

All-at-once and minimization based formulations of inverse problems and their regularization

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New Trends in Parameter Identification for Mathematical Models, IMPA



Outline

- examples of parameter identification problems in PDEs
- reduced versus all-at-once formulations
- minimization based formulations

examples

Parameter Identification in Differential Equations: Some Examples

- Identify spatially varying coefficients/source a, b, c in linear elliptic boundary value problem on $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$

$$-\nabla(a\nabla u) + cu = b \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

from boundary or (restricted) interior observations of u .

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- Identify parameter ϑ in initial value problem for ODE / PDE

$$\dot{u}(t) = f(t, u(t), \vartheta) \quad t \in (0, T), \quad u(0) = u_0$$

from discrete or continuous observations of u .

$$y_i = g_i(u(t_i)), \quad i \in \{1, \dots, m\} \text{ or } y(t) = g(t, u(t)), \quad t \in (0, T)$$

Abstract Formulation

Identify parameter q in (PDE or ODE) model

$$A(q, u) = 0$$

from observations of the state u

$$C(u) = y,$$

where $q \in X$, $u \in V$, $y \in Y$, $X, V, Y \dots$ Hilbert (Banach) spaces

$A : X \times V \rightarrow W^* \dots$ differential operator

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The Parameter-to-State Map S in some Examples

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- generally for model $A(q, u) = 0$:

$S : q \mapsto u$ solving $A(q, S(q)) = 0$

Motivation for All-at-once Formulation

- well-definedness of parameter-to-state map often requires restrictions on ...
 - ... parameters (e.g., $a \geq \underline{a} > 0$, $c \geq 0$ in $-\nabla(a \nabla u) + cu = b$)
 - ... models (e.g., monotonicity of ξ in $-\Delta u + \xi(u) = q$)

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- singular PDEs: parameter-to space map may exist only on a very restricted set, e.g. MEMS equation

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- it can make a difference in implementation and in the analysis (convergence conditions)
- for other all-at-once type approaches see, e.g.,
[Kupfer & Sachs '92, Shenoy & Heinkenschloss & Cliff '98,
Haber & Ascher '01, Burger & Mühlhuber '02,...]

reduced versus all-at-once formulations

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Tikhonov Regularization: reduced

$$\min_q \|F(q) - y^\delta\|^2 + \alpha \mathcal{R}(q)$$

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with $F = C \circ S$, S parameter-to-state map, $A(q, S(q)) = 0$,
equivalent to

$$\min_{q,u} \|C(u) - y^\delta\|^2 + \alpha \mathcal{R}(q) \quad \text{s.t. } A(q, u) = 0$$

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[Seidman&Vogel '89, Engl&Kunisch&Neubauer '89,...] in Hilbert space
[Burger& Osher'04, Resmerita & Scherzer'06, Scherzer et al. '08,
Hofmann&Pöschl&BK&Scherzer '07, Pöschl '09, Flemming '11, Werner
'12,...] in Banach space

Tikhonov Regularization: all-at-once

$$\min_{q,u} \|C(u) - y^\delta\|^2 + \|A(q, u)\|^2 + \alpha\mathcal{R}(q) + \alpha\tilde{\mathcal{R}}(u)$$

or

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i.e., (exact penalization) with ρ sufficiently large

$$\min_{q,u} \|C(u) - y^\delta\|^2 + \alpha\mathcal{R}(q) + \alpha\tilde{\mathcal{R}}(u) \text{ s.t. } A(q, u) = 0$$

i.e., **reduced Tikhonov**.

Regularized Gauss-Newton Method: reduced

q^k fixed, one Gauss-Newton step:

$$\min_q \|F(q^k) + F'(q^k)(q - q^k) - y^\delta\|^2 + \alpha_k \mathcal{R}_k(q)$$

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with $F = C \circ S$, S parameter-to-state map, $A(q, S(q)) = 0$,
equivalent to

$$\begin{aligned} \min_{q, u, \tilde{u}} & \|C(\tilde{u}) + C'(\tilde{u})(u - \tilde{u}) - y^\delta\|^2 + \alpha_k \mathcal{R}_k(q) \\ \text{s.t. } & A(q^k, \tilde{u}) + A'_u(q^k, \tilde{u})(u - \tilde{u}) + A'_q(q^k, \tilde{u})(q - q^k) = 0 \\ & \text{and } A(q^k, \tilde{u}) = 0 \end{aligned}$$

