Simultaneous Identification of Source, Initial Conditions and Asynchronous Sources in the Vibration Problem of Euler-Bernoulli Beams

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Abstract

In this article we show under what conditions it is possible to uniquely identify simultaneously the source and initial conditions in a vibrating Euler-Bernoulli beam, when the available data is the observation of the displacement of a point during an arbitrary small interval of time. A counterexample is also shown to indicate that if some conditions are not satisfied then the unique identification is impossible.

The equation

The equation that appears in this work is the Euler-Bernoulli equation that describes the motion of an elastic beam under dynamic loading.

The problem

$$\begin{cases} \rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = \sum_{j=1}^{J} g_j \otimes f_j, & \text{in }]0, T_0[\times]0, L[, \\ w(0) = w_0, & \text{in }]0, L[, \\ \frac{\partial w}{\partial t}(0, x) = v_0, & \text{in }]0, L[, \\ w(t, \xi) = \frac{\partial w}{\partial x}(t, \xi) = 0, & \forall t \in [0, T_0[, \ \forall \xi \in \{0, L\}, \end{cases} \tag{1}$$

where $\rho \in \mathcal{C}^\infty([0, L])$, $\rho > 0$, is the mass density, $EI \in \mathcal{C}^\infty([0, L])$, $EI > 0$, is the rigidity, $\{g_1, g_2, \ldots, g_J\} \subset \mathcal{C}^J[0, T_0[$ is such that

$$[G(0)] = \begin{bmatrix} 1 & g_1(0) & \cdots & g_J(0) \\ 0 & g_1'(0) & \cdots & g_J'(0) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & g_1^{(J)}(0) & \cdots & g_J^{(J)}(0) \end{bmatrix} \tag{2}$$

is invertible. The set of functions $\{f_1, \ldots, f_J\} \subset \mathrm{H}^{-2}(]0, L[)$ describe the spatial loading imposed to the beam and w is the displacement.

The problem

We will prove that if the initial velocity is known, the force spatial distribution $\{f_1, \dots, f_J\} \subset \mathrm{H}^{-2}(]0, L[)$ and the initial position can be simultaneously identified uniquely given the knowledge of the set

$$\Gamma = \{(w(t,x) : (t,x) \in [0,T] \times \Omega_0\}, \tag{3}$$

where $0 < T < T_0$ and $\Omega_0 \subset [0, L]$, non empty open set, can be arbitrarily small. Furthermore, the initial velocity $v_0 \in \mathrm{L}^2(]0, L[)$ can also be uniquely identified along with the initial position $w_0 \in \mathrm{L}^2(]0, L[)$ and the forcing terms $\{f_1, \dots, f_J\}$ if it is also available the final velocity (knowledge of the displacement profile is not necessary) of the beam at T_0 and the data (3).

Counter example

Let $u(t,x) = h(t)\varphi(x)$. Then it automatically satisfies

$$\rho\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) = \underbrace{\frac{\partial^2 h}{\partial t^2}\rho\varphi}_{g_1 f_1} + \underbrace{h\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \varphi}{\partial x^2}\right)}_{g_2 f_2}.$$

The initial conditions are

$$\begin{cases} u(0,x) = h(0)\varphi(x), & \forall x \in]0,L], \\ u_t(0,x) = h'(0)\varphi(x), & \forall x \in]0,L]. \end{cases}$$

Consider a situation in which $h(0) \neq 0$, $h'(0) \neq 0$, $\varphi \not\equiv 0$, but $\varphi|_{\Omega_0} = 0$. In this case, the forcing terms f_1, f_2 and the initial conditions are not null, but $w|_{[0,T]\times\Omega_0} = 0$. That is, the data (3) is insufficient to fix uniquely the loading $\{f_1, f_2\}$.

However, we are going to see that if the initial position is null, the set of functions g_1, g_2 satisfy a certain condition and $w|_{[0,T]\times\Omega_0} = 0$, then necessarily $f_1 = 0$ and $f_2 = 0$.

