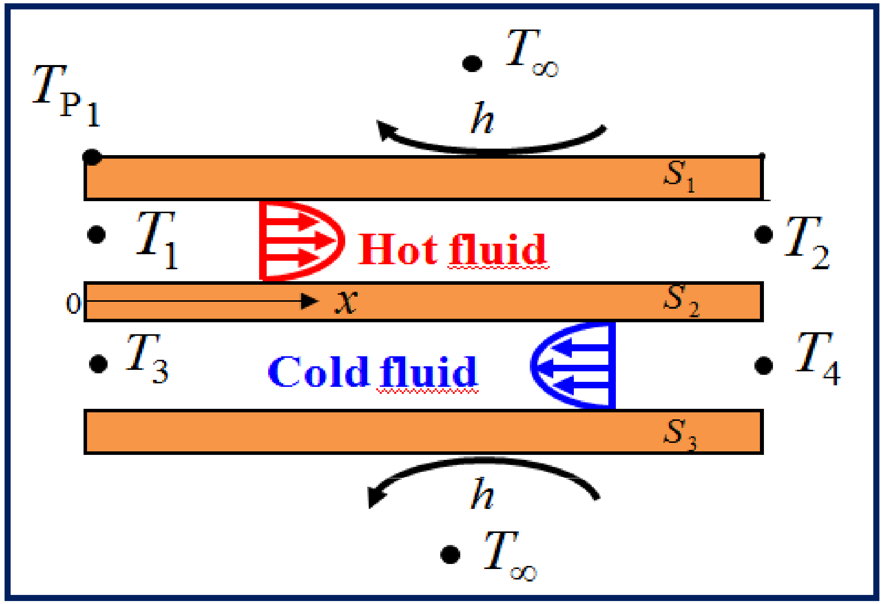
$$T^{ss} - T_{\infty} = Z^{ss} \; Q$$

Generalized resistance, no flux pipe

2. The calibration problem

2.1 Case of a heat exchanger

Assumptions



❑ Constant thermo-physical properties (fluid and walls) and velocities (LTI heat equation) :

$\boxed{\partial \beta / \partial t = 0}$ $\boxed{\beta \equiv u_{hot}, u_{cold}, \lambda, \rho, \ldots.}$

❑ Uniform initial conditions/initial steady state)

$T\,(\mathrm{P}, t=0) = T_{init}$ $T\,(\mathrm{P}, t=0) = T_{init}^{ss}\,(\mathrm{P})$

❑ Heat losses through convection/(linearized) radiation with environment through a uniform heat transfer coefficient h at temperature $T_\infty = T_{init}$

❑ One single heat source (inlet temperature increase) that starts at $t = 0^+$:

$\boxed{\theta_1(t) = T_1(t) - T_{init} \neq 0}$

Cause

$\boxed{\theta_4(t) = T_4(t) - T_{init} = 0}$

$\Rightarrow \Bigg\{$

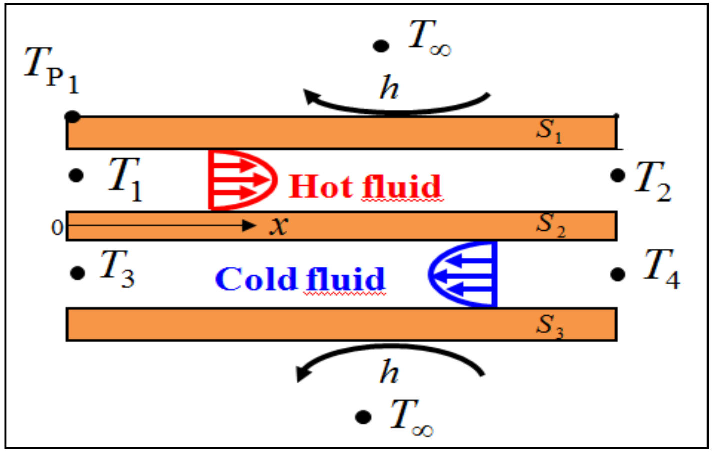
❑ Transient/unsteady thermal regime with observed responses at any point q :

$\boxed{\theta_q(t) = T_q(t) - T_{init}}$

$\boxed{\theta_q(t \leq 0) = 0 \quad \text{and} \quad \theta_q(t > 0) \neq 0}$

Consequences

Calculation of convolution products (transmittance case)
Parameterizing with piecewise constant functions, square case



response **transmittance** **unique pseudo source**

$$\theta(\mathrm{P},t) = W(\mathrm{P},t) * \theta_1(t)$$

$$= \int_0^t W(\mathrm{P},\ t-t')\ \theta_1(t')\ \mathrm{d}t'$$

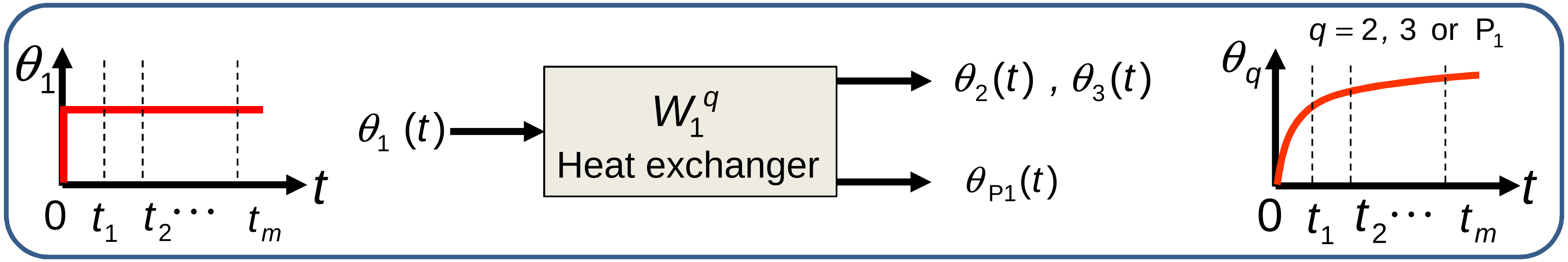
$$= \int_0^t \theta_1(\mathrm{P},\ t-t')\ W(\mathrm{P},t')\ \mathrm{d}t'$$

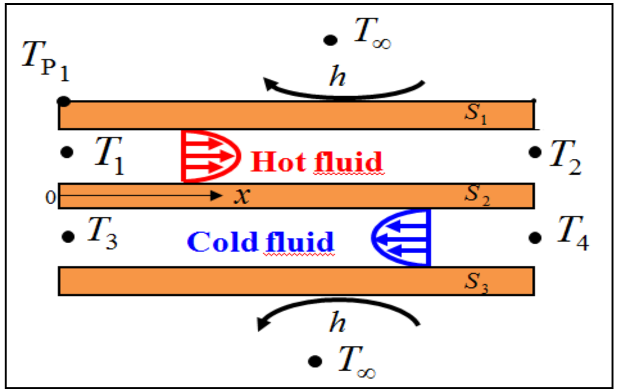
$$\theta(\mathrm{P},t_i) \approx \Delta t \sum_{j=1}^{m} \theta_{1,\,i-j+1}\ W_j(\mathrm{P})$$

sampled **averaged over 1 time interval**

$$t_0 = 0\ ;\ t_i = i\ \Delta t \quad \text{for} \quad i = 1 \text{ to } m\ ;\ \Delta t = t_{final}/m$$

$$z_i = \frac{1}{\Delta t}\int_{t_{i-1}}^{t_i} z(t)\ \mathrm{d}t \approx \frac{1}{2}\left(z(t_{i-1}) + z(t_i)\right) \quad \text{for} \quad z(t) = \theta_1 \ \text{ or } \ W(\mathrm{P})$$





$$\boxed{\mathbf{M}\ (\boldsymbol{z})\ \equiv\ \Delta t \begin{bmatrix} z_1 & & & & \\ z_2 & z_1 & & 0 & \\ z_3 & z_2 & z_1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ z_m & z_{m-1} & z_{m-2} & \cdots & z_1 \end{bmatrix}}$$

M (***z***) Lower triangular Toeplitz matrix function of a vector ***z***

$$z_i = \frac{1}{\Delta t}\int_{t_{i-1}}^{t_i} z\ (t)\ \mathrm{d}t \approx \frac{1}{2}\left(z\,(t_{i-1}) + z\,(t_i)\right)\ \text{for}\ \ z\,(t) = \theta_1\ \text{ or }\ W\,(\mathrm{P})$$

$$\boxed{\boldsymbol{\theta}_q = \mathbf{M}\,(\boldsymbol{\theta}_1)\ \boldsymbol{W}_1^{\ q} = \mathbf{M}\,(\boldsymbol{W}_1^{\ q})\ \boldsymbol{\theta}_1}$$

$$\boldsymbol{\theta}_q = \begin{pmatrix} \theta_q\,(t_1) \\ \theta_q\,(t_2) \\ \vdots \\ \theta_q\,(t_m) \end{pmatrix},q = 2,\ 3,\ 4\ \ \text{or}\ \ \mathrm{P}_1$$

instantaneous (sampled) values

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{1m} \end{pmatrix} \quad \boldsymbol{\theta}_1 = \begin{pmatrix} \theta_{1,1} \\ \theta_{1,2} \\ \vdots \\ \theta_{1,m} \end{pmatrix}$$

time averaged values

- **First experiment:**

  - Calibration (inverse) problem

$$\underset{\text{measured}}{\boldsymbol{\theta}_q} \longrightarrow \boxed{\underset{\text{measured}}{\left(\mathbf{M}\ (\boldsymbol{\theta}_1)\right)^{-1}}} \longrightarrow \underset{\text{estimated}}{\hat{\boldsymbol{W}}_1^{\ q}}$$

- **Next experiments:**

  - virtual sensor inverse problem (same as source estimation problem)

$$\underset{\text{measured}}{\boldsymbol{\theta}_q} \longrightarrow \boxed{\underset{\text{previously estimated}}{\left(\mathbf{M}\ (\boldsymbol{W}_1^{\ q})\right)^{-1}}} \longrightarrow \underset{\text{estimated}}{\hat{\boldsymbol{\theta}}_1}$$

Model for *W* identification calibration problem:

exact **unknown**

$$\boldsymbol{\theta}_q = \mathbf{M}(\boldsymbol{\theta}_1)\, \boldsymbol{W}^q$$

measured

$$\boldsymbol{\theta}_q^{exp} = \boldsymbol{\theta}_q + \boldsymbol{\varepsilon}_q \quad \text{and} \quad \boldsymbol{\theta}_1^{exp} = \boldsymbol{\theta}_1 + \boldsymbol{\varepsilon}_1$$

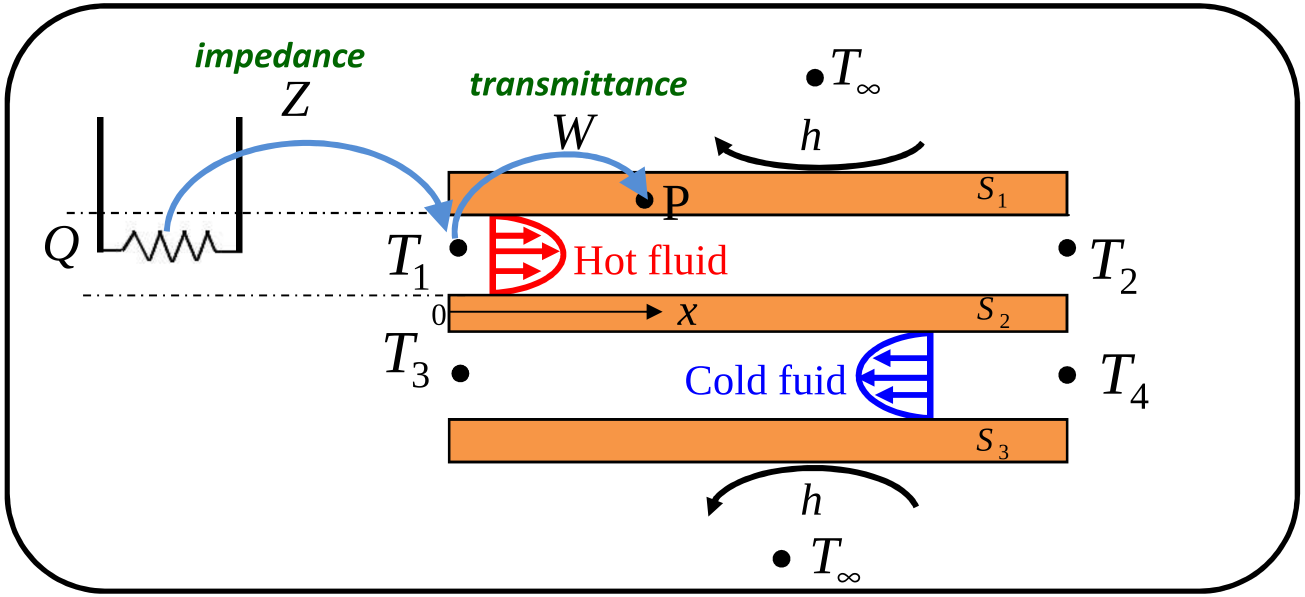
2 i.i.d. and independent noises

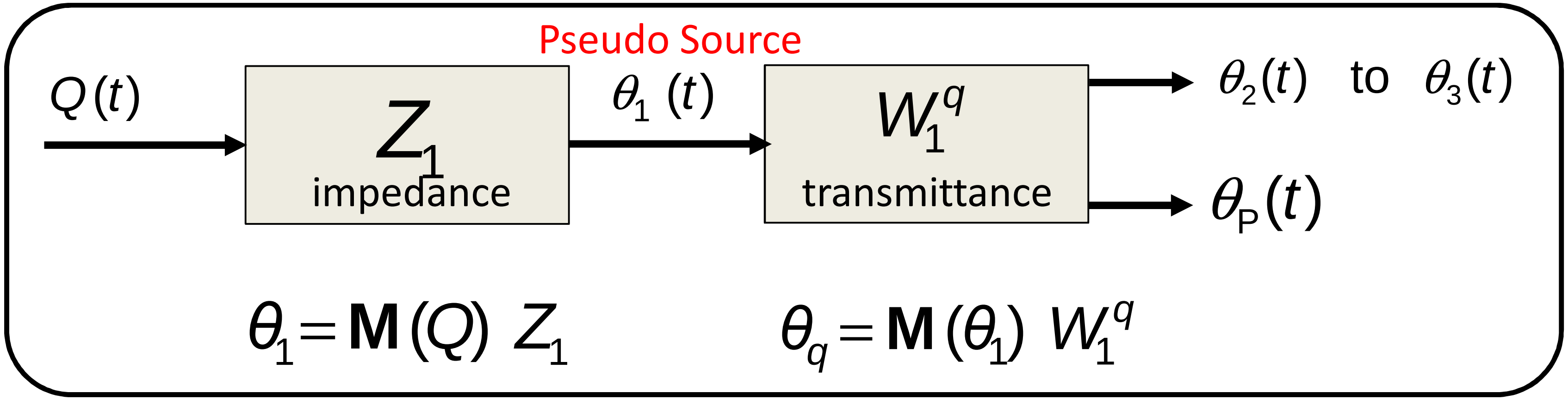
- Ordinary (linear) least squares: $\hat{\boldsymbol{W}}^q = \left(\mathbf{M}(\boldsymbol{\theta}_1^{exp})\right)^{-1} \boldsymbol{\theta}_q^{exp}$

