On the identification of piecewise constant coefficients in optical diffusion tomography by level set.

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Joint work with:

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1. Introduction
2. Inverse Problem
3. Numerical Examples
4. Conclusion

What is Diffuse Optical Tomography (DOT)?

- DOT is a non-invasive technique that utilize light in the near infrared spectral region to measure the optical properties of a physical body.
- The object under study has to be light-transmitting or translucent, so it works best on soft tissues such as breast and brain tissue.
- By monitoring variations in the light absorption and scattering of the tissue, spatial maps of properties such as total hemoglobin concentration, blood oxygen saturation and scattering can be obtained.
- DOT has been applied in breast cancer imaging, brain functional imaging, stroke detection, muscle functional studies, etc.

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The mathematical model

A *simplified* equation to model the light propagation is the following:

$$(DP)\begin{cases} -\nabla\cdot(a(x)\nabla u)+c(x)u=0 & \text{in } \Omega,\\ a(x)\dfrac{\partial u}{\partial \nu}=g & \text{on } \Gamma. \end{cases}$$

- u photon density.
- $a(x)$ diffusion coefficient.
- $c(x)$ absorption coefficient.
- g Neumann boundary data.
- Ω open, bounded and connected with Lipschitz boundary Γ.

Forward map

Parameter-to-measurement (forward) map

$$\begin{aligned} F := F_g : D(F) &\rightarrow H^{1/2}(\Gamma) \\ (a,c) &\mapsto h := u|_{\Gamma}, \end{aligned}$$

- where $u = u(g)$ is the unique solution of (DP) given the boundary data g and the pair (a,c).
- $D(F)$ is the set of piecewise constant functions $(a,c) \in [L^1(\Omega)]^2$ s.t.

$$\underline{a} \leq a(x) \leq \overline{a}, \quad \underline{c} \leq c(x) \leq \overline{c} \quad a.e. \text{ in } \Omega,$$

where $\underline{a}$, $\overline{a}$, $\underline{c}$ and $\overline{c}$ are known non negative real numbers.

Continuity of the forward map

Theorem

For each $g \in H^{-1/2}(\Gamma)$, the corresponding forward map $F_g : D(F) \to H^{1/2}(\Gamma)$ is continuous in the $[L^1(\Omega)]^2-$topology.

The proof is based on a generalization of Meyers' Theorem which prove that the solution u of (DP) belongs to $W^{1,p}(\Omega)$ for some $p > 2$ (therefore better than the standard regularity $u \in H^1(\Omega)$).

Inverse problem

- Since the optical properties within tissue are determined by the values of the diffusion and absorption coefficients, the problem of interest in DOT is the simultaneous identification of both coefficients from measurements of near-infrared diffusive light along the tissue boundary.
- Given a finite number of measurements h_m, corresponding to inputs $g_m = \frac{\partial u_m}{\partial \nu}$.

Find $(a, c) \in D(F)$ such that

$$F_m(a,c) = h_m, \quad \text{for } m = 1, \ldots, M. \tag{1}$$

Inverse problem

- Given the nature of the measurements, we can not expect that exact data h_m are available. Instead, one disposes only an approximate measured data h_m^δ satisfying

$$\left\| h_m - h_m^\delta \right\|_{L^2(\Gamma)} \leq \delta, \quad \text{for } m = 1, \ldots, M$$

where $\delta > 0$ is the noise level.

Find $(a, c) \in D(F)$ such that

$$F_m(a, c) = h_m^\delta, \quad \text{for } m = 1, \ldots, M. \tag{2}$$

Level set approach

- Level set functions $\phi^a, \phi^c \in H^1(\Omega)$ are chosen in such a way that discontinuities of the coefficients (a,c) are located "along" its zero level sets $\Gamma_{\phi^i} := \{x \in \Omega \,|\, \phi^i(x) = 0\}$.
- The diffusion and absorption coefficients can be written as

$$(a,c) \;=\; \big(a^2 + (a^1 - a^2)H(\phi^a),\, c^2 + (c^1 - c^2)H(\phi^c)\big) \;=:\; P(\phi^a, \phi^c)$$

- Inverse problem:

Find $(\phi^a, \phi^c) \in [H^1(\Omega)]^2$ such that

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Level set regularization

- A natural alternative to obtain stable solutions is to use a least-square approach combined with a Tikhonov-type regularization

$$\mathcal{F}_\alpha(\phi^a,\phi^c) := \sum_{m=1}^{M} \|F_m(P(\phi^a,\phi^c)) - h_m^\delta\|^2_{L^2(\Gamma)} + \alpha R(\phi^a,\phi^c) \qquad (3)$$

where

$$R(\phi^a,\phi^c) = \|\phi^a - \phi_0^a\|^2_{H^1(\Omega)} + \|\phi^c - \phi_0^c\|^2_{H^1(\Omega)} + \beta_a |H(\phi^a)|_{\mathrm{BV}(\Omega)} + \beta_c |H(\phi^c)|_{\mathrm{BV}(\Omega)}$$

- $\alpha > 0$ plays the role of a regularization parameter and β_j are scaling factors.
- The $H^1(\Omega)$-terms act as a control on the size of the norm of the level set function (key role to prove existence of minimizers).
- The $\mathrm{BV}(\Omega)$-seminorm terms penalize the length of the Hausdorff measure of the boundary of the sets $\Gamma_0^{\phi^i}$.