Solution of the direct problem

Associated eigenproblem

Consider the eigenvalue problem for $S_n \in H = \mathrm{H}_0^2(]0, L[)$:

$$\begin{cases} \frac{1}{\rho}\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 S}{\partial x^2}\right) = \lambda_n S, & \text{in }]0, L[, \\ S(0) = S(L) = S'(0) = S'(L) = 0. \end{cases} \tag{4}$$

Solution of the direct problem

Associated eigenproblem

With respect to the internal product,

$$\langle \phi_1, \phi_2 \rangle_H = \left(\frac{EI}{\rho} \frac{\partial^2 \phi_1}{\partial x^2}, \frac{\partial^2 \phi_2}{\partial x^2} \right)_{\mathrm{L}^2_\rho(0,L)}, \tag{5}$$

where

$$(\phi_1, \phi_2)_{\mathrm{L}^2_\rho(0,L)} = \int_0^L \rho(x)\phi_1(x)\phi_2(x)\,\mathrm{d}x,$$

the operator $\phi \overset{T}{\mapsto} \frac{1}{\rho}\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \phi}{\partial x^2}\right)$ is self adjoint. Then the set of eigenvectors of this problem forms an enumerable orthonormal basis $(S_n)_{n\in\mathbb{N}}$ of H that is also orthogonal in $\mathrm{L}^2_\rho(0, L)$. Furthermore, $\mathcal{O}(\lambda_n) = n^4$, $S_n \in \mathcal{C}^\infty([0, L])$, $\lambda_n > 0$ and

$$(S_n, S_n)_{\mathrm{L}^2_\rho(0,L)} = 1/\lambda_n, \quad \forall n \in \mathbb{N}. \tag{6}$$

Solution of the direct problem

Elements of the dual of H

Any $Q \in H^*$, can be expressed as

$$Q = \sum_{n\in\mathbb{N}} \beta_n \lambda_n S_n, \tag{7}$$

where $(\beta_n)_{n\in\mathbb{N}} \in \ell^2$. In fact, by Riez Theorem there is a $\sum_{n\in\mathbb{N}} \beta_n S_n \in H$, with $(\beta_n)_{n\in\mathbb{N}} \in \ell^2$, such that

$$Q(\phi) = \langle \phi\,, \sum_{n\in\mathbb{N}} \beta_n S_n \rangle_H = (\phi, \sum_{n\in\mathbb{N}} \lambda_n \beta_n S_n)_{\mathrm{L}^2_\rho}.$$

Then for any $Q \in H^*$ there is a sequence $(\beta_n)_{n\in\mathbb{N}} \in \ell^2$ such that $Q = \sum_{n\in\mathbb{N}} \lambda_n \beta_n S_n$.

Solution of the direct problem

Elements of the dual of H

Any component of the spatial force distribution $f_j \in H^*$, $j \in \{1, \dots J\}$ can be expressed as

$$\frac{f_j}{\rho} = \sum_{n \in \mathbb{N}} A_{j,n} \lambda_n S_n,$$

for $(A_{j,n}) \in \ell^2$.

Solution of the direct problem

Elements of the dual of H

The initial position $w_0 \in \mathrm{L}^2(]0, L[)$ and the initial velocity $v_0 \in \mathrm{L}^2(]0, L[)$ are represented respectively by

$$w_0 = \sum_{n \in \mathbb{N}} W_n \sqrt{\lambda_n} S_n, \;\; v_0 = \sum_{n \in \mathbb{N}} V_n \sqrt{\lambda_n} S_n,$$

where $W_n = (w_0, \sqrt{\lambda_n} S_n)_{\mathrm{L}^2_\rho}$. The expression for $(V_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ is analogous.

Solution of the direct problem

The solution

Using the Galerkin Method, we get a formal solution of (1) given by

$$
\begin{aligned}
w(t,x) = {} & \sum_{n\in\mathbb{N}} V_n \sin(\sqrt{\lambda_n} t) S_n(x) + \sum_{n\in\mathbb{N}} W_n \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} t) S_n(x) \\
& + \sum_{j=1}^{J} \int_0^t g_j(t-\tau) \sum_{n\in\mathbb{N}} A_{j,n} \sqrt{\lambda_n} \sin(\sqrt{\lambda_n}\tau)\, S_n(x)\, \mathrm{d}\tau.
\end{aligned}
\tag{8}
$$

By substitution we can see that (8) is a solution of the first equation of problem (1) in the sense of distributions. Note that al initial and boundary conditions are satisfied.