Ill-posed problem:
Inversion needs **regularization**
Here: Truncated SVD or 0 order Tikhonov

- with discrepancy principle (Morozov)

Practical way of making the inlet temperature vary





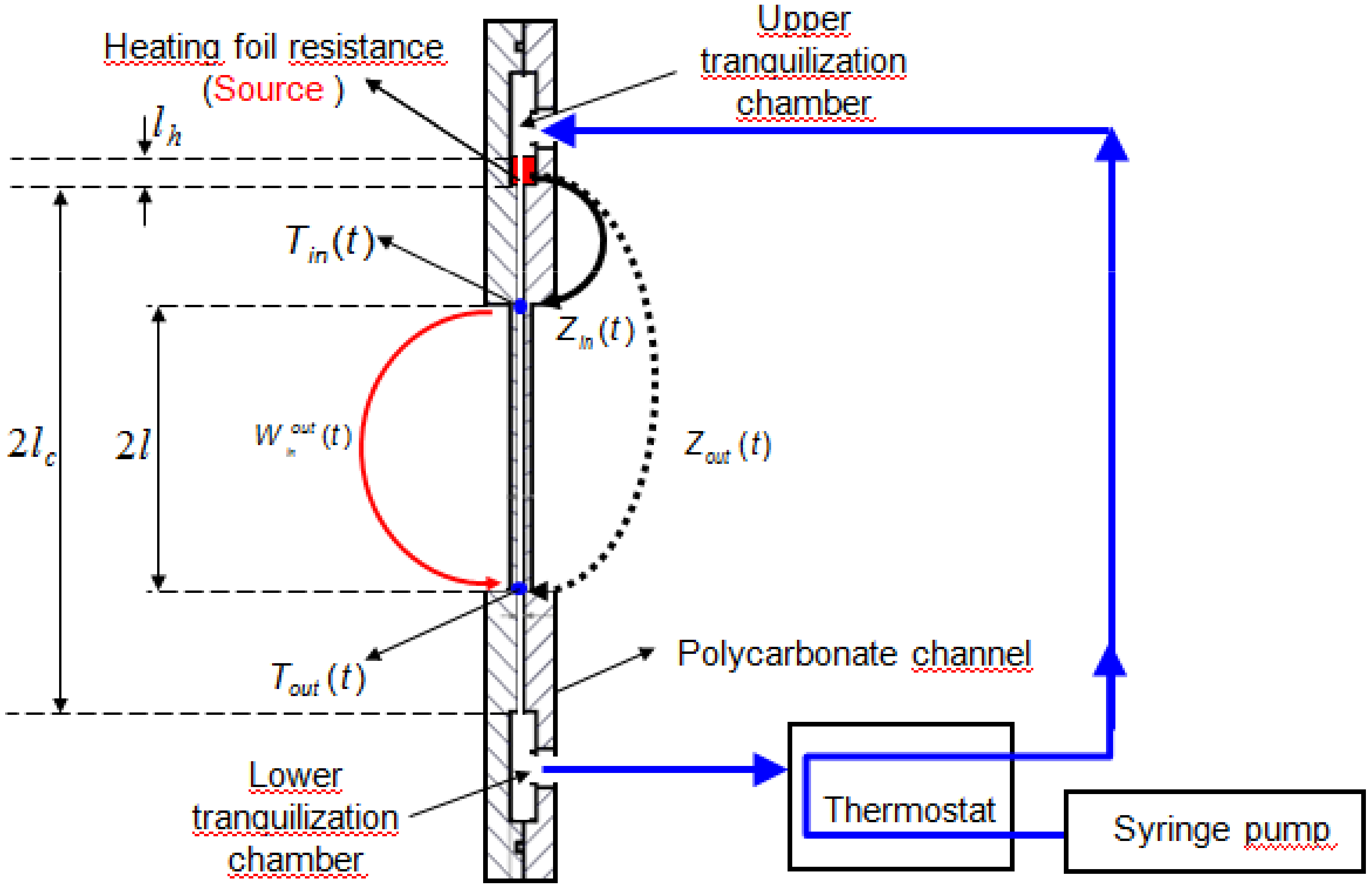
$$\theta_1 = \mathbf{M}(Q)\ Z_1 \qquad \theta_q = \mathbf{M}(\theta_1)\ W_1^q$$

2.2 – Experimental Impedance/transmittance estimation for a half heat exchanger

Identification of transmitance using experimental transient measurements (calibration)

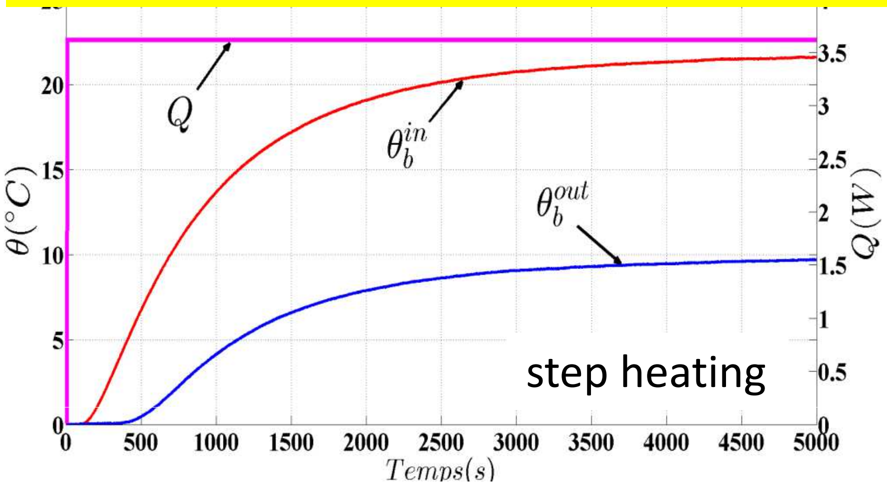
Response : thermocouples
d = 50.8 μm

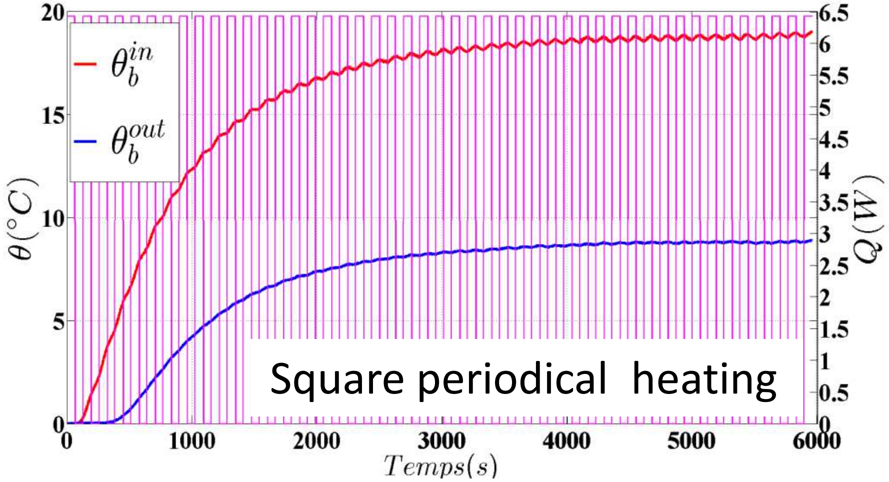
$e_1 = e_2(mm)$	$e_f(mm)$	$2l(mm)$	$2l_c(mm)$	$w(mm)$	$l_h(mm)$
2	1	65	120	50	10

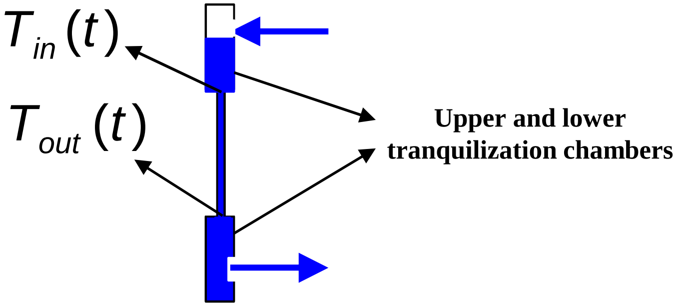


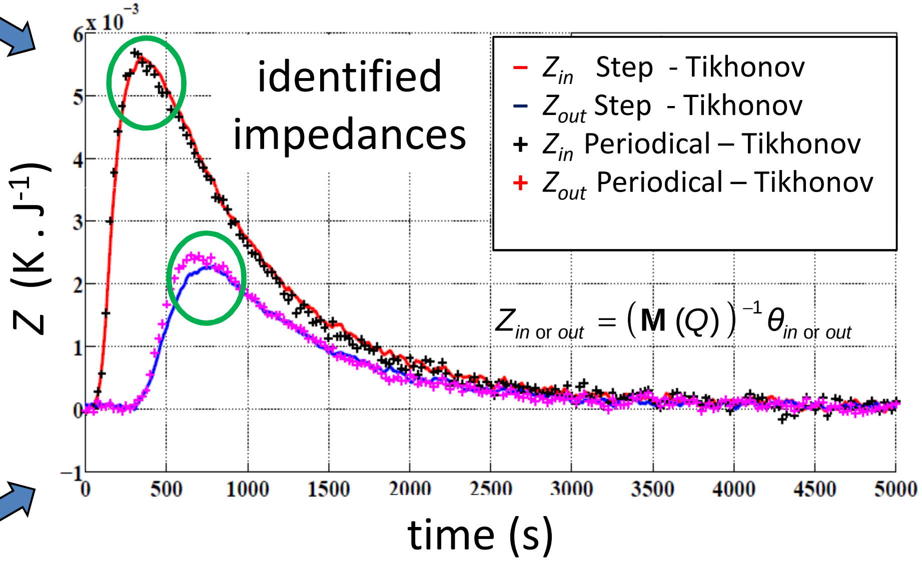
Identification of transfer function using experimental temperature recording:

Comparison of identified Z : step or periodical heating



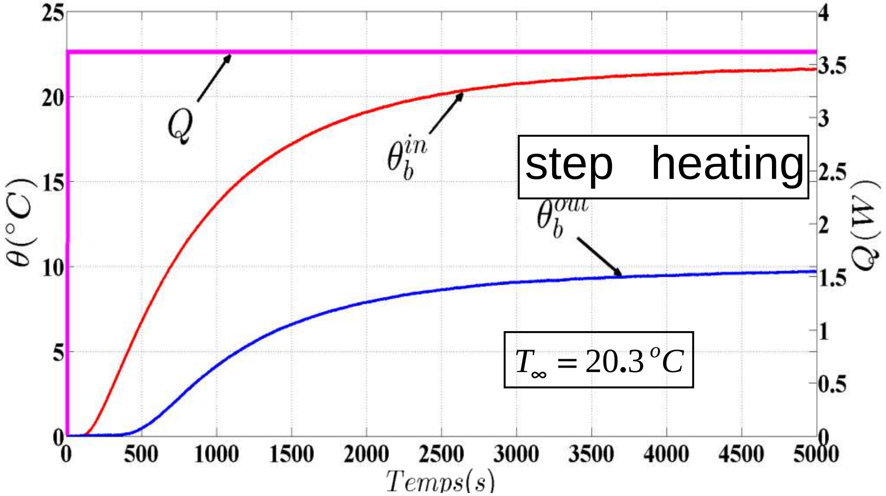


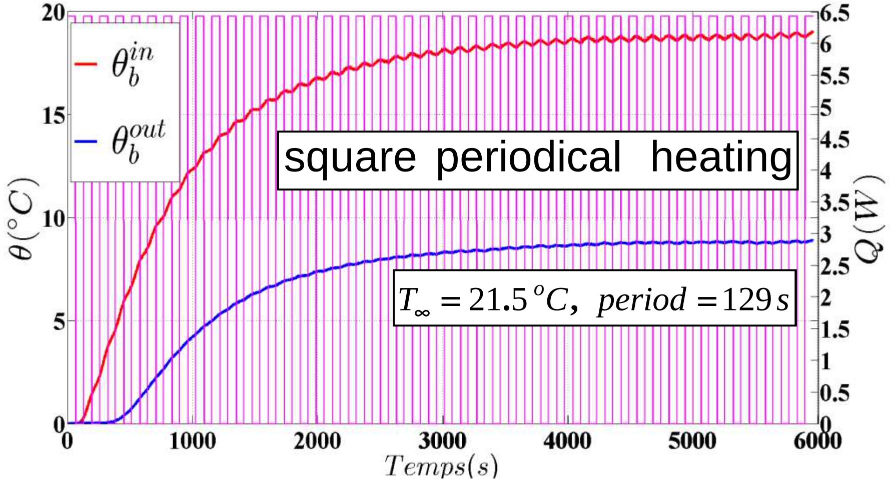


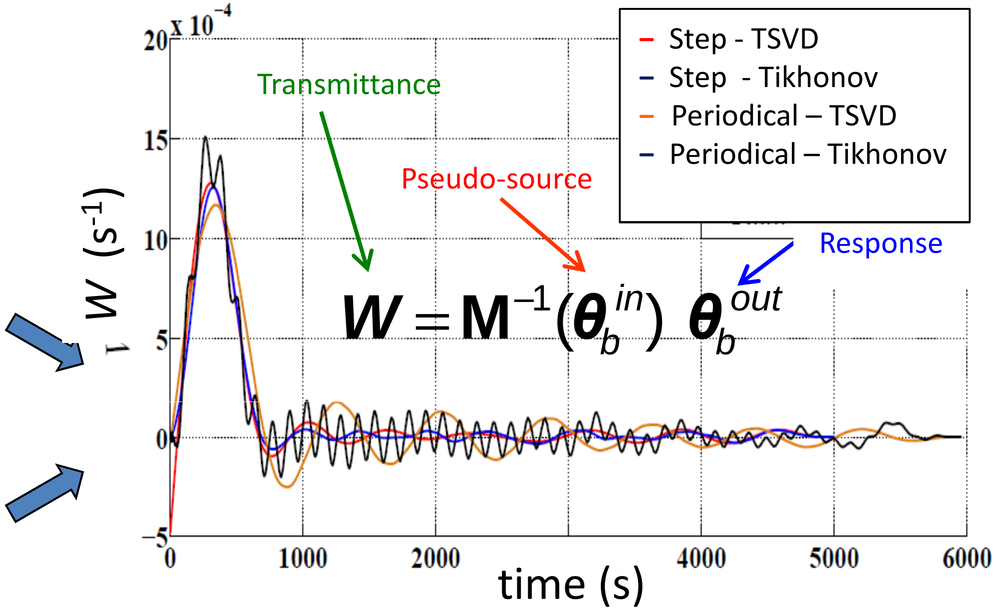


$$Z_{in\ or\ out} = \left(\mathbf{M}\,(Q)\right)^{-1}\theta_{in\ or\ out}$$

Comparison of identified transmittance *W* (outlet/inlet): step or periodical heating







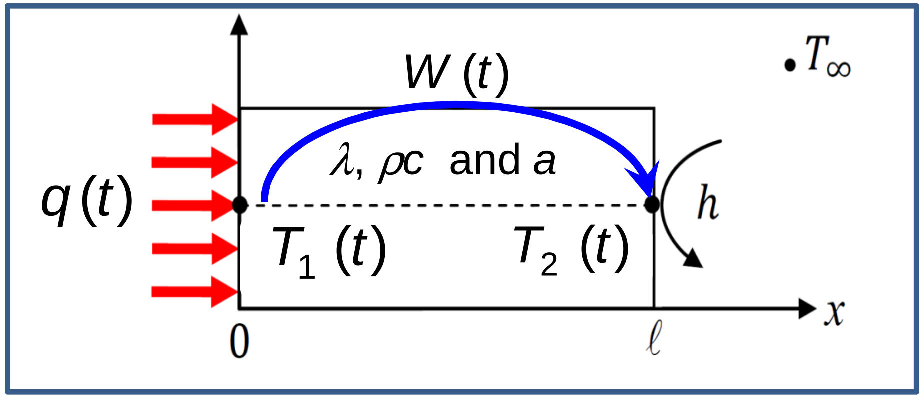
$$\boldsymbol{W} = \mathbf{M}^{-1}(\boldsymbol{\theta}_b^{in})\ \boldsymbol{\theta}_b^{out}$$

Oscillations past first peak and for long times, zero initial level hard to recover :

Estimation of transmittance *W* (noisy output and input) more difficult than *Z* (noisy output only)

3. Analysis of deconvolution deadlocks

3.1 Reference case: 1D transient conduction



Heat eq. $$\left\{ \frac{\partial^2 \theta}{\partial x^2} = \frac{1}{a}\frac{\partial \theta}{\partial t} \right.$$

boundary $$\begin{cases} \varphi = -\lambda \dfrac{\partial \theta}{\partial t} = q(t) & \text{at } x = 0 \quad \text{for} \quad t > 0\,; \\ \varphi = -\lambda \dfrac{\partial \theta}{\partial t} = h\ \theta & \text{at } x = \ell \quad \text{for} \quad t > 0 \end{cases}$$

$$\theta(t) = T(t) - T_\infty$$

change of variables

initial $$\{\ \theta = 0 \qquad \text{at } \ t = 0 \quad \text{for} \quad 0 \le x \le \ell$$

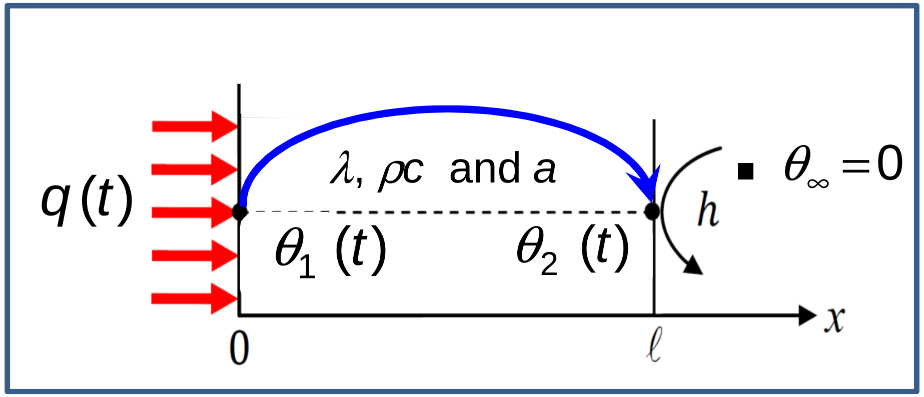
$$H \equiv W\ ,\quad u \equiv \theta_1,\quad y \equiv \theta_2$$

Laplace transform $$\bar{\psi}\,(x, p) \equiv \int_0^t \psi(x, t) \exp(-p\,t)\,\mathrm{d}t \qquad \text{for} \qquad \psi = \theta \ \text{ or } \ \varphi$$

$$\bar{\theta}_2\,(p) = \bar{W}(p)\ \bar{\theta}_1\,(p) \qquad \Leftrightarrow \qquad \theta_2\,(t) = W(t) * \theta_1\,(t)$$

Transmittance $$\boxed{W\,(t) = L^{-1}\left[\frac{1}{\cosh(\beta\,\ell) + h \sinh(\beta\,\ell) / (\lambda\,\beta)} \right]}$$ with $\beta^2 = p/a$

Comparison: analytical W and identified W from synthetic profiles (COMSOL)



Analytical Laplace W + numerical inversion of Laplace

$$\boxed{W\ (t) = L^{-1}\left[\frac{1}{\cosh\ (\beta\,\ell) + h\ \sinh\ (\beta\,\ell) / (\lambda\,\beta)}\right]}$$

versus

Identified W by OLS and TLS

$$\boxed{\boldsymbol{W} \text{ from } \boldsymbol{\theta}_1 \text{ and } \boldsymbol{\theta}_2}$$

Numerical Inversion of Laplace Transforms by Hoog's algorithm

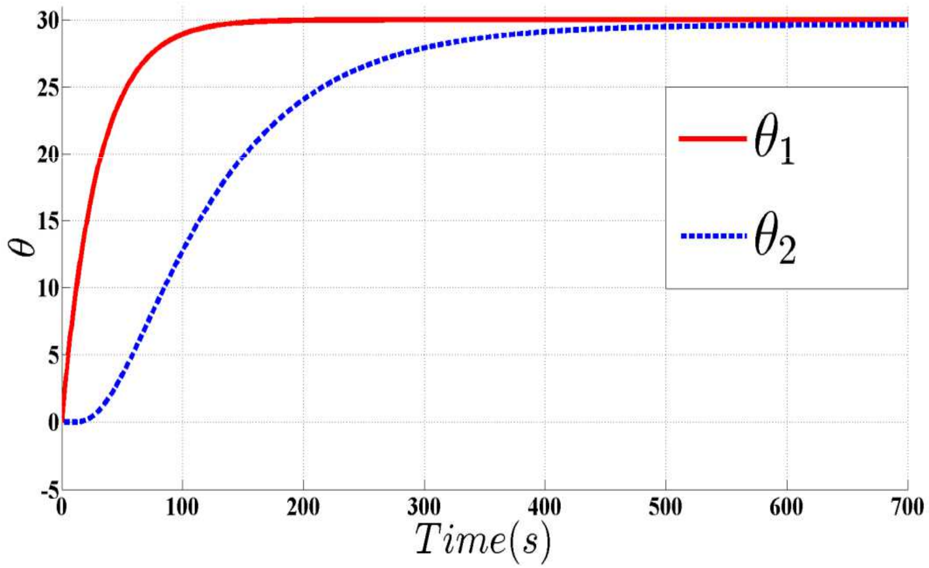
Input : $\theta_1(t) = \left(1 - e^{-\frac{t}{\tau}}\right)\theta_1^{ss}$ with $\tau = 30s$; $\theta_1^{ss} = 30\ °C$ and $\Delta t = 0.5s$

t_f (s)	ℓ (mm)	h (W.m^{-2}.K^{-1})	λ (W.m^{-1}.K^{-1})	ρc (kJ.m^{-3}.K^{-1})
700	50	10	43	3666

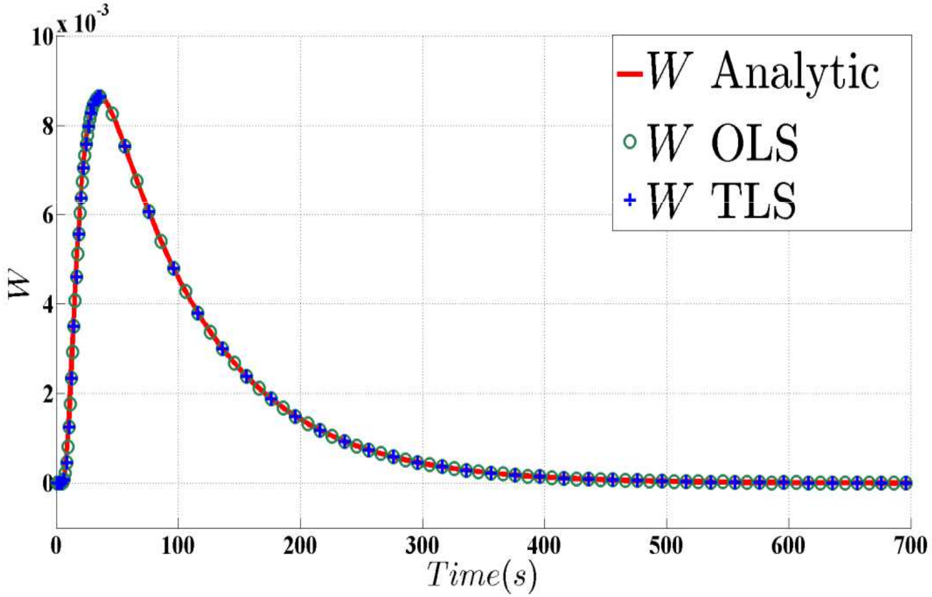
Comparison: analytical W and identified W from synthetic profiles (COMSOL)

Comparison without noise:

Validation without committing an INVERSE CRIME



Temperature profiles (COMSOL)



Analytical and identified W
(without regularization)

3.2 Noisy matrix and Total Least Squares

Calibration problem: how to get the best transfer function *H* (*t*) at a point P?