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Continuous operator

- In general, variational minimization techniques involve compact embedding arguments and continuity of the operator on the set of admissible functions to guarantee the existence of minimizers.
- We are dealing with the Heaviside operator H and consequently the operator P is discontinuous.
- For each $\varepsilon > 0$, we consider the smooth approximations
  - $H_\varepsilon(t) := \begin{cases} 1 + t/\varepsilon & \text{for } t \in [-\varepsilon, 0] \\ H(t) & \text{for } t \in \mathbb{R} \setminus [-\varepsilon, 0] \end{cases}$
  - $P_\varepsilon(\phi^a, \phi^c) := \left(a^2 + (a^1 - a^2) H_\varepsilon(\phi^a),\ c^2 + (c^1 - c^2) H_\varepsilon(\phi^c)\right)$

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The concept of generalized minimizers

- A **vector** $(\phi^a, \phi^c, z^a, z^c) \in [H^1(\Omega)]^2 \times [L^\infty(\Omega)]^2$ is called **admissible** if there exist sequences $\{\phi_k^j\}$ of H^1-functions and a sequence $\{\varepsilon_k\} \in \mathbb{R}^+$ converging to zero such that

$$\lim_{k\to\infty} \|\phi_k^j - \phi^j\|_{L^2(\Omega)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|H_{\varepsilon_k}(\phi_k^j) - z^j\|_{L^1(\Omega)} = 0 .$$

- A **generalized minimizer** of the functional $\mathcal{F}_\alpha$ in (3) is any admissible vector $(\phi^a, \phi^c, z^a, z^c)$ minimizing

$$\hat{\mathcal{F}}_\alpha(\phi^a, \phi^c, z^a, z^c) := \sum_{m=1}^{M} \|F_m(Q(z^a, z^c)) - h_m^\delta\|_{L^2(\Gamma)}^2 + \alpha\rho(\phi^a, \phi^c, z^a, z^c),$$

  - $Q(z^a, z^c) := (a^2 + (a^1 - a^2)z^a, c^2 + (c^1 - c^2)z^c)$,
  - $\rho(\phi^a, \phi^c, z^a, z^c) := \inf\left\{\liminf_{k\to\infty} \sum_{j=1}^2 \left(\beta_j |H_{\varepsilon_k}(\phi_k^j)|_{BV} + \|\phi_k^j - \phi_0^j\|_{H^1}^2\right)\right\}$

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Convergence Analysis

Theorem

1. **[Well-posedness]** $\hat{\mathcal{F}}_\alpha$ attains minimizers on the set of admissible vectors.
2. **[Convergence for exact data]** Assume that $h^\delta = h$. For every $\alpha > 0$ denote by $(\phi^a_\alpha, \phi^c_\alpha, z^a_\alpha, z^c_\alpha)$ a minimizer of $\hat{\mathcal{F}}_\alpha$. Then, for every sequence of positive numbers $\{\alpha_k\}$ converging to zero there exists a subsequence, denoted again by $\{\alpha_k\}$, such that $(\phi^a_{\alpha_k}, \phi^c_{\alpha_k}, z^a_{\alpha_k}, z^c_{\alpha_k})$ is strongly convergent in $[L^2(\Omega)]^2 \times [L^1(\Omega)]^2$. Moreover, the limit is a solution of (1), that is $F_m(Q(\bar{z}^a, \bar{z}^c)) = h_m$, $m = 1, \dots, M$.
3. **[Convergence for noisy data]** Let $\alpha = \alpha(\delta)$ be a function satisfying

$$\lim_{\delta \to 0} \alpha(\delta) = 0 \quad \text{and} \quad \lim_{\delta \to 0} \delta^2 \alpha(\delta)^{-1} = 0.$$

   Moreover, let $\{\delta_k\}$ be a sequence of positive numbers converging to zero and $\{h^{\delta_k}\}$ be corresponding noisy data. Then, there exists a subsequence, denoted again by $\{\delta_k\}$, and a sequence $\{\alpha_k := \alpha(\delta_k)\}$ such that $(\phi^a_{\alpha_k}, \phi^c_{\alpha_k}, z^a_{\alpha_k}, z^c_{\alpha_k})$ converges in $[L^2(\Omega)]^2 \times [L^1(\Omega)]^2$ to a solution of (1).