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[Bakushinskii '92, Hohage '97, BK&Neubauer&Scherzer '97,...] in Hilbert space

e.g., [Bakushinskii&Kokurin'04, BK&Schöpfer&Schuster '08, Jin '12, Hohage&Werner '13,...] in Banach space

Regularized Gauss-Newton Method: all-at-once

(q^k, u^k) fixed, one Gauss-Newton step:

$$\min_{q,u} \|C(u^k) + C'(u^k)(u - u^k) - y^\delta\|^2 + \alpha_k \mathcal{R}(q) + \alpha_k \tilde{\mathcal{R}}(u) \\ + \|A(q^k, u^k) + A'_u(q^k, u^k)(u - u^k) + A'_q(q^k, u^k)(q - q^k)\|^2$$

or

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or (q^k, u^k) fixed, one Gauss-Newton step:

$$\min_{q,u} \|C(u^k) + C'(u^k)(u - u^k) - y^\delta\|^2 + \alpha_k \mathcal{R}_k(q) + \alpha_k \tilde{\mathcal{R}}(u) \\ + \rho \|A(q^k, u^k) + A'_u(q^k, u^k)(u - u^k) + A'_q(q^k, u^k)(q - q^k)\|$$

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Comparison of optimality conditions for reduced and all-at-once Newton

reduced:

$$\begin{cases} A(q^k, \tilde{u}) = 0 & (\text{nonlinear decoupled state equation}) \\ A'_u(q^k, \tilde{u})(u - \tilde{u}) = -A(q^k, \tilde{u}) - A'_q(q^k, \tilde{u})(q - q^k) & (\text{linear state eq.}) \\ A'_q(q^k, \tilde{u})^* p + \alpha \partial \mathcal{R}(q) = 0 & (\text{gradient equation}) \\ A'_u(q^k, \tilde{u})^* p = -C'(\tilde{u})^*(C(\tilde{u}) + C'(\tilde{u})(u - \tilde{u}) - y^\delta) & (\text{adjoint equation}) \end{cases}$$

all-at-once:

$$\begin{cases} A'_u(q^k, u^k)(u - u^k) = -A(q^k, u^k) - A'_q(q^k, u^k)(q - q^k) & (\text{linear state eq.}) \\ A'_q(q^k, u^k)^* p + \alpha \partial \mathcal{R}(q) = 0 & (\text{gradient equation}) \\ A'_u(q^k, u^k)^* p = -C'(u^k)^*(C(u^k) + C'(u^k)(u - u^k) - y^\delta) - \alpha \partial \tilde{\mathcal{R}}(u) & (\text{adj.eq.}) \end{cases}$$

Gradient Methods: reduced

gradient steps for

$$\min_q \|F(q) - y^\delta\|^2$$

\rightsquigarrow Landweber iteration (steepest descent, minimal error)

$$q^{k+1} = q^k - \mu^k F'(q^k)^*(F(q^k) - y^\delta)$$

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$$\begin{aligned} q^{k+1} &= q^k - \mu^k F'(q^k)^* (F(q^k) - y^\delta) \\ &= q^k - \mu^k (C'(S(q^k)) S'(q^k))^* (C(S(q^k)) - y^\delta) \\ &= q^k + \mu^k A'_q(q^k, \tilde{u})^* p \end{aligned}$$

where

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[Hanke&Neubauer&Scherzer '95,...] in Hilbert space

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(q^k, u^k) fixed, one Landweber step for $\mathbf{F} \begin{pmatrix} q \\ u \end{pmatrix} = \begin{pmatrix} A(q, u) \\ C(u) \end{pmatrix}$:

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i.e.

$$\begin{cases} q^{k+1} = A'_q(q^k, u^k)^* A(q^k, u^k) \\ u^{k+1} = C'(u^k)^* (C(u^k) - y^\delta) + A'_u(q^k, u^k)^* A(q^k, u^k) \end{cases}$$

completely explicit, no model to solve!