Model

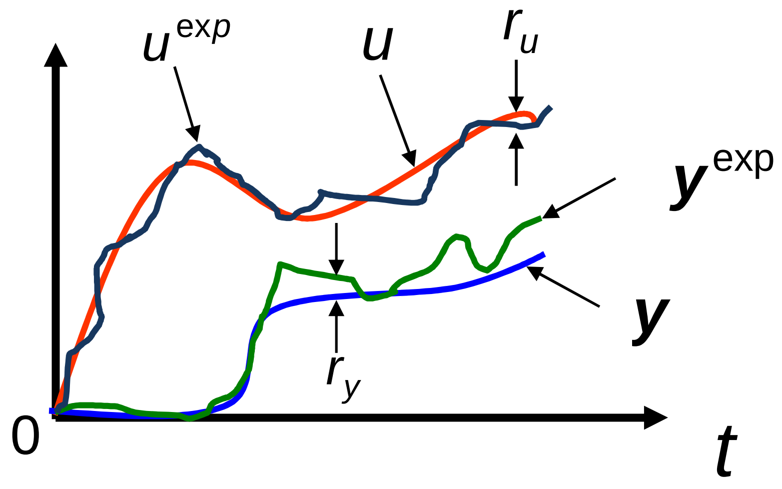
$$y\,(\mathrm{P}, t) = u\,(t) * H\,(\mathrm{P}, t) = \int_0^t u\,(, t-t')\,H\,(t-t')\,\mathrm{d}t'$$

sampling **parameterization**

$$\boldsymbol{y} = \mathbf{M}(\boldsymbol{u})\,\boldsymbol{H} \quad \text{or} \quad \boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{H} \quad \text{with} \quad \boldsymbol{A} = \mathbf{M}(\boldsymbol{u})$$

Measurements

Available information:
discrete noisy values
of *y* (P, *t*) and u (*t*)



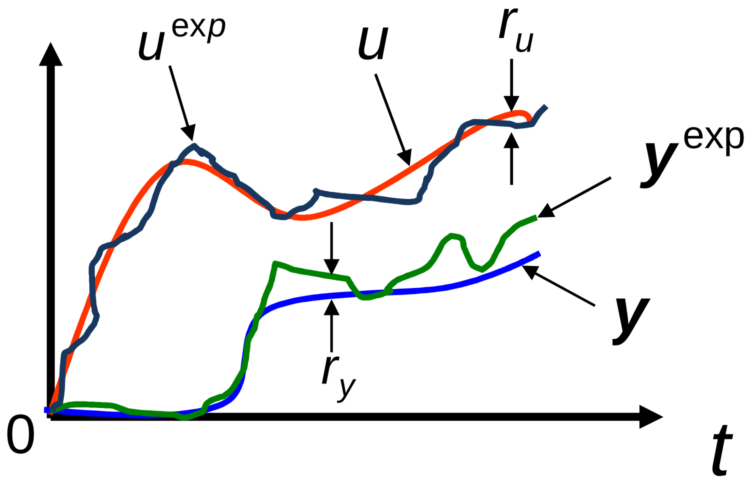
$$y^{exp}\,(t_i) = y\,(t_i) + \varepsilon_i$$

noise

$$u^{exp}\,(t_i) = u\,(t_i) + \tau_i \quad \Rightarrow \quad \boldsymbol{A}^{exp} = \boldsymbol{A} + \boldsymbol{\varepsilon}_u \quad \text{with} \quad \boldsymbol{\varepsilon}_u = \mathbf{M}\,(\tau)$$

noisy matrix

H Identification from *u* and *y* measurements : $\boldsymbol{y} = \mathbf{M}\,(\boldsymbol{u})\;\boldsymbol{H}$



Total Least Squares (TLS) solution :

augmented matrix $\boldsymbol{G} = \left[\ \mathbf{M}(\boldsymbol{u})\ \middle|\ \boldsymbol{y}\right]$

concatenation

$\hat{\boldsymbol{H}}$ such as minimum residuals $\boldsymbol{r}_y$ and $\boldsymbol{r}_u$

Frobenius norm: $J_{TLS}(\boldsymbol{H}) = \left\| \boldsymbol{r}_G(\boldsymbol{H}) \right\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{m+1} g_{ij}^2$

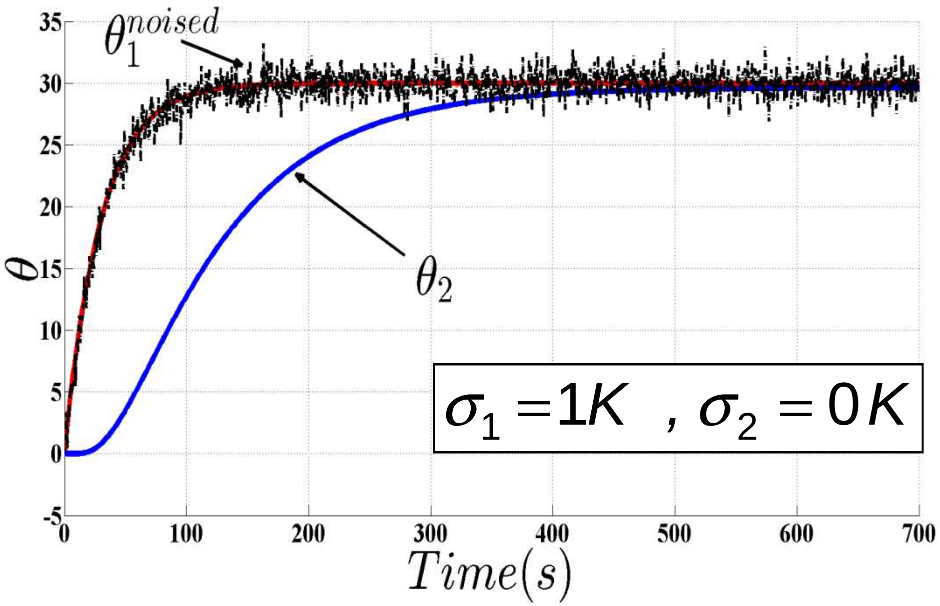
with $\boldsymbol{r}_G(\boldsymbol{H}) = \boldsymbol{G}^{exp} - \boldsymbol{G}$

SVD Form of G ⟶ **ill-posed** ⟶ **Regularized form: Truncated TLS**

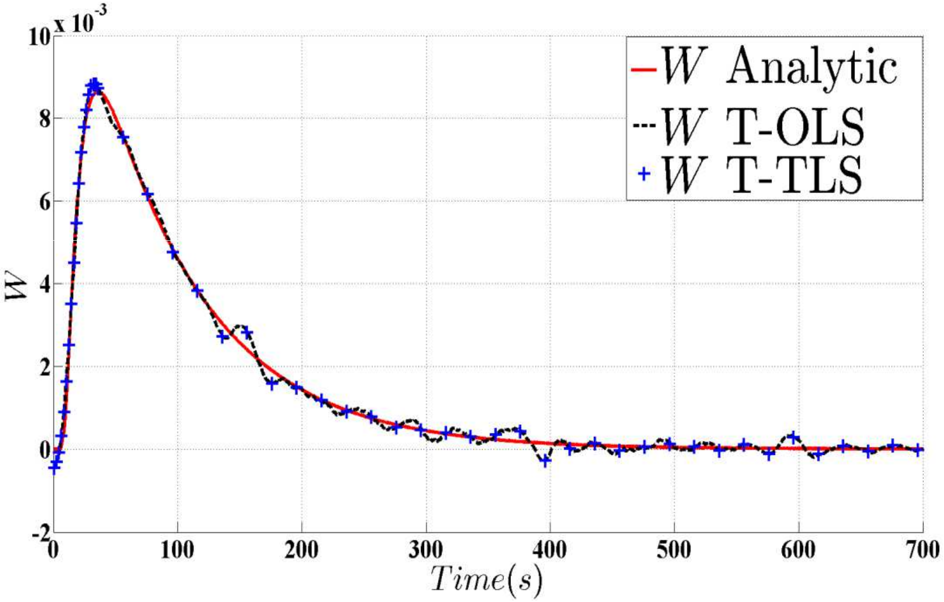
3.3 Comparison of calibration methods

Comparison: analytical W and identified W from synthetic profiles (COMSOL)

Comparison with noise: $\boxed{\theta_1 \text{ noised}}$



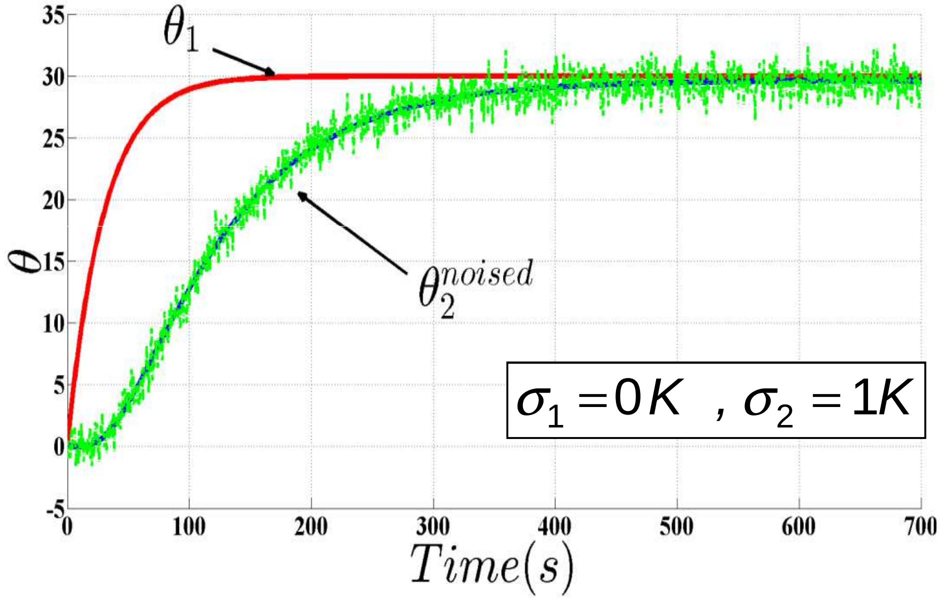
Temperature profiles (COMSOL)

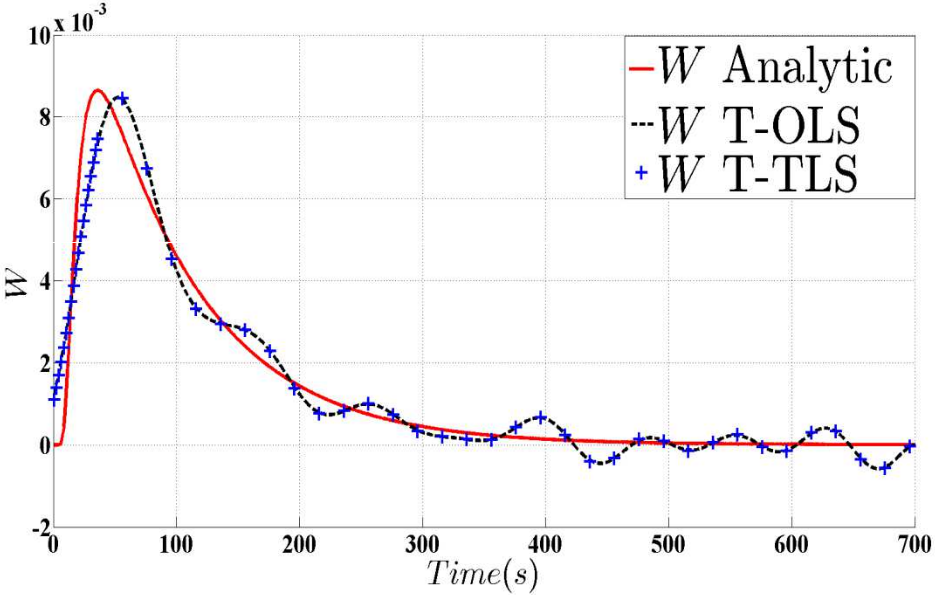


Analytical and identified W
(with regularization)

Comparison: analytical W and identified W from synthetic profiles (COMSOL)

Comparison with noise: $\boxed{\theta_2 \text{ noised}}$





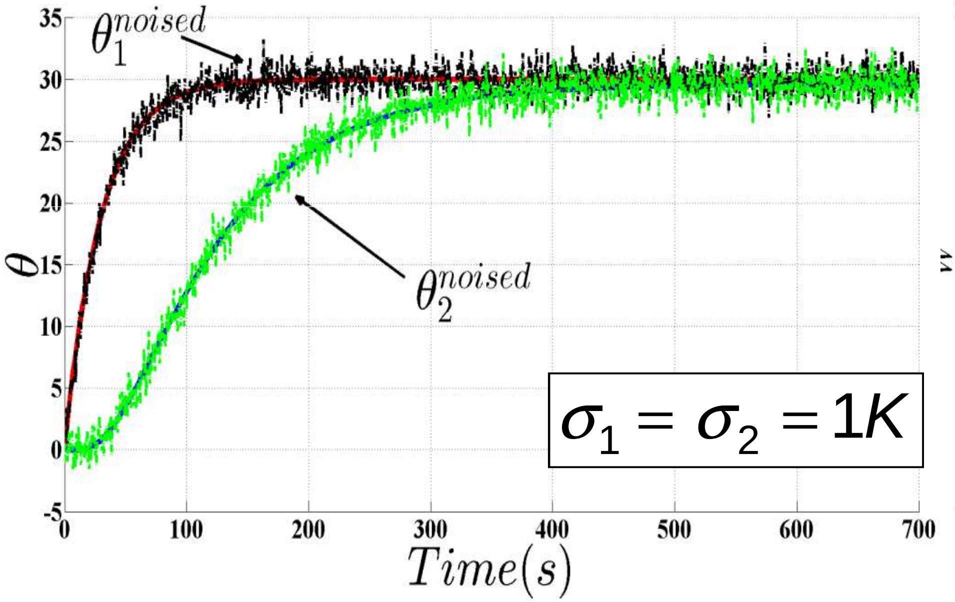
Temperature profiles (COMSOL)

Analytical and identified W
(with regularization)

