Monotonicity-based regularization of inverse coefficient problems

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Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

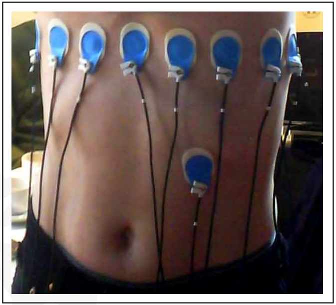
$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\}?$$

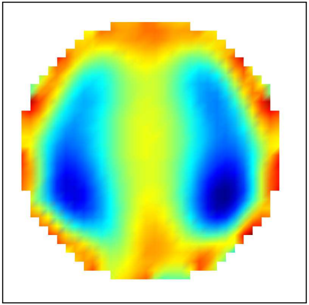
Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Application: Electrical impedance tomography





- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Inversion of $\sigma \mapsto \Lambda(\sigma)$?

Generic solvers for non-linear inverse problems:

- Linearize and regularize:

$$\Lambda_{\text{meas}} \approx \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve using linear regularization method, repeat for Newton-type algorithm.)

- Regularize and linearize:

  E.g., minimize non-linear Tikhonov functional

$$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \to \text{min!}$$

Advantages of generic optimization-based solvers:

- Very flexible, additional data/unknowns easily incorporated
- Problem-specific regularization can be applied

  (e.g., total variation penalization, stochastic priors, etc.)

Inversion of $\sigma \mapsto \Lambda(\sigma)$?

Problems with generic optimization-based solvers

- High computational cost
  - Evaluations of $\Lambda(\cdot)$ and $\Lambda'(\cdot)$ require PDE solutions.
  - PDE solutions too expensive for real-time imaging
- Convergence unclear
  (Validity of TCC/Scherzer-condition is a long-standing open problem for EIT.)
  - Convergence against true solution for exact meas. Λ_{meas}?
    (in the limit of infinite computation time)
  - Convergence against true solution for noisy meas. $\Lambda_{\text{meas}}^{\delta}$?
    (in the limit of $\delta \to 0$ and infinite computation time)
  - Global convergence? Resolution estimates for realistic noise?

Is there any specific problem structure that we can use to derive convergent algorithms?

Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from *(Kang/Seo/Sheen 1997, Ikehata 1998)*

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial\Omega} g\left(\Lambda(\sigma_0) - \Lambda(\sigma_1)\right) g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial\Omega} = g.$$

Converse monotonicity relation can be shown by controlling $|\nabla u_0|^2$.

*(Localized Potentials: **H.**, Inverse Probl. Imaging 2008)*

Sample inclusion detection problem (for ease of presentation)

- $\sigma_0 = 1$
- $\sigma_1 = 1 + \chi_D$
- D open, $\overline{D} \subseteq \Omega$, $\Omega \setminus \overline{D}$ connected

All of the following also holds for

- σ_0 pcw. analytic and known,
- $\sigma_1 = \sigma_0 + \kappa\chi_D$ with $\kappa \in L^\infty_+(D)$,
- in any dimension $n \geq 2$,
- for partial boundary data on open subset $\Gamma \subseteq \partial\Omega$.

Monotonicity method

H./Ullrich, SIAM J. Math. Anal. 2013:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma)$$

- Yields theoretical uniqueness result
- Simple to implement, no PDE solutions
- Similar comput. cost as single Newton (linearization) step
- Rigorously detects unknown shape for exact data
- Convergence for noisy data $\Lambda^\delta_{\text{meas}} \to \Lambda(\sigma) - \Lambda(1)$:

$$R(\Lambda^\delta_{\text{meas}}, \delta, B) := \begin{cases} 1 & \text{if } \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda^\delta_{\text{meas}} - \delta I \\ 0 & \text{else.} \end{cases}$$

Then $R(\Lambda^\delta_{\text{meas}}, \delta, B) \to 1$ iff $B \subseteq D$.

Monotonicity method

Quantitative, pixel-based variant of monotonicity method:

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Quantitative monotonicity tests

$$\beta_k \in [0,\infty] \text{ max. values s.t. } \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$
$$\beta_k^\delta \in [0,\infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1)\chi_{P_k} \geq \Lambda_{\text{meas}}^\delta - \delta I$$

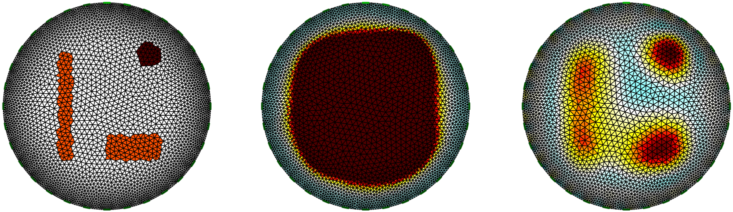
"Raise conductivity in each pixel until monotonicity test fails."