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- Existence of minimizers, stability, convergence, rates under (variational, approximate) source conditions follow as corollaries of existing results for Tikhonov, IRGNM, Landweber, when regularizing with respect to q and u

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- Rates for Tikhonov and IRGNM so far only in case of regularization of both q and u
- Conditions on nonlinearity of F , A , C , e.g., tangential cone or Scherzer condition: often weaker in all-at-once setting (additional freedom in choosing the model equation space W^*)

numerical results

Numerical Tests

nonlinear inverse source problem:

$$-\Delta u + \zeta u^3 = q \text{ in } \Omega = (0, 1) \quad \& \text{ homogeneous Dirichlet BC}$$

Identify q from distributed measurements of u in Ω

Comparison of reduced and all-at-once Landweber

ζ	it _{aao}	it _{red}	cpu _{aao}	cpu _{red}	$\frac{\ b_{k_*(\delta), \text{aao}}^\delta - b^\dagger\ _X}{\ b^\dagger\ _X}$	$\frac{\ b_{k_*(\delta), \text{red}}^\delta - b^\dagger\ _X}{\ b^\dagger\ _X}$
0.5	5178	2697	2.97	18.07	0.0724	0.1047
5	$> 2 \cdot 10^6$	48510	1293.60	482.19	0.7837	0.1633
10	$> 2 \cdot 10^6$	$> 10^5$	1257.50	639.87	0.9621	0.1632
-0.5	10895	2016	8.85	14.55	0.1406	0.2295
-1	18954	-	11.42	-	0.2313	-

(1% Gaussian noise)

Comparison of reduced and all-at-once IRGNM

ζ	it _{aao}	it _{red}	cpu _{aao}	cpu _{red}	$\frac{\ b_{k_*(\delta), \text{aao}}^\delta - b^\dagger\ _X}{\ b^\dagger\ _X}$	$\frac{\ b_{k_*(\delta), \text{red}}^\delta - b^\dagger\ _X}{\ b^\dagger\ _X}$
0	34	32	0.14	0.10	0.0149	0.0151
10	43	43	0.20	0.55	0.0996	0.1505
100	55	56	0.28	0.82	0.0721	0.0770
1000	68	68	0.42	1.07	0.0543	0.0588
-0.5	33	32	0.13	0.35	0.1174	0.2165
-1.	35	-	0.23	-	0.2023	-
-10	44	-	0.23	-	0.0768	-
-100	77	-	0.59	-	0.2246	-
-1000	70	-	0.49	-	0.0321	-

(1% Gaussian noise)

Numerical Tests in 2-d with Adaptive Discretization

nonlinear inverse source problem:

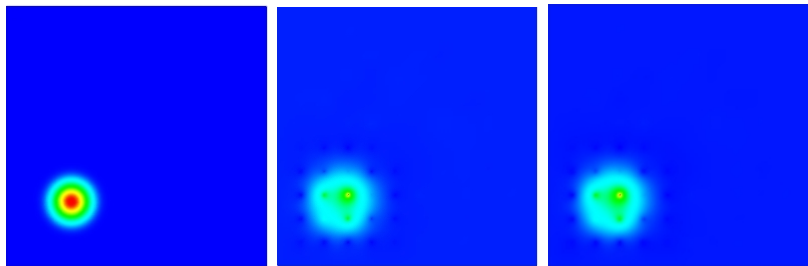
$$-\Delta u + \zeta u^3 = q \text{ in } \Omega = (0, 1)^2 \quad \& \text{ homogeneous Dirichlet BC}$$

Identify q from distributed measurements of u at 10×10 points in Ω

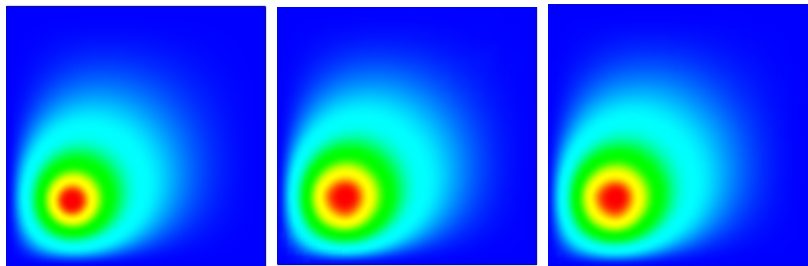
$$q^\dagger = \frac{c}{2\pi\sigma^2} \exp \left(-\frac{1}{2} \left(\left(\frac{sx - \mu}{\sigma} \right)^2 + \left(\frac{sy - \mu}{\sigma} \right)^2 \right) \right)$$

with $c = 10$, $\mu = 0.5$, $\sigma = 0.1$, and $s = 2$.