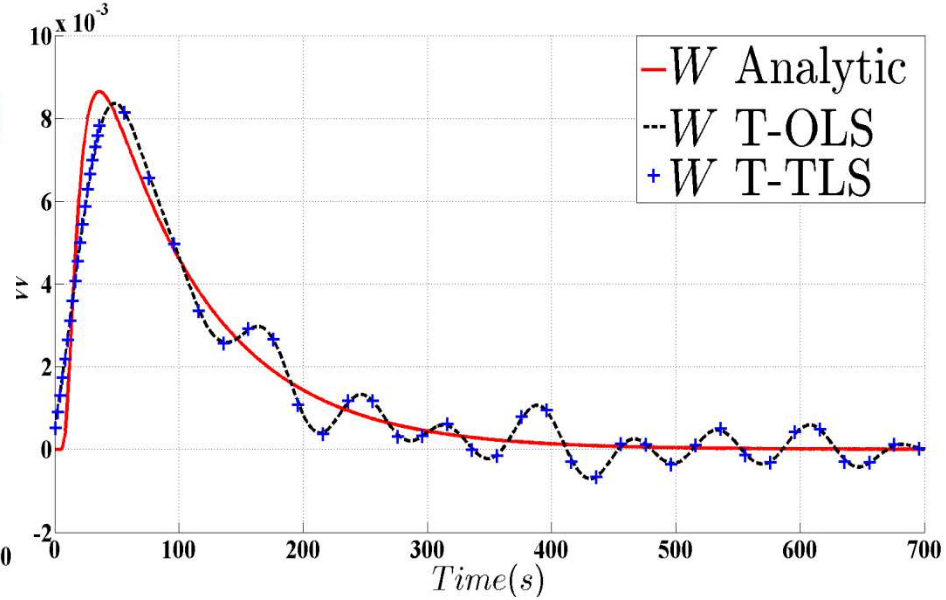
Noise over signal ratio: $\left\| \boldsymbol{\theta}_2^{noised} - \boldsymbol{\theta}_2^{\text{exact}} \right\| / \left\| \boldsymbol{\theta}_2^{noised} \right\| = 3.90\ \%$

Comparison: analytical W and identified W from synthetic profiles (COMSOL)

Comparison with noise: $\boxed{\theta_1,\ \theta_2\ \text{noised}}$

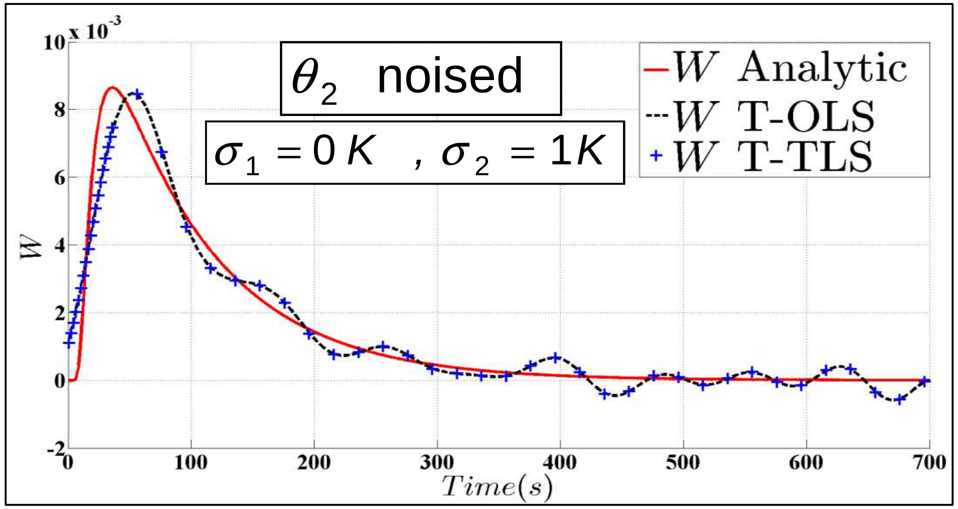


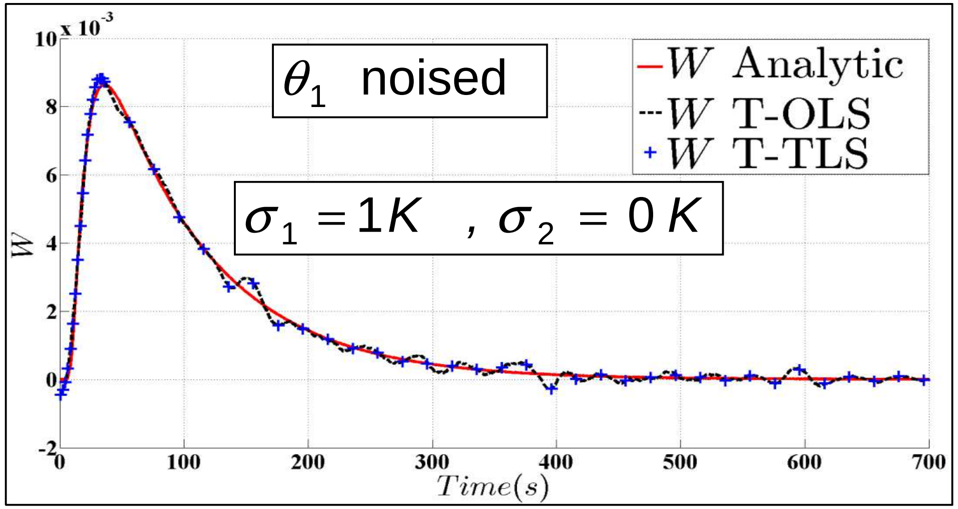
Temperature profiles (COMSOL)



Analytical and identified W
(with regularization)

Conclusion of this comparison of deconvolution techniques

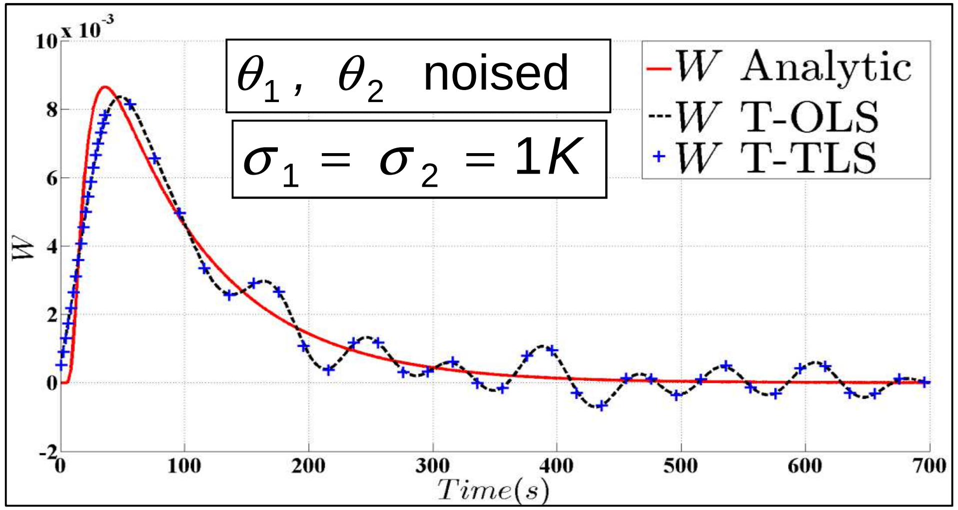




- Noise on the response θ2 more penalizing than noise on the source θ1.
- The truncated total least squares do not take into account the convolutive structure of the matrix M (θ1)

  ⇨ no improvement of the estimate
- Important to improve short times values of the identified TF in a calibration experiment:

  → largest impact on posterior inverse input estimation experiments.

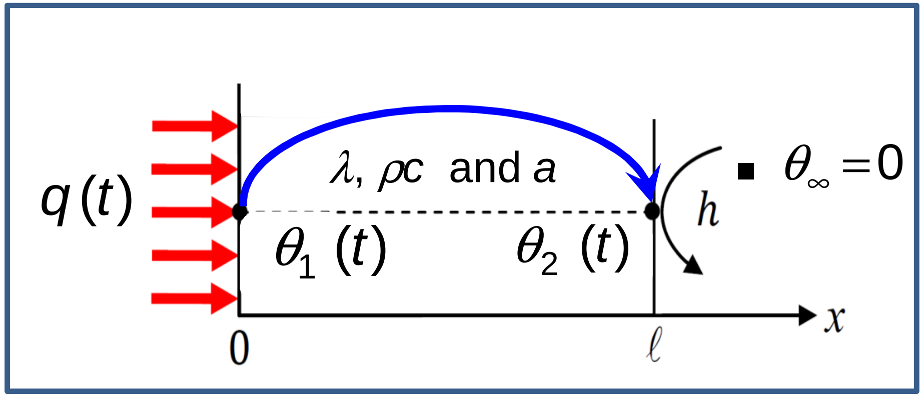


4. Rectangular deconvolution

Calibration problem :

$$\boldsymbol{\theta}_2 = \boldsymbol{A}\,\boldsymbol{W} \quad \text{with} \quad \boldsymbol{A} = \mathbf{M}\,(\boldsymbol{\theta}_1)$$

m output data — *n* =*m* unknowns — *m* input data

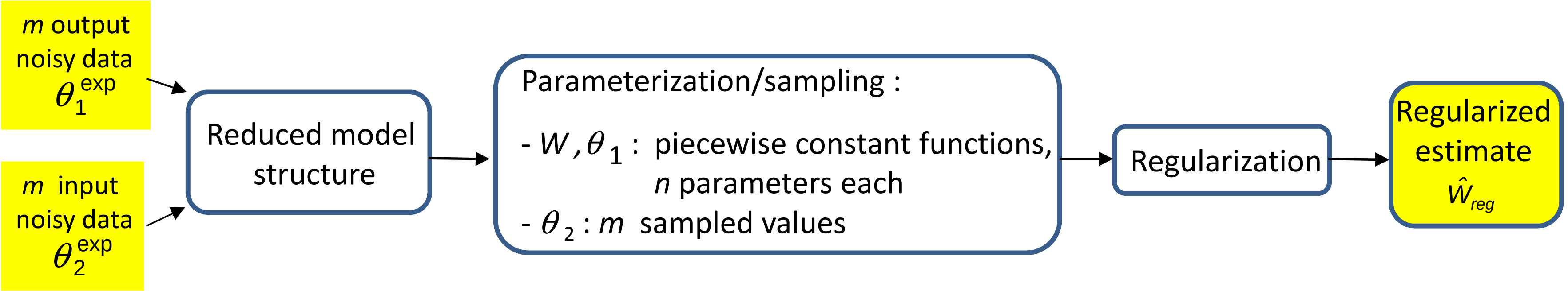


Can the ill-posed problem be more **parsimonious** ?

→ Less many unknowns *n* than output data: $n \ll m$

Calibration = model identification for a given structure

Idea: tayloring the definition of the estimate


m output noisy data θ_1^{exp}

m input noisy data θ_2^{exp}

Reduced model structure

Parameterization/sampling :

- W, θ_1 : piecewise constant functions, *n* parameters each
- θ_2 : *m* sampled values

Regularization

Regularized estimate $\hat{W}_{reg}$

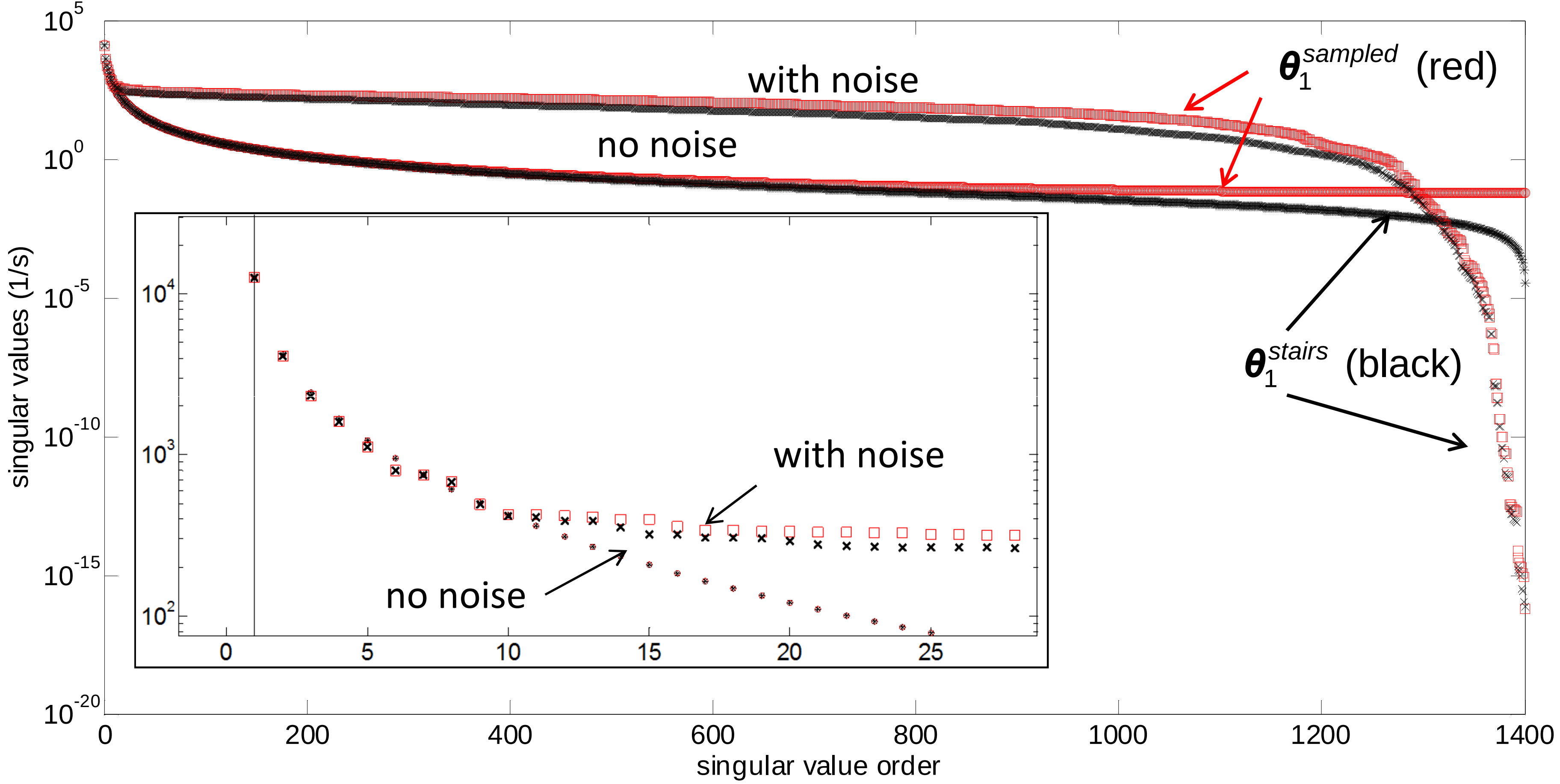
2 paths deserve to be investigated **BEFORE** regularization:

- Rectangular parametrization using piecewise constant parametrization for *W*
- ARX model construction (perspective)

4.1. Point versus averaged values for input and unknown

Singular values of $\mathbf{M}(\boldsymbol{\theta}_1)$ depending on type of parameterization of $\boldsymbol{\theta}_1$ and on its noise

$$\mathbf{M}(\boldsymbol{\theta}_1) = \boldsymbol{U}\,\boldsymbol{S}\,\boldsymbol{V}^T$$



$$s_1\left(\mathbf{M}\,(\boldsymbol{\theta}_{1,\alpha}^{stairs,\sigma_1})\right) \approx s_1\left(\mathbf{M}\,(\boldsymbol{\theta}_{1,\alpha}^{\text{sampled},\sigma_1})\right)$$

$$\text{cond}\left(\mathbf{M}\,(\boldsymbol{\theta}_{1,\alpha})\right) = s_1 / s_\alpha \quad \text{for} \quad 1 \leq \alpha \leq m$$

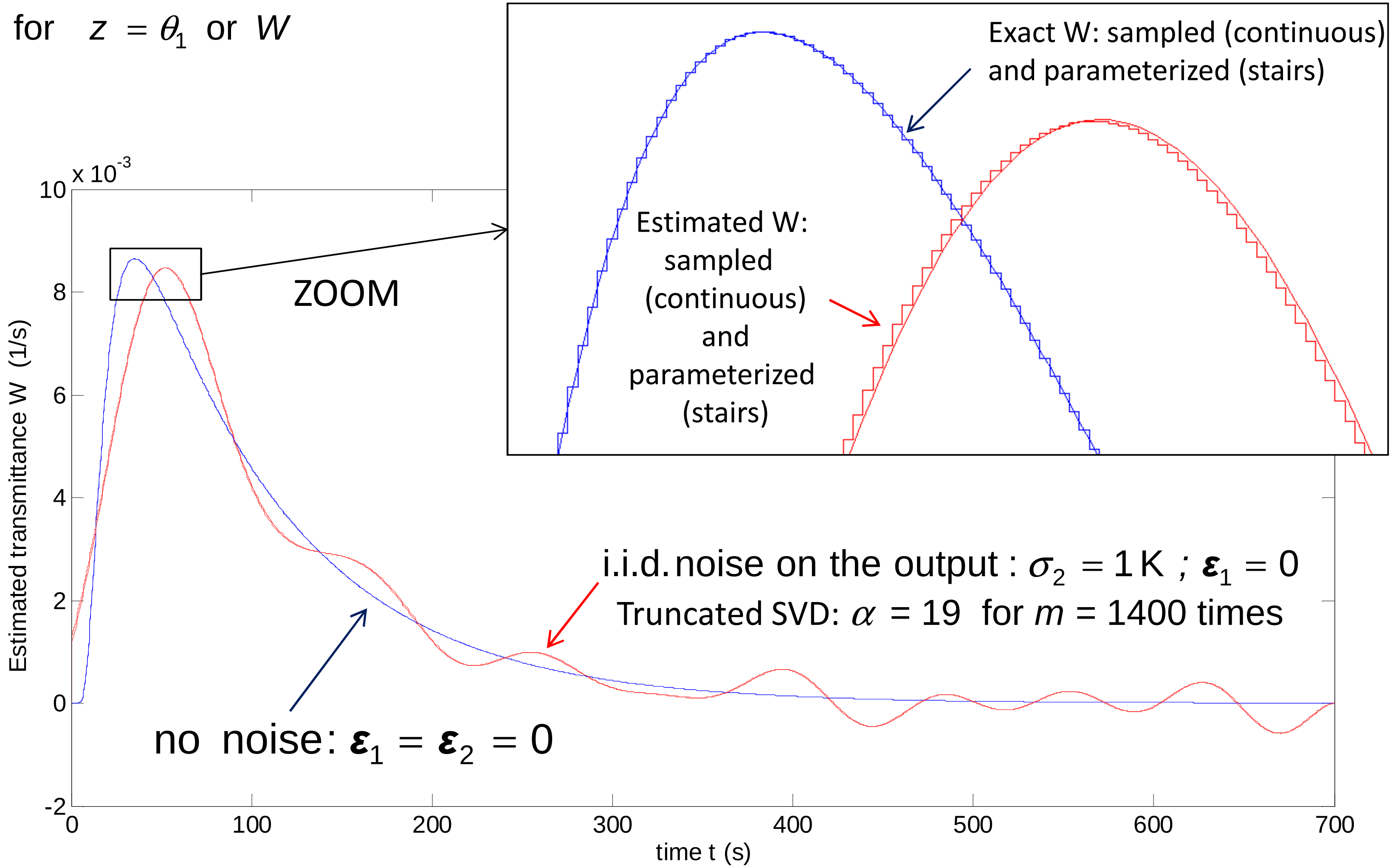
α : truncation order - σ_1 = noise level

$$\text{cond}\left(\mathbf{M}\,(\boldsymbol{\theta}_{1,\alpha}^{stairs,\sigma_1})\right) \geq \text{cond}\left(\mathbf{M}\,(\boldsymbol{\theta}_{1,\alpha}^{\text{sampled},\sigma_1})\right)$$

Effect of the type of parameterization: noise on θ_2 only

Sampled = parameters = instantaneous value (parameterization over a basis of « hat » functions) : $z_i = z(t_i)$
Parameterized = averaged value (parameterization over a base of « doors » functions) : $z_i = 0.5\ (z(t_i) + z(t_{i-1}))$

for $z = \theta_1$ or W



Exact W: sampled (continuous) and parameterized (stairs)

Estimated W: sampled (continuous) and parameterized (stairs)

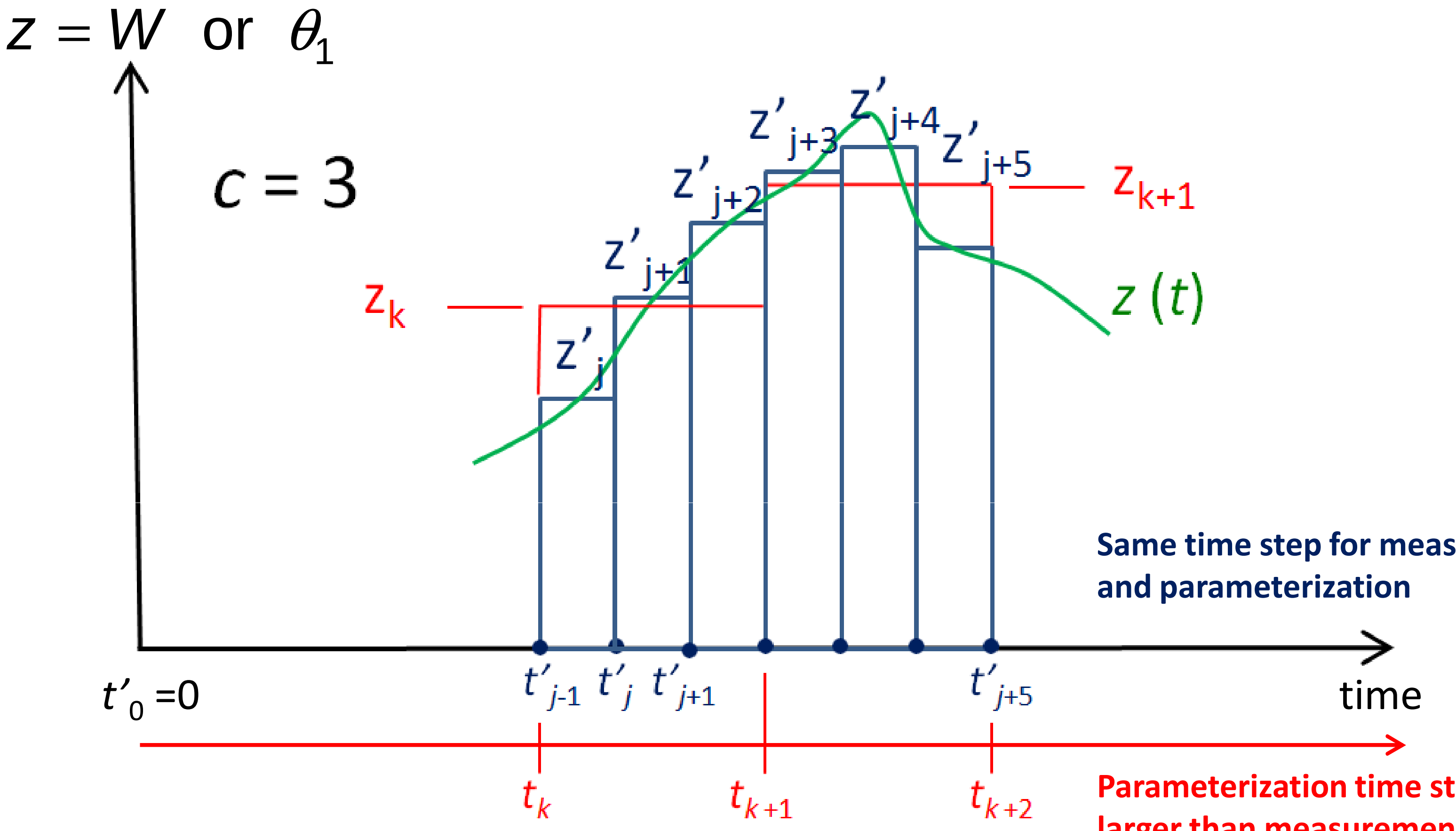
i.i.d. noise on the output : $\sigma_2 = 1\,\text{K}\ ;\ \boldsymbol{\varepsilon}_1 = 0$

Truncated SVD: $\alpha = 19$ for $m = 1400$ times

no noise: $\boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2 = 0$

$$\text{bias}(\boldsymbol{\beta}) = \text{E}\left(\hat{\boldsymbol{\beta}}\right) - \boldsymbol{\beta}^{exact} \qquad \left\|\text{bias}\left(\hat{\boldsymbol{W}}_{\alpha_{opt}}^{stairs,\ \sigma_2}\right)\right\| \leq \left\|\text{bias}\left(\hat{\boldsymbol{W}}_{\alpha_{opt}}^{sampled,\ \sigma_2}\right)\right\|$$

4.2 Rectangular deconvolution (using « stairs » parameterizing)



Same time step for measurements and parameterization

Parameterization time step larger than measurement one

- Less many parameters than measurement times: $n << m$
- Simplest method = use of a basis of n piecewise constant functions : $m/n = c$ (integer)

Square model :

$$\boldsymbol{\theta}_2^{square} = \mathbf{M}\ (\boldsymbol{\theta}_1)\ \boldsymbol{W}^{\ square} = \boldsymbol{A}\ \ \boldsymbol{W}^{\ square}$$

m x 1 m x m m x 1 m x m

$$\boldsymbol{A} = \left[\boldsymbol{A}_1\quad \boldsymbol{A}_2\quad \cdots\quad \boldsymbol{A}_n\right]$$

sensitivity matrix **sensitivity vectors**

Rectangular model :

$$\boldsymbol{\theta}_2^{rectangular} = \mathbf{M}\ (\boldsymbol{\theta}_1)\ \boldsymbol{W}^{\ square} = \boldsymbol{X}\ \ \boldsymbol{W}$$

m x 1 m x n m x 1 m x n n x 1

repetition of some lines

$$\boldsymbol{W}^{\ square} = \boldsymbol{G}\ \ \boldsymbol{W}$$

m x 1 m x n n x 1

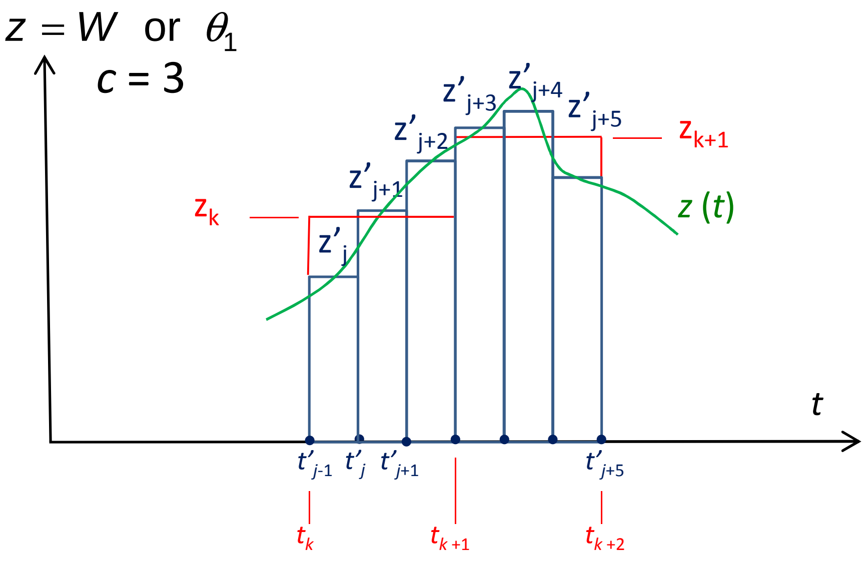
local averaging

$$\boldsymbol{\theta}_2^{rectangular} = \boldsymbol{A}\ \boldsymbol{G}\ \boldsymbol{W}\ = \boldsymbol{X}\ \boldsymbol{W}$$

m x 1 m x n m x n n x 1

$$\boldsymbol{X} = \left[\boldsymbol{X}_1\quad \boldsymbol{X}_2\quad \cdots\quad \boldsymbol{X}_n\right]$$