- By theory of monotonicity method:

$$\beta_k^\delta \to \beta_k \quad \text{and} \quad \beta_k \text{ fulfills} \begin{cases} \beta_k = 0 & \text{if } P_k \not\subseteq D \\ \beta_k \geq \frac{1}{2} & \text{if } P_k \subseteq D \end{cases}$$

Plotting β_k^δ shows true inclusions up to pixel partition.

Realistic example (32 electrodes, 1% noise)



True image | Monotonicity method | Standard linearized method

- Monotonicity method rigorously converges for $\delta \to 0$ ...
- ... but the heuristic standard linearized method works much better for realistic scenarios.

Can we improve the monotonicity method without loosing convergence?

Monotonicity-based regularization

- Standard linearized methods for EIT: Minimize

$$\|\Lambda'(1)\kappa-(\Lambda(\sigma)-\Lambda(1))\|^2+\alpha\|\kappa\|^2 \to \text{min!}$$

Choice of norms heuristic. No convergence theory!

- Monotonicity-based regularization: Minimize

$$\|\Lambda'(1)\kappa-(\Lambda(\sigma)-\Lambda(1))\|_{\mathsf{F}} \to \text{min!}$$

under the constraint $\kappa|_{P_k} = \text{const.},\ 0 \leq \kappa|_{P_k} \leq \min\{\frac{1}{2},\beta_k\}$.

($\|\cdot\|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

Theorem (**H.**/Mach, Inverse Problems 2016)

- There exists unique minimizer $\hat{\kappa}$ and

$$P_k \subseteq \operatorname{supp}\hat{\kappa} \quad \Longleftrightarrow \quad P_k \subseteq \operatorname{supp}(\sigma-1).$$

- Minimizer fulfills $\hat{\kappa} = \sum_{k=1}^{m} \min\{1/2,\beta_k\}\chi_{P_k}$

Monotonicity-based regularization

For noisy measurements $\Lambda_{\text{meas}}^{\delta} \approx \Lambda(\sigma) - \Lambda(1)$:

- Use regularized monotonicity tests

$$\beta_k^{\delta} \in [0,\infty] \text{ max. values s.t. } \beta_k^{\delta} \Lambda'(1)\chi_{P_k} \geq \Lambda_{\text{meas}}^{\delta} - \delta I$$

($\delta > 0$: noise level in $\mathcal{L}(L^2_\diamond(\partial\Omega))$-norm)

- Minimize

$$\|\Lambda'(1)\kappa^{\delta} - \Lambda_{\text{meas}}^{\delta}\|_{\text{F}} \to \text{min!}$$

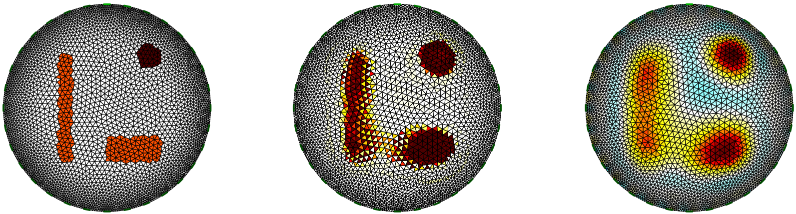
under the constraint $\kappa^{\delta}|_{P_k} = \text{const.}$, $0 \leq \kappa^{\delta}|_{P_k} \leq \min\{\frac{1}{2}, \beta_k^{\delta}\}$.

Theorem (**H.**/Mach, Inverse Problems 2016)

- There exist minimizers κ^{δ} and $\kappa^{\delta} \to \hat{\kappa}$ for $\delta \to 0$.

Monotonicity-regularized solutions converge against correct shape.

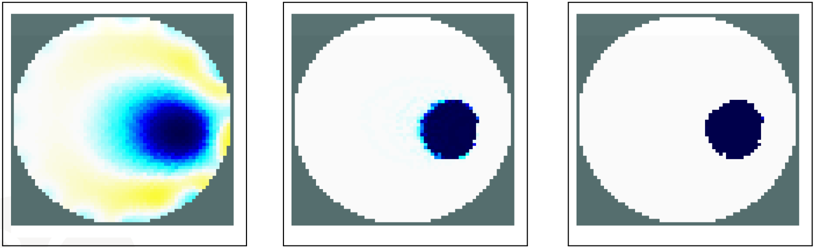
Realistic example (32 electrodes, 1% noise)



True image | Monotonicity regularized method | Standard linearized method

- Monotonicity regularized method rigorously converges and is up to par with (outperforms?) heuristic standard linearized method.

Phantom data example



standard | monoton.-regularized (Matlab quadprog) | monoton.-regularized (cvx package)

Monotonicity-regularization vs. community standard

(**H.**/Mach, Trends Math., to appear)

- EIDORS: `http://eidors3d.sourceforge.net` (Adler/Lionheart)
- EIDORS standard solver: linearized method with Tikhonov regularization
- Dataset: `iirc_data_2006` (Woo et al.): 2cm insulated inclusion in 20cm tank
  - using interpolated data on active electrodes (**H.**, Inverse Problems 2015)

Stochastic Calderón problem

Deterministic Calderón Problem: Can we recover σ from NtD

$$\Lambda(\sigma): L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves $\nabla\cdot(\sigma\nabla u) = 0$ in Ω with $\sigma\partial_\nu u|_{\partial\Omega} = g$?