Solution of the direct problem

Uniqueness for the direct problem

From the fact that $(S_n)_{n\in\mathbb{N}}$ forms an orthonormal basis in H and $\| S_n \|^2_{\mathrm{L}^2_\rho} = 1/\lambda_n$, $\forall n \in \mathbb{N}$, we obtain the following proposition.

Proposition

$w \in \mathcal{C}([0, T_0], \mathrm{H}^2(]0, L[)) \cap \mathcal{C}^1([0, T_0], H^*)$. *Besides,* $w(0), w'(0) \in \mathrm{L}^2_\rho(0, L)$.

Using a method analogous to the energy method applied to the wave equation found, for example, in [Evans(1991)], we can see that (1) admits at most one solution in $\mathcal{C}([0, T_0], \mathrm{H}^2(]0, L[)) \cap \mathcal{C}^1([0, T_0], H^*)$.

Preparation for the solution of the inverse problem

Rewritting the solution of the direct problem

The solution (8) can be rewritten in another form as

$$w(t,x) = \sum_{n\in\mathbb{N}} W_n \sqrt{\lambda_n} S_n(x)$$
$$+ \int_0^t 1 \times \Big(\sum_{n\in\mathbb{N}} V_n \sqrt{\lambda_n} \cos(\sqrt{\lambda_n}\tau) S_n(x) - W_n \lambda_n \sin(\sqrt{\lambda_n}\tau) S_n(x)\Big) \, \mathrm{d}\tau$$
$$+ \sum_{j=1}^{J} \int_0^t g_j(t-\tau) \sum_{n\in\mathbb{N}} A_{j,n} \sqrt{\lambda_n} \sin(\sqrt{\lambda_n}\tau) \, S_n(x) \, \mathrm{d}\tau.$$

Preparation for the solution of the inverse problem

Rewritting the solution of the direct problem

Defining

$$\begin{aligned}
F_{0,V}(t,x) &= \sum_{n\in\mathbb{N}} V_n \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} t) S_n(x), \\
F_{0,W}(t,x) &= \sum_{n\in\mathbb{N}} W_n \, \lambda_n \sin(\sqrt{\lambda_n} t) S_n(x), \\
F_0(t,x) &= F_{0,V}(t,x) + F_{0,W}(t,x), \\
F_j(t,x) &= \sum_{n\in\mathbb{N}} A_{j,n} \sqrt{\lambda_n} \sin(\sqrt{\lambda_n}\tau) \, S_n(x), \, j \in \{1,\dots,J\}
\end{aligned}$$

The last equation can be put in the final form

$$\begin{aligned}
w(t,x) = &\sum_{n\in\mathbb{N}} W_n \, \sqrt{\lambda_n} S_n(x) \\
&+ \int_0^t 1 \times F_0(\tau,x) \, \mathrm{d}\tau + \sum_{j=1}^{J} \int_0^t g_j(t-\tau) F_j(t,x) \, \mathrm{d}\tau.
\end{aligned} \tag{9}$$

Preparation for the solution of the inverse problem

Sequences

Before the next lemma we recall that a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ is uniformly discrete if there is $\delta > 0$ such that $|\lambda_n - \lambda_m| \geq \delta$, for every $m, m \in \mathbb{N}$ with $m \neq n$.

Preparation for the solution of the inverse problem

Paley-Wiener space

Given a bounded set $S \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with positive measure, the Paley-Wiener space PW_S is defined as

$$PW_S = \left\{ \hat{F} \, : \, F \in \mathrm{L}^2; \mathrm{supp}(F) \subset S \right\}.$$

Here we are interested only in the case when $d = 1$.