sensitivity matrix **sensitivity vectors**

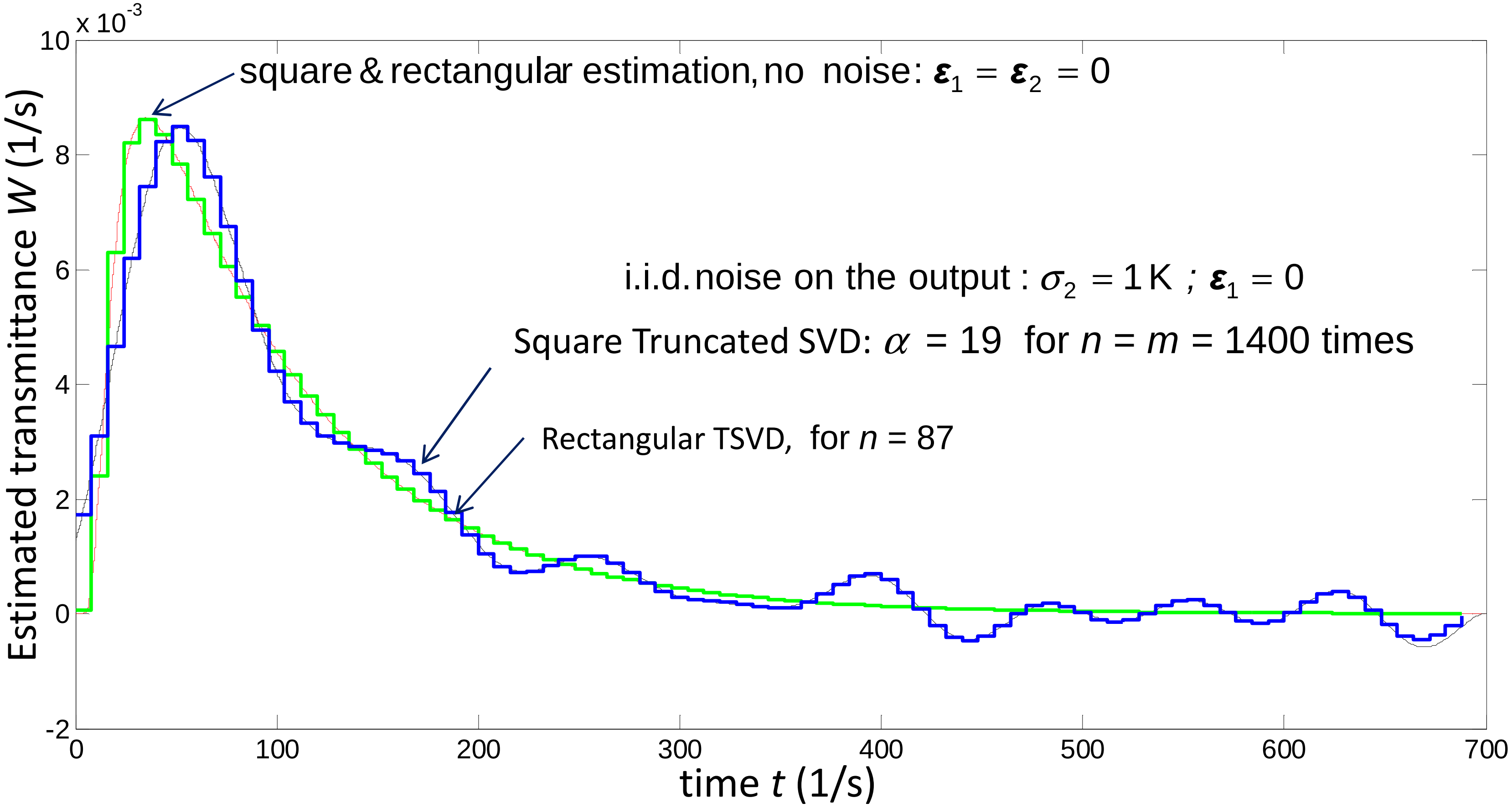


$$\boldsymbol{X}_k = \frac{1}{c}\sum_{j=(k-1)\,c+1}^{k\,c} \boldsymbol{A}_j$$

$$\boldsymbol{A}_j = \Delta t \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \theta_{1,\,j} \\ \theta_{1,\,j+1} \\ \vdots \\ \theta_{1,\,m-j+1} \end{bmatrix}$$

The *n* rectangular sensitivity vectors are simply the averaged values of the *m* square sones

Effect of rectangular inversion: 1 unknown every *c* =16 time steps



square & rectangular estimation, no noise: $\boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2 = 0$

i.i.d. noise on the output : $\sigma_2 = 1\,\mathrm{K}\ ;\ \boldsymbol{\varepsilon}_1 = 0$

Square Truncated SVD: α = 19 for *n* = *m* = 1400 times

Rectangular TSVD, for *n* = 87

Rectangular case with *n* = 87 unknowns $\quad \mathrm{cond}(\boldsymbol{A}) = 2530$

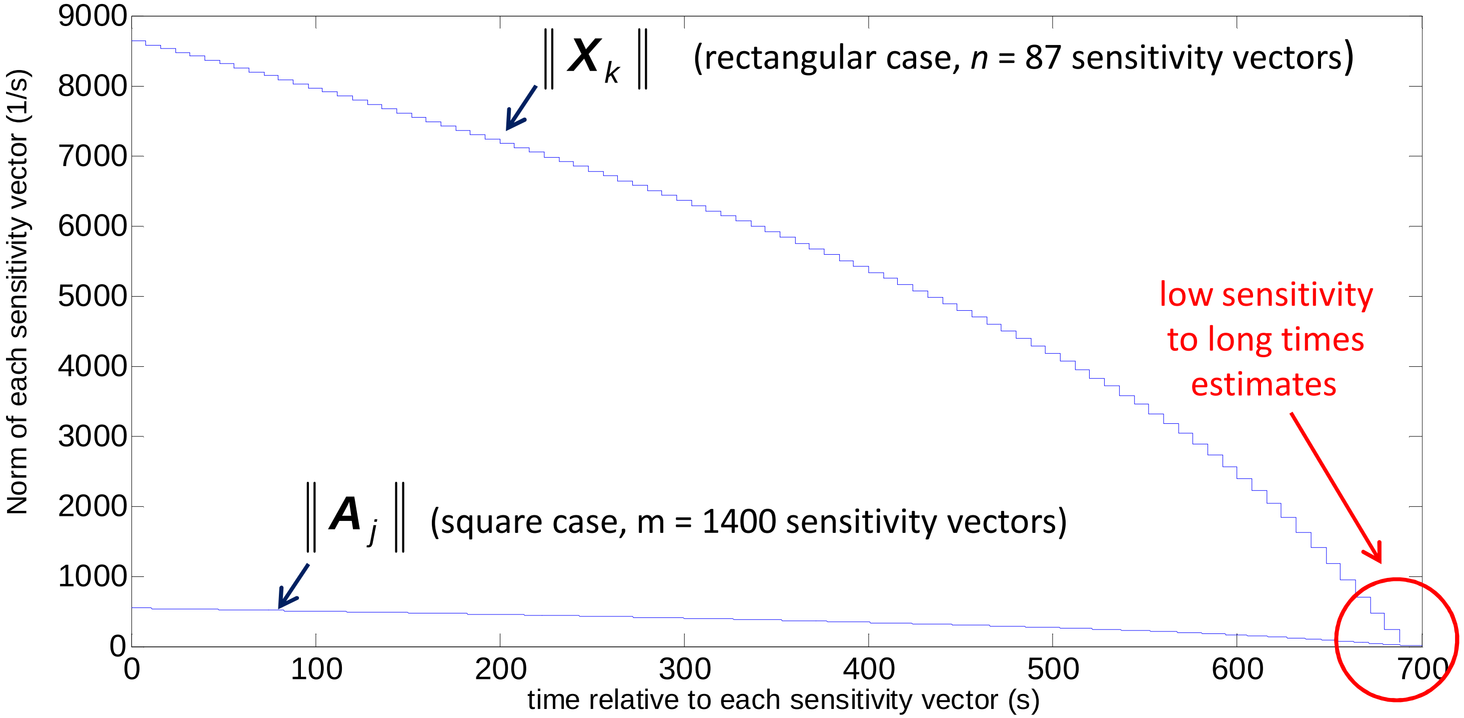
Same relative RMS residual

$$\left\| \boldsymbol{\theta}_2^{\sigma_2} - \boldsymbol{\theta}_2^{square,recalculated}(\hat{\boldsymbol{W}}_{\alpha_{opt}}) \right\| / \left\| \boldsymbol{\theta}_2^{\sigma_1} \right\| = 3.88\ \%$$

$$\left\| \boldsymbol{\theta}_2^{\sigma_2} - \boldsymbol{\theta}_2^{rec\tan gular,\ recalculated}(\hat{\boldsymbol{W}}_{OLS}) \right\| / \left\| \boldsymbol{\theta}_2^{\sigma_1} \right\| = 3.88\ \%$$

No gain in term of estimation bias

Study of the norm (length) of the sensitivity vectors



4.3 Rectangular estimation with $n < m$ non uniform (NU) time steps

$$\boldsymbol{X}_k^{NU} = \frac{1}{c_k} \sum_{a_{k-1}+1}^{a_k} \boldsymbol{A}_j \quad \text{where} \quad a_k = \sum_{k'=1}^{k-1} c_k$$

Different number of elementary time steps

Each of the n rectangular sensitivity vectors are simply the averaged values of c_k square ones

$$\boldsymbol{A}_j = \Delta t \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \theta_{1,j} \\ \theta_{1,j+1} \\ \vdots \\ \theta_{1,m-j+1} \end{bmatrix}$$

$$\underset{m \times 1}{\boldsymbol{\theta}_2^{rectangular,\,NU}} = \underset{m \times n}{\boldsymbol{X}^{NU}} \; \underset{n \times 1}{\boldsymbol{W}} \quad \text{where} \quad \boldsymbol{X}^{NU} \equiv \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \cdots & \boldsymbol{X}_n \end{bmatrix}$$

$$\text{with} \quad \underset{m \times n}{\boldsymbol{X}^{NU}} = \underset{m \times n}{\boldsymbol{G}^{NU}} \; \underset{m \times m}{\boldsymbol{A}} = \boldsymbol{G}^{NU} \; \mathbf{M}\,(\boldsymbol{\theta}_1)$$

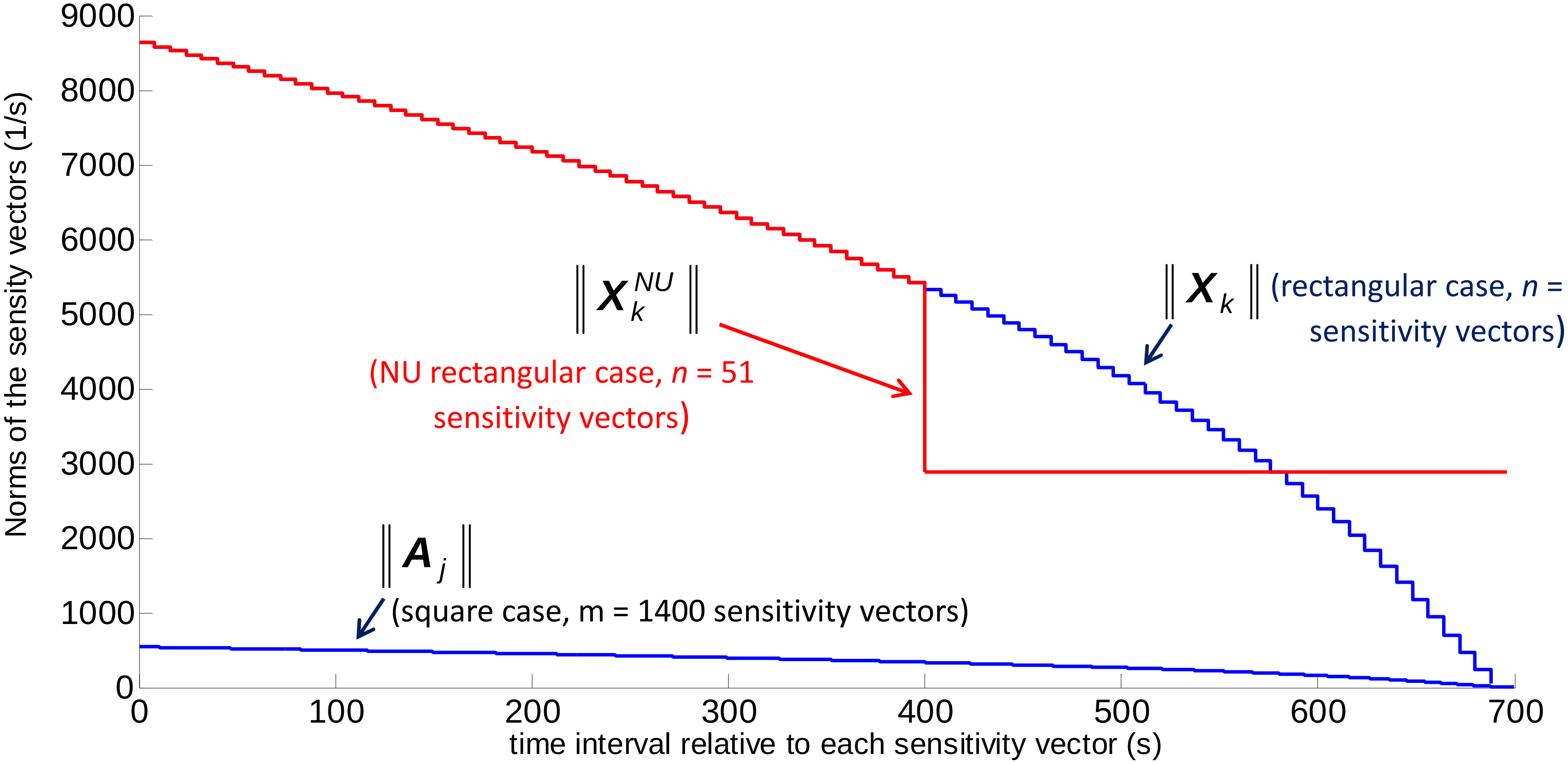
local averaging

lower triangular Toeplitz matrix (square)

Question: how to chose the n limits of the n different time steps ?