- Stochastic Calderón problem:

  Can we recover $\mathbb{E}(\sigma)$ *from* $\mathbb{E}(\Lambda(\sigma))$*?*

- Stochastic inclusion detection in hom. background ($\sigma_0 = 1$):

  Can we recover $\operatorname{supp}(\mathbb{E}(\sigma) - 1)$ *from* $\mathbb{E}(\Lambda(\sigma))$*?*

- (Possible) Application: Biomedical anomaly detection from temporally averaged measurements.

NtD-operator is of finite expectation

Theorem *(Barth/**H.**/Hyvönen/Mustonen, Inverse Problems 2017)*

If $\sigma, \sigma^{-1} \in L^1(W, L^\infty_+(\Omega))$, W probability space, then

- $\Lambda(\sigma) \in L^1(W, L^\infty_+(\Omega))$,
- $\mathbb{E}(\Lambda(\sigma))$ is well-defined,
- $\mathbb{E}(\Lambda(\sigma)) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega)$ is compact and self-adjoint.

Proof.

- $\Lambda(\sigma) : W \to \mathcal{L}(L^2_\diamond(\partial\Omega))$ is concatenation of strongly meas. function and continuous function and thus strongly measurable.
- Integrability bound on $\Lambda(\sigma)$ follows from monotonicity inequality.

Detecting stochastic inclusions

Theorem *(Barth/**H.**/Hyvönen/Mustonen, Inverse Problems 2017)*

If

- $\sigma = \begin{cases} 1 & \text{in } \Omega \setminus D, \\ \sigma_D(x,\omega) & \text{in } D, \end{cases}$
- $\sigma_D : \Omega \to L^\infty_+(D)$, W probability space,
- $\sigma_D, \sigma_D^{-1} \in L^1(W, L^\infty_+(D))$,
- $\exists \alpha > 0: \quad \mathbb{E}(\sigma_D) > 1 + \alpha,$ and $\mathbb{E}(\sigma_D^{-1})^{-1} > 1 + \alpha,$

Then D is uniqu. determined by Monoton. Meth. applied to $\mathbb{E}(\Lambda(\sigma))$

(and also by the similar Factorization Method).

Stochastic uncertainty in σ behaves like deterministic uncertainty in
$[\mathbb{E}(\sigma^{-1})^{-1}, \mathbb{E}(\sigma)]$.

Monotonicity for stochastic inclusions

Main idea of the proof. Monotonicity for stochastic inclusions:

For deterministic σ_0 and stochastic σ:

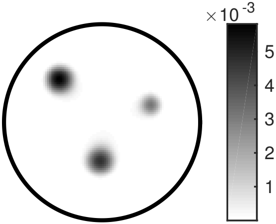
$$\int_\Omega (\mathbb{E}(\sigma) - \sigma_0)) |\nabla u_0|^2 \, \mathrm{d}x \geq \int_{\partial\Omega} g\big(\Lambda(\sigma_0) - \mathbb{E}(\Lambda(\sigma))\big) g \, \mathrm{d}s$$

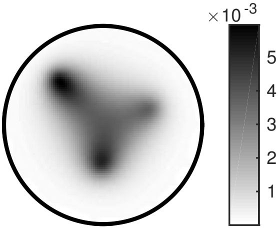
$$\geq \int_\Omega \sigma_0^2 \left(\sigma_0^{-1} - \mathbb{E}(\sigma^{-1})\right) |\nabla u_0|^2 \, \mathrm{d}x.$$

In particular,

$$\sigma_0 \leq \mathbb{E}(\sigma) \text{ and } \sigma_0 \leq \mathbb{E}(\sigma^{-1})^{-1} \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \mathbb{E}(\Lambda(\sigma))$$

Example





- Background conductivity $\sigma_0 = 1$
- Inclusions conductivity uniformly distributed in $[0.5, 3.5]$

$$\mathbb{E}(\sigma_D) \geq \mathbb{E}(\sigma_D^{-1})^{-1} \approx 1.54 > 1 = \sigma_0$$

- Images show result of Factorization Method applied to $\mathbb{E}(\sigma)$ (Left Image: no noise, Right Image: 0.1% noise)

Conclusions

Inverse coeff. problems such as EIT are highly ill-posed & non-linear.

- Convergence of generic solvers unclear.
- Often heuristic regularization without theor. justification is used.

Monotonicity and localized potentials yield

- theoretical uniqueness results,
- convergent inclusion detection methods,
- rigorous regularizers for residuum-based methods.

Approach can be extended

- to partial boundary data, independently of dimension $n \geq 2$,
- to stochastic settings,
- to other linear elliptic problems *(diffuse optical tomography, magnetostatics)*
- at least partially to closely related problems

  (eddy-current equations, p-Laplacian, inverse scattering, fractional diffusion)