Preparation for the solution of the inverse problem

Sequences

Definition

The upper uniform density of a uniformly discrete set Λ *is defined by*

$$D(\Lambda) = \lim_{c \to +\infty} \max_{a \in \mathbb{R}} \frac{\#(\Lambda \cap]a, a+c[)}{c}.$$

Definition

An indexed set $\Lambda \doteq (\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ *is an interpolation set for* PW_S, $S \subset \mathbb{R}$ *bounded with positive measure, if for every sequence* $(c_n)_{n \in \mathbb{N}} \subset \ell^2(\mathbb{N})$ *there is* $\phi \in PW_S$ *such that* $\phi(\lambda_n) = c_n$, $\forall \lambda_n \in \Lambda$.

In the case of an interval $]a_1, a_2[\subset \mathbb{R}$, Kahane [Kahane(1957)] (see also [Olevskii and Ulanovskii(2009)]) proved that

$$D(\Lambda) < \frac{1}{2\pi}(a_2 - a_1) \Rightarrow \Lambda \text{ is an interpolating set of } PW_{]a_1, a_2[}. \tag{10}$$

Preparation for the solution of the inverse problem

A fundamental lemma: Statement

Now a lemma directly related to the result we are seeking:

Lemma

Consider $F(t,x) = \sum_{n\in\mathbb{N}} A_n \mathrm{e}^{-i\sqrt{\lambda_n}t} S_n(x)$, *with* $(A_n)_{n\in\mathbb{N}} \subset \ell^2$. *The sequences* $(\lambda_n)_{n\in\mathbb{N}}$ *and* $(S_n)_{n\in\mathbb{N}}$ *are as described above.*
If there are $T \in]0, T_0[$ *and* $\Omega_0 \subset [0, L]$ *such that* $F(t,x) = 0$ *in* $]0, T[\times\Omega_0$, *then* $F \equiv 0$.

Preparation for the solution of the inverse problem

A fundamental lemma: Proof

Observe that

$$\langle F(\cdot, x)\,,\, \hat{\hat{\varphi}} \rangle = \sum_{n\in\mathbb{N}} A_n S_n(x) \hat{\varphi}(\sqrt{\lambda_n}),\ \forall \varphi \in \mathcal{C}_c^\infty(]0, T_0[).$$

Note that if $\varphi \in \mathcal{C}_c^\infty(]0, T[)$ then $\hat{\varphi} \in PW_{]0,T[}$.
Now, since $D((\sqrt{\lambda_n})_{n\in\mathbb{N}}) = 0$, from the density of $\mathcal{C}_c^\infty(]0, T[)$ in $PW_{]0,T[}$, we obtain that $A_n S_n(x) = 0$, $\forall n \in \mathbb{N}$. But since $(S_n)_{n\in\mathbb{N}}$ is a Hilbert basis, for any $n \in \mathbb{N}$, there is always $\phi \in \mathcal{C}_c^\infty(\Omega_0)$ such that $S_n(\phi) \neq 0$. Therefore $A_n = 0$, $\forall n \in \mathbb{N}$, and $F \equiv 0$.

$\blacksquare$

Preparation for the solution of the inverse problem

Non analyticity

Observation

If we test $F(t,x) = \sum_{n\in\mathbb{N}} A_n \mathrm{e}^{-i\sqrt{\lambda_n}t} S_n(x)$*, as in Lemma (2) against* $\phi \in \mathcal{C}_c^\infty(\Omega_0)$*, we get a function* $t \mapsto \tilde{F}(t) \doteq \langle F(t,\cdot)\,,\,\phi\rangle$*, which is not a real analytic function in general. This can be seen as we do the following calculation using integration by parts.*

$$\int_0^t h''(t-\tau)\tilde{F}(\tau)\,\mathrm{d}\tau - \int_0^t h(t-\tau)\tilde{F}''(\tau)\,\mathrm{d}\tau \\ + h(0)\tilde{F}'(t) + h'(0)\tilde{F}(t) = \tilde{F}'(0)h(t).$$

Inverse problem

The main theorem follows.

Inverse problem

The main theorem: statement

Theorem

Let $J \in \mathbb{N}$, $\{g_j : j = 1, \ldots, g_J\} \subset \mathcal{E}'([0, +\infty)) \cap \mathcal{C}^J([0, T[)$. Suppose that the matrix

$$[G(0)] = \begin{bmatrix} g_1(0) & \cdots & g_J(0) \\ g_1'(0) & \cdots & g_J'(0) \\ \vdots & \vdots & \vdots \\ g_1^{(J-1)}(0) & \cdots & g_J^{(J-1)}(0) \end{bmatrix}$$

is invertible.