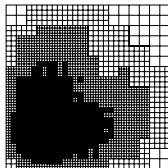
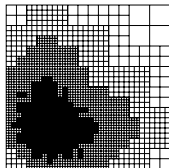
- goal-oriented, dual weighted residual estimators
- computations with *Gascoigne* and *RoDoBo*
- joint work with Alana Kirchner and Boris Vexler (TU Munich)



left: exact source q^\dagger ,
middle: reconstruction by reduced Tikhonov (RT),
right: reconstruction by all-at-once Gauss-Newton (AGN),
with $\zeta = 100$, 1% noise



left: exact state u^\dagger ,
middle: reconstruction by reduced Tikhonov (RT),
right: reconstruction by all-at-once Gauss-Newton (AGN),
with $\zeta = 100$, 1% noise



adaptively refined meshes,
left: by **reduced Tikhonov (RT)**,
right: by **all-at-once Gauss-Newton (AGN)**,
with $\zeta = 100$, 1% noise

Table : all-at-once Gauss-Newton (AGN) versus reduced Tikhonov (RT) for different choices of ζ with 1% noise.

ctr: Computation time reduction using (AGN) in comparison to (RT)

ζ	RT			AGN			ctr
	error	β	# nodes	error	β	# nodes	
1	0.418	2985	2499	0.412	4600	3873	-65%
10	0.417	3194	2473	0.411	4918	3965	-59%
100	0.408	5014	6653	0.417	6773	9813	39%
500	0.418	9421	11851	0.404	13756	821	97%
1000	0.439	11486	44391	0.426	16355	793	99%

minimization based formulations

Abstract formulation of inverse problems

- reduced

$$F(x) = y$$

where x ...searched for parameter, y ...observed data,

$F : X \rightarrow Y$...forward operator

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- all-at-once

$$A(x, u) = 0 \text{ model}$$

$$C(u) = y \text{ observations}$$

$A : X \times V \rightarrow W$...model operator,

$C : V \rightarrow Y$...observation operator

Abstract formulation of inverse problems

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$F : X \rightarrow Y$...forward operator

$F = C \circ S$ with $A(x, S(x)) = 0$

$S : X \rightarrow V$... parameter-to-state map

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$$\min_{x,u} \mathcal{J}(x, u; y) \text{ s.t. } (x, u) \in M_{ad}(y)$$

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- [Kindermann '17] (reduced type formulation),
[BK '17] (avoid parameter-to-state map)

Reduced as special case of minimization based formulation

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where x ...searched for parameter, y ...observed data,

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$S : X \rightarrow V$... parameter-to-state map

- equivalent to

$$\min_{(x,u) \in X \times V} \underbrace{\mathcal{S}(C(u), y) + \mathcal{I}_{\{0\}}(A(x, u))}_{\mathcal{J}(x,u;y)} \quad \text{s.t. } (x, u) \in \underbrace{X \times V}_{M_{ad}(y)},$$

where $\mathcal{S} : Y \times Y \rightarrow \overline{\mathbb{R}}$ is a positive definite functional

$$\forall y_1, y_2 \in Y : \quad \mathcal{S}(y_1, y_2) \geq 0 \quad \text{and} \quad (y_1 = y_2 \Leftrightarrow \mathcal{S}(y_1, y_2) = 0).$$

$$\text{and } \mathcal{I}_M(w) = \begin{cases} 0 & \text{if } w \in M \\ +\infty & \text{else} \end{cases} \quad \dots \text{indicator function}$$