First try: constant level past time t = 400 s (steady state reached)

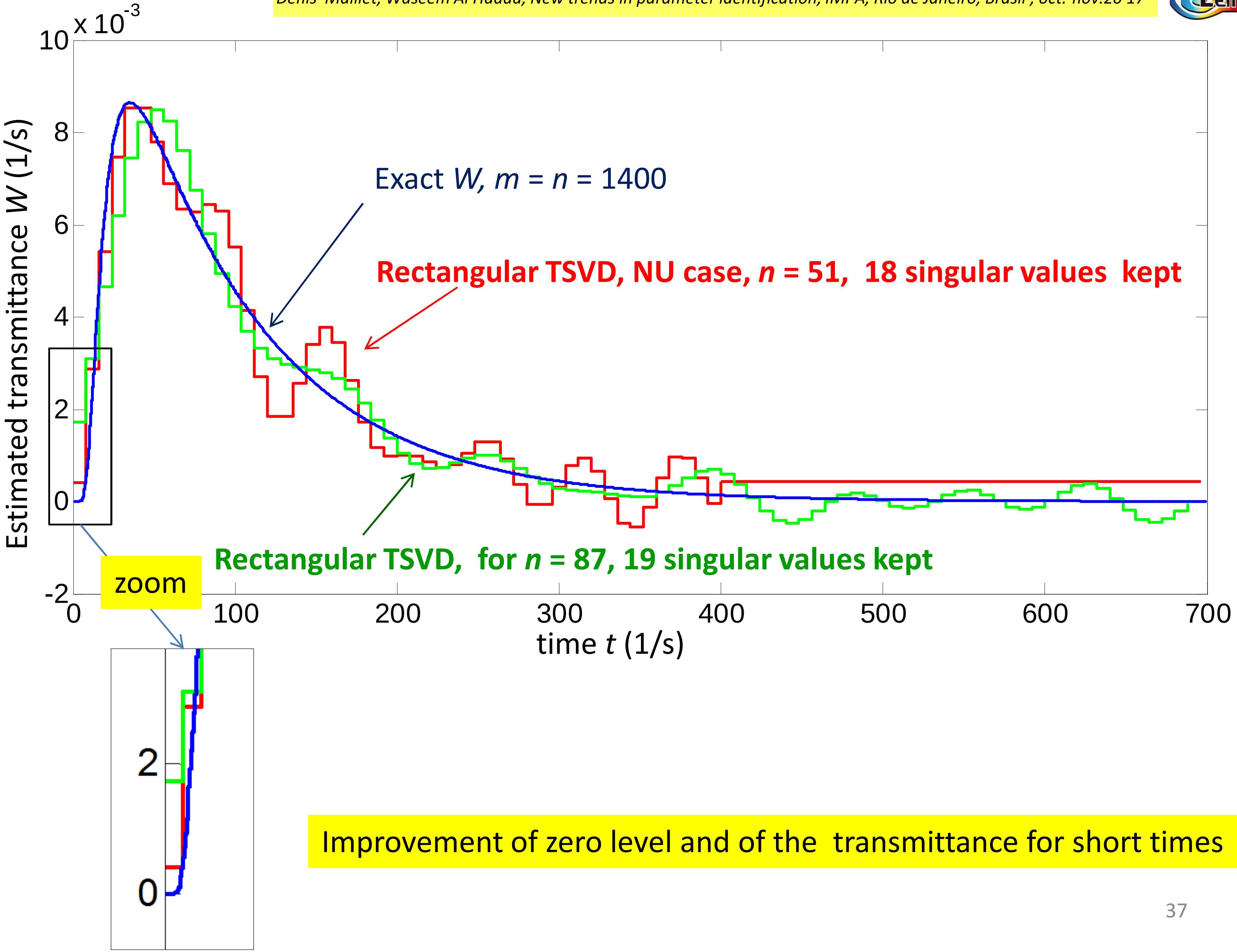
Study of the norm (length) of the sensitivity vectors



$\left\| \boldsymbol{X}_k^{NU} \right\|$ (NU rectangular case, n = 51 sensitivity vectors)

$\left\| \boldsymbol{X}_k \right\|$ (rectangular case, n = 87 sensitivity vectors)

$\left\| \boldsymbol{A}_j \right\|$ (square case, m = 1400 sensitivity vectors)



Exact W, $m = n = 1400$

Rectangular TSVD, NU case, n = 51, 18 singular values kept

Rectangular TSVD, for n = 87, 19 singular values kept

Improvement of zero level and of the transmittance for short times

5. Conclusions/perspectives

- Importance and applicability of **transfer functions** (impedances, transmittances, …) in (exact) **reduced** convolutive model structures for **Linear Time Invariant** physical systems (detailled model = PDE, integro-differential equations, …)

- Convolution products can be given a **commutative** vector/matrix form in discrete time

  ⇨ ill-posed inverse problems: identification problem (calibration) **first**, inverse input problem (source estimation) or inverse (or direct) virtual sensor, or use in a Non Destructive Testing procedure **next**

- Path to improve the quality of estimation of transmittance: **rectangular deconvolution** and pertinent way of **tayloring** its unknown parameters

- Perspective: use of **ARX structures** (AutoRegressive models with eXternal inputs) for better estimation of transfer functions

Muito obrigado!

Model for *W* identification calibration problem: →

$$\overset{\text{exact}}{\boldsymbol{\theta}_q} = \mathbf{M}(\overset{\text{exact}}{\boldsymbol{\theta}_1})\ \overset{\text{unknown}}{\boldsymbol{W}^q}$$

measured

$$\boldsymbol{\theta}_q^{exp} = \boldsymbol{\theta}_q + \boldsymbol{\varepsilon}_q \quad \text{and} \quad \boldsymbol{\theta}_1^{exp} = \boldsymbol{\theta}_1 + \boldsymbol{\varepsilon}_1$$

2 i.i.d. and independent noises

- Ordinary least squares: $\hat{\boldsymbol{W}}^q = \left(\mathbf{M}(\boldsymbol{\theta}_1^{exp})\right)^{-1} \boldsymbol{\theta}_q^{exp}$

Ill-posed problem:
Inversion needs **regularization**
Here: Truncated SVD or 0 order Tikhonov

- SVD decomposition of square sensitivity matrix:

$$\mathbf{M}\,(\theta_1^{exp}) = \boldsymbol{U}\ \boldsymbol{S}\ \boldsymbol{V}^T \qquad \text{with} \quad \boldsymbol{S} = \text{diag}\,\underbrace{\left(s_1 \quad s_2 \quad \cdots \quad s_m\right)}_{\text{singular values}}$$

-TSVD: $\hat{\boldsymbol{W}}_\alpha^q = \boldsymbol{V}\ \boldsymbol{S}_\alpha^{-1} \boldsymbol{U}^T\ \boldsymbol{\theta}_q^{exp}$ with $\boldsymbol{S}_\alpha^{-1} = \text{diag}\,(1/\,s_1 \quad 1/\,s_2 \quad \cdots \quad 1/\,s_\alpha \quad 0 \quad \cdots \quad 0)$

- Zero order Tikhonov: $\hat{\boldsymbol{W}}_\mu^q = \text{Arg}\left(\min_{\boldsymbol{W}}\left(\underbrace{\left\|\boldsymbol{r}(\boldsymbol{W})\right\|_2^2}_{\substack{\text{ordinary}\\\text{least squares sum}}} + \mu \left\|\boldsymbol{W}\right\|_2^2\right)\right)$ where $\underbrace{\boldsymbol{r}\,(\boldsymbol{W})}_{\text{residual vector}} \equiv \boldsymbol{\theta}_q^{exp} - \mathbf{M}\,(\boldsymbol{\theta}_1^{exp})\ \boldsymbol{W}$

or: $\hat{\boldsymbol{W}}_\mu^q = \boldsymbol{V}\ \boldsymbol{S}_\mu^{-1} \boldsymbol{U}^T\ \boldsymbol{\theta}_q^{exp}$ with $\boldsymbol{S}_\mu^{-1} = \boldsymbol{F}_\mu \boldsymbol{S}$ where $\boldsymbol{F}_\mu = \text{diag}\left(\dfrac{s_1^2}{\mu^2 + s_1^2} \quad \dfrac{s_2^2}{\mu^2 + s_2^2} \quad \cdots \quad \dfrac{s_m^2}{\mu^2 + s_m^2}\right)$

- Choice of the hyperparameters $\gamma = \alpha$ or μ by discrepancy principle (Morozov) : $\left\|\boldsymbol{r}(\hat{\boldsymbol{W}}_\gamma)\right\|_2^2 \approx m\ \sigma^2$

standard deviation of $\boldsymbol{\varepsilon}_q$

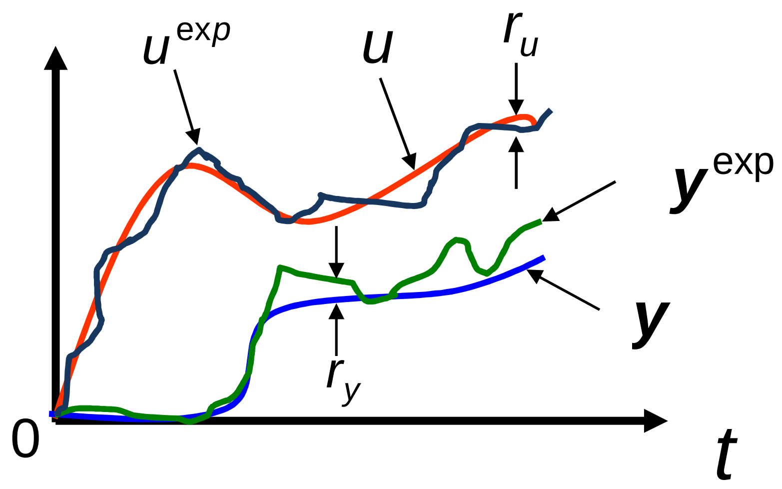
Total Least Squares (TLS) solution : augmented matrix $\boldsymbol{G} = \left[\, \mathbf{M}(\boldsymbol{u}) \;\middle|\; \boldsymbol{y} \,\right]$

concatenation

$\hat{\boldsymbol{H}}$ such as minimum residuals $\boldsymbol{r}_y$ and $\boldsymbol{r}_u$

$$J_{TLS}(\boldsymbol{H}) = \left\| \boldsymbol{r}_G(\boldsymbol{H}) \right\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{m+1} g_{ij}^2$$

with $\boldsymbol{r}_G(\boldsymbol{H}) = \boldsymbol{G}^{exp} - \boldsymbol{G}$



SVD form

ill-posed

Regularized form: T-TSVD

$$\hat{\boldsymbol{H}}_{TLS} = -\boldsymbol{V}_{12}\, v_{22}^{-1}$$

with $\boldsymbol{G}^{\exp} = \boldsymbol{U} \Sigma \boldsymbol{V}^T$

and $\boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & v_{22} \end{bmatrix}$

$$\hat{\boldsymbol{H}}_{T-TLS} = -\boldsymbol{V}_{12}^{\alpha_2} \left(\boldsymbol{V}_{22}^{\alpha_2} \right)^T \Big/ \left\| \boldsymbol{V}_{22}^{\alpha_2} \right\|^2$$

with $\boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_{11}^{\alpha_2} & \boldsymbol{V}_{12}^{\alpha_2} \\ \boldsymbol{V}_{21}^{\alpha_2} & \boldsymbol{V}_{22}^{\alpha_2} \end{bmatrix}$

4.1. Point versus averaged values for input and unknown

Back to phe parameterization problem

$$\boldsymbol{\theta}_2 = \mathbf{M}\ (\boldsymbol{\theta}_1)\ \boldsymbol{W}$$

$$\mathbf{M}\ (\boldsymbol{z})\ \equiv\ \Delta t \begin{bmatrix} z_1 & & & & \\ z_2 & z_1 & & 0 & \\ z_3 & z_2 & z_1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ z_m & z_{m-1} & z_{m-2} & \cdots & z_1 \end{bmatrix}$$

- $z\ (t) = W\ (t)$ or $\theta_1\ (t)$: defined on a basis of m piecewise constant functions

$$z\ (t) = \sum_{j=1}^{m} z_j\ \left[\mathrm{H}\ (t - t_{j-1})\ \text{-}\ \mathrm{H}\ (t - t_j)\right]$$

Heaviside function

- θ_2 : vector of m sampled values

$$z_i\ = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} z\ (t)\ \mathrm{d}t \approx \frac{1}{2} \left(z\,(t_{i-1}) + z\,(t_i)\right) \quad \Rightarrow\ \text{cond}\left(\mathbf{M}(\boldsymbol{\theta}_1^{stairs})\right) = 3.64\ 10^8$$

for $z\ = \theta_1$ or W

Other choice (sampled values)

$$z_i\ = z\,(t_i) \quad \text{for} \quad z\ = \theta_1\ \text{ or }\ W \qquad \Rightarrow\ \text{cond}\left(\mathbf{M}(\boldsymbol{\theta}_1^{sampled})\right) = 2.05\ 10^5$$