Inverse problem

The main theorem: statement

Theorem (Cont.)

Suppose also that $F_j \in \mathcal{C}^\infty([0,+\infty),\mathcal{S}')$ *satisfies* $F_j(0,x)=0$, $\forall x \in [0,L]$, $\forall j \in \{1,\dots,J\}$.
If

$$w(t,x) = C(x) + \sum_{j=1}^{J} \int_0^t g_j(t-\tau)\, F_j(\tau,x)\,\mathrm{d}t, \quad \forall (t,x) \in]0,T[\times\Omega_0, \tag{11}$$

where C *is a distribution that does not depend on* t, *is the solution of* (1), *then the existence of* $T \in]0,T_0[$ *and* $\Omega_0 \in]0,L[$ *such that* $w|_{]0,T[\times\Omega_0} = 0$ *implies* $F_j = 0$ *in* $]0,T[\times\Omega_0$, $\forall j \in \{1,\dots,J\}$.

Inverse problem

The main theorem: Proof

We can derivate (11) it with respect to t to obtain

$$\sum_{j=1}^{J} g_j(0) F_j(t,x) + \sum_{j=1}^{J} \int_0^t g_j'(t-\tau)\, F_j(\tau,x)\, \mathrm{d}\tau = 0, \quad \forall t \in]0,T[, \forall x \in \Omega_0.$$

Now we test both sides of the last equation, with respect to the spatial variable x, with $\hat{\phi} \in \mathcal{C}_c^\infty(\Omega_0)$, $\phi \in PW_{\Omega_0}$. We obtain

$$\left\langle \sum_{j=1}^{J} g_j(0) \hat{F}_j(t,\cdot) + \int_0^t g_j'(t-\tau)\, \hat{F}_j(\tau,\cdot)\, \mathrm{d}\tau \,,\, \phi \right\rangle = 0, \quad \forall t \in]0,T[. \quad (12)$$

Inverse problem

The main theorem: Proof

Derivating (12) with respect to the time variable t and using the elementary fact that

$$\hat{F}'(t,\xi) = \int_0^t \hat{F}''(\tau,\xi)\,\mathrm{d}\tau + \hat{F}'(0,\xi), \tag{13}$$

we obtain that $\forall t \in]0,T[$,

$$\begin{aligned} &\left\langle \sum_{j=1}^{J} g_j'(0)\hat{F}_j(t,\cdot) + \sum_{j=1}^{J} \int_0^t \left[g_j(0)\xi^4 + g_j''(t-\tau)\right] \hat{F}_j(\tau,\cdot)\,\mathrm{d}\tau \,,\, \phi \right\rangle \\ &= -\left\langle \sum_{j=1}^{J} g_j(0)\widehat{F_j'}(0,\cdot)\,,\, \phi \right\rangle . \end{aligned} \tag{14}$$

Realizing that the left hand side of this equation is zero for $t = 0$ and the right hand side does not depend on t, we have necessarily that it must be necessarily null.

Inverse problem

The main theorem: Proof

Derivating again the expression just obtained with respect to the time variable and using (13), we obtain for all $t \in]0, T]$,

$$\left\langle \sum_{j=1}^{J} [g_j(0)\xi^4 + g_j''(0)] \hat{F}_j(t,\cdot) + \sum_{j=1}^{J} \int_0^t [g_j'(0)\xi^4 + g_j'''(t-\tau)]\, \hat{F}_j(\tau,\cdot)\, \mathrm{d}\tau \,,\, \phi \right\rangle$$
$$= - \left\langle \sum_{j=1}^{J} g_j'(0) \widehat{F}_j'(0,\cdot) \,,\, \phi \right\rangle . \tag{15}$$

Again, the left hand side $\sum_{j=1}^{J} g_j'(0)\widehat{F}_j'(0,\xi)$ must be null, because it does not depend on t and for $t = 0$ the left hand side is zero.