All-at-once as special case of minimization based formulation

- all-at-once

$$A(x, u) = 0 \text{ model}$$

$$C(u) = y \text{ observations}$$

$A : X \times V \rightarrow W \dots$ model operator,

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All-at-once as special case of minimization based formulation

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- equivalent to

$$\min_{(x,u) \in X \times V} \underbrace{\mathcal{S}(C(u), y) + \mathcal{Q}(A(x, u))}_{\mathcal{J}(x, u; y)} \text{ s.t. } (x, u) \in \underbrace{X \times V}_{M_{ad}(y)},$$

where $\mathcal{S} : Y \times Y \rightarrow \overline{\mathbb{R}}$, $\mathcal{Q} : W \rightarrow \overline{\mathbb{R}}$ are positive definite functionals

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$$\forall w \in W : \quad \mathcal{Q}(w) \geq 0 \quad \text{and} \quad (w = 0 \Leftrightarrow \mathcal{Q}(w) = 0).$$

Regularized minimization

inverse problem:

$$(x, u) \in \operatorname{argmin}\{\mathcal{J}(x, u; y) : (x, u) \in M_{ad}(y)\}$$

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y^δ ... perturbed measured data

inverse problem is ill-posed:

minimizer does not depend continuously on y

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y^δ ... perturbed measured data

inverse problem is ill-posed:

minimizer does not depend continuously on y

\leadsto regularized inverse problem:

$$(x_\alpha^\delta, u_\alpha^\delta) \in \operatorname{argmin}\{\mathcal{J}(x, u; y^\delta) + \alpha \cdot \mathcal{R}(x, u) : (x, u) \in M_{ad}^\delta(y^\delta)\}$$

regularize by
adding penalties (Tikhonov type) and/or by
imposing constraints (Ivanov type)

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regularize by

adding penalties (Tikhonov type) and/or by
imposing constraints (Ivanov type)

treat data misfit by

penalty term in cost function (Tikhonov type) or
constraint (Morozov type)

Regularization with data misfit penalization

inverse problem (IP):

$$\begin{aligned} \min_{(x,u) \in X \times V} \quad & \mathcal{S}(C(u), y) + \mathcal{Q}(A(x, u)) \\ \text{s.t.} \quad & (x, u) \in M_{ad}(y) = X \times V, \end{aligned}$$

regularization (RdmP):

$$\begin{aligned} \min_{(x,u) \in X \times V} \quad & \mathcal{S}(C(u), y^\delta) + \mathcal{Q}(A(x, u)) + \alpha \cdot \mathcal{R}(x, u) \\ \text{s.t.} \quad & (x, u) \in M_{ad}^\delta(y^\delta) = \{(x, u) \in X \times V : \tilde{\mathcal{R}}(x, u) \leq \rho\}. \end{aligned}$$

where $\mathcal{S} : Y \times Y \rightarrow \overline{\mathbb{R}}$, $\mathcal{Q} : W \rightarrow \overline{\mathbb{R}}$ are positive definite functionals

$$\forall y_1, y_2 \in Y : \quad \mathcal{S}(y_1, y_2) \geq 0 \quad \text{and} \quad (y_1 = y_2 \Leftrightarrow \mathcal{S}(y_1, y_2) = 0),$$

$$\forall w \in W : \quad \mathcal{Q}(w) \geq 0 \quad \text{and} \quad (w = 0 \Leftrightarrow \mathcal{Q}(w) = 0).$$

Regularization with constraint on data misfit

inverse problem (IP):

$$\begin{aligned} \min_{(x,u) \in X \times V} \quad & Q(A(x, u)) \\ \text{s.t.} \quad & (x, u) \in M_{ad}(y) = \{(x, u) \in X \times V : C(u) = y\}, \end{aligned}$$

regularization (RdmC):

$$\begin{aligned} \min_{(x,u) \in X \times V} \quad & Q(A(x, u)) + \alpha \cdot \mathcal{R}(x, u) \\ \text{s.t.} \quad & (x, u) \in M_{ad}^\delta(y^\delta) = \{(x, u) \in X \times V : \mathcal{S}(C(u), y^\delta) \leq \tau\delta \\ & \text{and } \tilde{\mathcal{R}}(x, u) \leq \rho\}. \end{aligned}$$

where $\mathcal{S} : Y \times Y \rightarrow \overline{\mathbb{R}}$, $Q : W \rightarrow \overline{\mathbb{R}}$ are positive definite functionals

$$\forall y_1, y_2 \in Y : \quad \mathcal{S}(y_1, y_2) \geq 0 \quad \text{and} \quad (y_1 = y_2 \Leftrightarrow \mathcal{S}(y_1, y_2) = 0),$$

$$\forall w \in W : \quad Q(w) \geq 0 \quad \text{and} \quad (w = 0 \Leftrightarrow Q(w) = 0).$$

Assumptions

$(x^\dagger, u^\dagger) \in X \times V \dots$ exact solution, $y \in Y \dots$ exact data.

- $\mathcal{R}(x^\dagger, u^\dagger) < \infty$ and \mathcal{R} bounded from below.
- $\tilde{\mathcal{R}}(x^\dagger, u^\dagger) \leq \rho$
- a topology \mathcal{T} on $X \times V$ exists such that
 - for all $z \in Y$, $c > 0$, the mapping $X \times V \rightarrow \overline{\mathbb{R}}^4$,
 $(x, u) \mapsto (\mathcal{S}(C(u), z), \mathcal{Q}(A(x, u)), \mathcal{R}(x, u), \tilde{\mathcal{R}}(x, u))$
is \mathcal{T} coercive and component wise \mathcal{T} lower semicontinuous
 - the family of mappings $(\mathcal{S}(z, \cdot) : Y \rightarrow \mathbb{R})_{z \in Z}$ is uniformly continuous on $Z = \{C(u) : \exists x \in X : \tilde{\mathcal{R}}(x, u) \leq \rho\}$ at y , i.e.
 $\lim_{\hat{y} \rightarrow y} \sup_{z \in Z} |\mathcal{S}(z, \hat{y}) - \mathcal{S}(z, y)| = 0$.

Well-definedness and Convergence

$$\mathcal{S}(y, y^\delta) \leq \delta \text{ and } \|y^\delta - y\|_Y \rightarrow 0 \text{ as } \delta \rightarrow 0,$$

Theorem

For each $\alpha > 0$ a minimizer of the regularized problem with data misfit penalization (RdmP) or constraint (RdmC) exists.

Theorem

Choose $\alpha = \alpha(\delta, y^\delta)$ such that
$$\begin{cases} \alpha(\delta, y^\delta) \rightarrow 0 \text{ and } \frac{\delta}{\alpha(\delta, y^\delta)} \leq c \text{ as } \delta \rightarrow 0 \text{ for (RdmP)} \\ \alpha(\delta, y^\delta) \rightarrow 0 & \text{as } \delta \rightarrow 0 \text{ for (RdmC)} \end{cases}$$

Then, as $\delta \rightarrow 0$, $y^\delta \rightarrow y$, the family $(x_{\alpha(\delta, y^\delta)}^\delta, u_{\alpha(\delta, y^\delta)}^\delta)_{\delta \in (0, \bar{\delta}]}$ has a \mathcal{T} convergent subsequence and the limit of every \mathcal{T} convergent subsequence solves (IP). If the solution (x^\dagger, u^\dagger) to (IP) is unique then $(x_{\alpha(\delta, y^\delta)}^\delta, u_{\alpha(\delta, y^\delta)}^\delta) \xrightarrow{\mathcal{T}} (x^\dagger, u^\dagger)$.

The variational approach to EIT

see, e.g., [Kohn&Vogelius'87, Kohn&McKenny'90, Knowles'98]

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Identify spatially distributed conductivity σ in $\Omega \subseteq \mathbb{R}^2$

$$\nabla \cdot J_i = 0, \quad \nabla^\perp \cdot E_i = 0, \quad J_i = \sigma E_i \quad \text{in } \Omega, \quad i = 1, \dots, I,$$

(with $\nabla^\perp \psi = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})^T$ so that $\nabla^\perp \cdot = \text{curl}$)

from observations of boundary currents j_i and voltages v_i .