Inverse problem

The main theorem: Proof

By induction we prove that for all $t \in]0, T]$,

$$\left\langle \sum_{j=1}^{J} [\sum_{m=1}^{\frac{n+1}{2}} \xi^{4(\frac{n+1}{2}-m)} g_j^{(2(m-1))}(0)] \hat{F}_j(t, \cdot) \right. \\ \left. + \int_0^t \sum_{j=1}^{J} \left[g_j^{(n)}(t-\tau) + \sum_{m=1}^{\frac{n-1}{2}} \xi^{4(\frac{n+1}{2}-m)} g_j^{(2m-1)}(0) \right] \hat{F}_j(\tau, \cdot)\, \mathrm{d}\tau, \phi \right\rangle = 0, \tag{16}$$

for n odd,

Inverse problem

The main theorem: Proof

and

$$\Bigg\langle \sum_{j=1}^{J}[\sum_{m=1}^{\frac{n}{2}} \xi^{4(\frac{n}{2}-m)} g_j^{(2m-1)}(0)]\hat{F}_j(t,\cdot) + \int_0^t \sum_{j=1}^{J} \left[g_j^{(n)}(t-\tau) + \sum_{m=0}^{\frac{n}{2}-1} \xi^{4(\frac{n}{2}-m)} g_j^{(2m)}(0) \right] \hat{F}_j(\tau,\cdot)\,\mathrm{d}\tau, \phi \Bigg\rangle = 0, \tag{17}$$

for n even.

Inverse problem

The main theorem: Proof

Based on (16) and (17) we define the matrices

$$[\tilde{G}(0)] = [\tilde{G}_{nj}(0)], \text{ and } [\tilde{\mathcal{G}}(t)] = [\tilde{\mathcal{G}}_{nj}(t)], \tag{18}$$

where

$$\tilde{G}_{nj}(0) = \begin{cases} \sum_{m=1}^{\frac{n+1}{2}} \xi^{4(\frac{n+1}{2}-m)} g_j^{(2(m-1))}(0), & \text{if } n \text{ is odd,} \\ \sum_{m=1}^{\frac{n}{2}} \xi^{4(\frac{n}{2}-m)} g_j^{(2m-1)}(0), & \text{if } n \text{ is even} \end{cases}$$

and

$$\tilde{\mathcal{G}}_{nj}(t) = \begin{cases} g_j^{(n)}(t) + \sum_{m=1}^{\frac{n-1}{2}} \xi^{4(\frac{n+1}{2}-m)} g_j^{(2m-1)}(0), & \text{if } n \text{ is odd,} \\ g_j^{(n)}(t) + \sum_{m=0}^{\frac{n}{2}-1} \xi^{4(\frac{n}{2}-m)} g_j^{(2m)}(0). & \text{if } n \text{ is even} \end{cases}$$

Inverse problem

The main theorem: Proof

We also define

$$[\hat{F}(t,\xi)] = [\hat{F}_1(t,\xi) \cdots \hat{F}_J(t,\xi)]^t, \;\; [\phi(\xi)] = [\phi_1(\xi) \cdots \phi_J(\xi)]^t.$$

In this way, (16) and (17) can be written in matrix form

$$\left\langle [\tilde{G}(0)][\hat{F}(t,\cdot)] + \int_0^t [\mathcal{G}(t-\tau)][\hat{F}(\tau,\cdot)]\,\mathrm{d}\tau \,,\, [\phi] \right\rangle = [0], \, \forall t \in]0,T[. \quad (19)$$

Inverse problem

The main theorem: Proof

Now we recognize the fact that the n-line of the matrix $\tilde{G}(0)$ is the result of replacing the n-line of the matrix

$$[G(0)] = \begin{bmatrix} g_1(0) & \cdots & g_J(0) \\ g_1'(0) & \cdots & g_J'(0) \\ \vdots & \vdots & \vdots \\ g_1^{(J-1)}(0) & \cdots & g_J^{(J-1)}(0) \end{bmatrix}$$

by its n-line added to linear combinations of other lines. Since by hypothesis $\det[G(0)] \neq 0$, we have also $\det[\tilde{G}(0)] \neq 0$. Then, given the existence of its inverse, we can write (19) as a Volterra integral equation of the second kind.