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Using potentials ϕ_i and ψ_i for J_i and E_i

$$J_i = -\nabla^\perp \psi_i, \quad E_i = -\nabla \phi_i, \quad i = 1, \dots, I,$$

we can rewrite the problem as

$$\sqrt{\sigma} \nabla \phi_i = \frac{1}{\sqrt{\sigma}} \nabla^\perp \psi_i \quad \text{in } \Omega, \quad \psi_i = \gamma_i, \quad \phi_i = v_i \quad \text{on } \partial\Omega, \quad i = 1, \dots, I,$$

where $\gamma_i(x(s)) = -\int_0^s j_i(x(r)) dr$ for $\partial\Omega = \{x(s) : s \in (0, \text{length}(\partial\Omega))\}$.

The variational approach to EIT

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equivalent to

$$\begin{aligned} \min_{\sigma, \underline{\phi}, \underline{\psi}} \quad & \sum_{i=1}^l \frac{1}{2} \int_{\Omega} \left| \sqrt{\sigma} \nabla \phi_i - \frac{1}{\sqrt{\sigma}} \nabla^\perp \psi_i \right|^2 dx \\ \text{s.t.} \quad & \psi_i = \gamma_i, \quad \phi_i = v_i \quad \text{on } \partial\Omega, \quad i = 1, \dots, l \end{aligned}$$

The variational approach to EIT

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equivalent to (since $\int_{\Omega} \nabla \phi_i \cdot \nabla^\perp \psi_i dx = \int_{\partial\Omega} v_i j_i dx$)

$$\begin{aligned} \min_{\sigma, \underline{\phi}, \underline{\psi}} \sum_{i=1}^l \frac{1}{2} \int_{\Omega} \left(\sigma |\nabla \phi_i|^2 + \frac{1}{\sigma} |\nabla^\perp \psi_i|^2 \right) dx \\ \text{s.t. } \psi_i = \gamma_i, \quad \phi_i = v_i \quad \text{on } \partial\Omega, \quad i = 1, \dots, l \end{aligned}$$

Regularized variational EIT

inverse problem (EIT):

$$\min_{\sigma, \underline{\phi}, \underline{\psi}} \sum_{i=1}^l \frac{1}{2} \int_{\Omega} |\sqrt{\sigma} \nabla \phi_i - \frac{1}{\sqrt{\sigma}} \nabla^{\perp} \psi_i|^2 dx$$

s.t. $\psi_i = \gamma_i, \phi_i = v_i$ on $\partial\Omega, \quad i = 1, \dots, l$

regularization (RegEIT):

$$\min_{\sigma, \underline{\phi}, \underline{\psi}} \sum_{i=1}^l \left\{ \frac{1}{2} \int_{\Omega} |\sqrt{\sigma} \nabla \phi_i - \frac{1}{\sqrt{\sigma}} \nabla^{\perp} \psi_i|^2 dx + \frac{\alpha}{2} (\|\phi_i\|_{H^{\frac{3}{2}-\epsilon}(\Omega)}^2 + \|\psi_i\|_{H^{\frac{3}{2}-\epsilon}(\Omega)}^2) \right\}$$

s.t. $\underline{\sigma} \leq \sigma \leq \bar{\sigma}$ on $\Omega,$

$$\left. \begin{aligned} v_i^{\delta} - \tau\delta &\leq \phi_i \leq v_i^{\delta} + \tau\delta, \\ \gamma_i^{\delta} - \tau\delta &\leq \psi_i \leq \gamma_i^{\delta} + \tau\delta, \end{aligned} \right\} \quad \text{on } \partial\Omega, \quad i = 1, \dots, l.$$

\rightsquigarrow special case of regularization with constraint on data misfit
(RdmC)

Regularized variational EIT: Function space setting

$$x = \sigma, \quad u = (\phi_1, \dots, \phi_I, \psi_1, \dots, \psi_I), \quad y = (v_1, \dots, v_I, \gamma_1, \dots, \gamma_I)$$

$$X = L^\infty(\Omega)$$

$$Y = L^\infty(\partial\Omega)^I \times W^{1,1}(\partial\Omega)^I$$

$$V = \{(\phi_1, \dots, \phi_I, \psi_1, \dots, \psi_I) \in H^1(\Omega)^{2I} : \text{tr}_{\partial\Omega}^I(\phi_1, \dots, \phi_I, \psi_1, \dots, \psi_I) \in Y\}$$

$$W = L^2(\Omega)^I$$

$$A(x, u) = \left(\sqrt{\sigma} \nabla \phi_1 - \frac{1}{\sqrt{\sigma}} \nabla^\perp \psi_1, \dots, \sqrt{\sigma} \nabla \phi_I - \frac{1}{\sqrt{\sigma}} \nabla^\perp \psi_I \right),$$

$$C = \text{tr}_{\partial\Omega}^{2I}$$

$$\mathcal{Q}(w) = \frac{1}{2} \|w\|_{L^2(\Omega)^I}^2$$

$$\mathcal{R}(x, u) = \mathcal{R}(u) = \sum_{i=1}^I \left(\|\phi_i\|_{H^{\frac{3}{2}-\epsilon}(\Omega)}^2 + \|\psi_i\|_{H^{\frac{3}{2}-\epsilon}(\Omega)}^2 \right)$$

$$\tilde{\mathcal{R}}(x, u) = \tilde{\mathcal{R}}(x) = \left\| \sigma - \frac{\bar{\sigma} + \sigma}{2} \right\|_{L^\infty(\Omega)}, \quad \rho = \frac{\bar{\sigma} - \sigma}{2}$$

$$\mathcal{S}(y, \tilde{y}) = \max_{i \in \{1, \dots, I\}} \|v_i - \tilde{v}_i\|_{L^\infty(\partial\Omega)} + \|\gamma_i - \tilde{\gamma}_i\|_{L^\infty(\partial\Omega)}$$

Regularized variational EIT: well-definedness, convergence

$$(\sigma_n, \Phi_n, \Psi_n) \xrightarrow{\mathcal{T}} (\sigma, \Phi, \Psi) \Leftrightarrow \begin{cases} \sigma_n \xrightarrow{*} \sigma \text{ and } \frac{1}{\sigma_n} \xrightarrow{*} \frac{1}{\sigma} \text{ in } L^\infty(\Omega), \\ (\Phi_n, \Psi_n) \rightarrow (\Phi, \Psi) \text{ in } H^{2-3\epsilon/2}(\Omega)^{2l}, \\ (\Phi_n, \Psi_n) \rightharpoonup (\Phi, \Psi) \text{ in } H^{3/2-\epsilon}(\Omega)^{2l}, \\ \text{tr}(\Phi_n, \Psi_n) \rightarrow \text{tr}(\Phi, \Psi) \text{ in } L^\infty(\partial\Omega)^{2l} \end{cases}$$

Corollary

For each $y^\delta \in Y$ and $\alpha > 0$ a minimizer of (RegEIT) exists.

Let $\mathcal{S}(y, y^\delta) \leq \delta$ and $\|y^\delta - y\|_Y \rightarrow 0$ as $\delta \rightarrow 0$,

$\underline{\sigma} \leq \sigma^\dagger \leq \bar{\sigma}$ a.e. in Ω

and choose $\alpha = \alpha(\delta, y^\delta)$ such that $\alpha(\delta, y^\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Then, as $\delta \rightarrow 0$, $y^\delta \rightarrow y$, the family

$(\sigma_{\alpha(\delta, y^\delta)}^\delta, \Phi_{\alpha(\delta, y^\delta)}^\delta, \Psi_{\alpha(\delta, y^\delta)}^\delta)_{\delta \in (0, \bar{\delta}]}$ has a \mathcal{T} convergent subsequence

and the limit of every \mathcal{T} convergent subsequence solves (EIT).

Conclusions and Outlook

- reduced versus all-at-once formulations:
 - Tikhonov:
reduced \sim all-at-once
 - Newton:
reduced: solve nonlinear and linear models in each step
all-at-once: only solve linearized models
 - Landweber:
reduced: solve nonlinear and linear models in each step
all-at-once: never solve models!

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 - Landweber:
reduced: solve nonlinear and linear models in each step
all-at-once: never solve models!
- minimization based formulations:
 - generalizes reduced and all-at-once formulations
 - regularization by penalization and/or constraints
 - comprises variational approach to EIT



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B. Kaltenbacher.

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Thank you for your attention!