$$\left\langle [\hat{F}(t,\cdot)] + \int_0^t [\tilde{G}(0)]^{-1}[\mathcal{G}(t-\tau)][\hat{F}(\tau,\cdot)]\,\mathrm{d}\tau\,,\ [\phi] \right\rangle = [0],\ \forall t \in]0,T[. \tag{20}$$

Inverse problem

The main theorem: Proof

The conclusion is that $\langle F_j(t,\cdot)\,,\,\hat{\phi}\rangle = 0$, $\forall j \in \{1,\ldots,J\}$, $\forall t \in]0,T[$, $\forall \hat{\phi} \in \mathcal{C}_c^\infty(\Omega_0)$.
Finally, we apply Lemma 2 again to conclude that $F_n \equiv 0$, $\forall n \in \{1,\ldots J\}$.

$\blacksquare$

Inverse problem

Corollary: Statement

Corollary

In problem (1), *if* $v_0 \equiv 0$, *and the matrix* (2) *is invertible, then the initial position* $w_0 \in \mathrm{L}^2(]0, L[)$ *and the force spatial distribution* $\{f_1, \dots, f_J\} \subset \mathrm{H}^{-2}(]0, L[)$ *can be simultaneously identified uniquely given the knowledge of the set* Γ *described in* (3).

Inverse problem

Corollary: Proof

Given $T \in]0, T_0[$ and $\emptyset \neq \Omega_0 \subset]0, L[$, Problem 1 defines a linear operator

$$(w_0, (f_j)_{j=1}^{J}) \mapsto w\big|_{]0,T_0[\times\Omega_0}.$$

To prove that the information contained in Γ uniquely determines w_0 and $(f_j)_{j=1}^{J}$, it suffices to prove that

$$w\big|_{]0,T_0[\times\Omega_0} = 0 \Rightarrow (w_0, (f_j)_{j=1}^{J}) = (0, 0).$$

Inverse problem

Corollary: Proof

Suppose then that

$$w\big|_{]0,T_0[\times\Omega_0} = 0. \tag{21}$$

With the hypothesis posed in this Corollary, all the conditions for the application of Theorem 4 are fulfilled. We conclude then that

$$F_j(t) = 0, \; \forall t \in]0, T_0[, \forall j \in 0, \ldots, J.$$

From Lemma 2 we conclude that the sequences $(A_{n,j})_{n\in\mathbb{N}}$, $(W_n)_{n\in\mathbb{N}}$ are all null, that is, $(w_0, (f_j)_{j=1}^{J}) = (0, 0)$.

■

Inverse problem

Corollary: Statement

If we have also information concerning a final observation, then we can say a little more.

Corollary

In problem (1)*, if the matrix* (2) *is invertible, the initial position* $w_0 \in \mathrm{L}^2(]0, L[)$*, the initial velocity* $v_0 \in \mathrm{L}^2(]0, L[)$ *and the force spatial distribution* $\{f_1, \dots, f_J\} \subset \mathrm{H}^{-2}(]0, L[)$ *can be simultaneously identified uniquely given the knowledge of the set* Γ *described in* (3) *and the measurement of the velocity distribution at* $t = T_0$*.*

It is enough to revert the time arrow. Equation (1) is invariant if t is substituted for $-t$. Then the knowledge of the velocity at $t = T_0$ becomes the new initial velocity. Then the conclusion follows as an immediate consequence of Corollary 6.

Conclusion

The Counter example presented in the beginning is important because it shows a case on which data (3) is not sufficient for the unique determination of the right hand side of equation (1). In fact, over Ω_0 where the data is taken, the displacement is null for every $t \geq 0$. Corollary 6 shows that if the initial condition $v_0 = 0$ is imposed then not only this counter example is excluded but also in this case data (3) enables unique determination of the right hand side of equation (1) together with the initial position $w(0)$. Using reversal of time in Corollary 7, from the final conditions at $t = T_0$, the initial position and velocity are obtained.

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