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Level set regularization: numerical realization.

- In this case, the energy functional is:

$$\mathcal{F}_{\alpha,\varepsilon}(\phi^a,\phi^c) := \sum_{m=1}^{M} \|F_m(P_\varepsilon(\phi^a,\phi^c)) - h_m^\delta\|^2_{L^2(\Gamma)} + \alpha R_\varepsilon(\phi^a,\phi^c)$$

where

$$R_\varepsilon(\phi^a,\phi^c) = |H_\varepsilon(\phi^a)|_{\mathrm{BV}(\Omega)} + |H_\varepsilon(\phi^c)|_{\mathrm{BV}(\Omega)} + \|\phi^a - \phi_0^a\|^2_{H^1(\Omega)} + \|\phi^c - \phi_0^c\|^2_{H^1(\Omega)}$$

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2. Let α be given. For each $\varepsilon > 0$ denote by $(\phi^a_{\varepsilon,\alpha}, \phi^c_{\varepsilon,\alpha})$ a minimizer of $\mathcal{F}_{\alpha,\varepsilon}$. There exists a sequence of positive numbers $\{\varepsilon_k\}$ converging to zero such that $(\phi^a_{\varepsilon_k,\alpha}, \phi^c_{\varepsilon_k,\alpha}, H_{\varepsilon_k}(\phi^a_{\varepsilon_k,\alpha}), H_{\varepsilon_k}(\phi^c_{\varepsilon_k,\alpha}))$ converges strongly in $[L^2(\Omega)]^2 \times [L^1(\Omega)]^2$ and the limit is a generalized minimizer of $\mathcal{F}_\alpha$.

- Differently from $\mathcal{F}_\alpha$, the minimizers of $\mathcal{F}_{\alpha,\varepsilon}$ can be computed.
- Derive the first order optimality condition for a minimizer of $\mathcal{F}_{\alpha,\varepsilon}$.

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- First order optimality condition: $\frac{\partial \mathcal{F}_{\alpha,\varepsilon}}{\partial \phi^j}(h) = 0 \quad \forall h \in H^1(\Omega)$.

$$\begin{aligned} \alpha(\Delta - I)(\phi^j - \phi^j_0) &= L^j_{\varepsilon,\alpha}(\phi^a, \phi^c) \quad \text{in } \Omega \\ \frac{\partial}{\partial \nu}(\phi^j - \phi^j_0) &= 0 \quad \text{on } \Gamma. \end{aligned}$$

$$\begin{aligned} L^a_{\varepsilon,\alpha}(\phi^a, \phi^c) = & (a^1 - a^2) H'_\varepsilon(\phi^a) \left[\sum_{m=1}^{M} \left(\frac{\partial F_m(P_\varepsilon(\phi^a, \phi^c))}{\partial \phi^a} \right)^* (F_m(P_\varepsilon(\phi^a, \phi^c)) - h^\delta_m) \right] \\ & - \alpha \beta_a \left[H'_\varepsilon(\phi^a) \nabla \cdot \left(\frac{\nabla H_\varepsilon(\phi^a)}{|\nabla H_\varepsilon(\phi^a)|} \right) \right] \end{aligned}$$

Iterative regularization algorithm

1. Evaluate the residual

$$r_{k,m} := F_m(P_\varepsilon(\phi_k^a, \phi_k^c)) - h_m = u_{k,m_{|\Gamma}} - h_m, \qquad m = 1, \dots, M.$$

2. Evaluate $\left(\frac{\partial F_m(P_\varepsilon(\phi_k^a,\phi_k^c))}{\partial \phi_k^a}\right)^*$ and $\left(\frac{\partial F_m(P_\varepsilon(\phi_k^a,\phi_k^c))}{\partial \phi_k^c}\right)^*$ $\quad m = 1, \dots, M.$

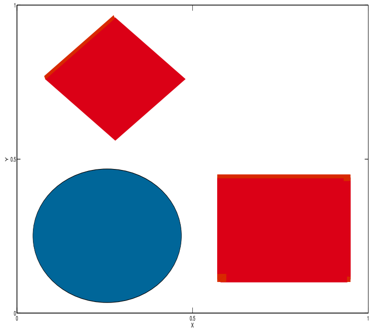
3. Calculate $\delta\phi_k^a$ and $\delta\phi_k^c$ solutions of the BVP

$$\begin{cases} (\Delta - I)\delta\phi_k^i = L_{\varepsilon,\alpha}^i(\phi_k^a, \phi_k^c) & \text{in } \Omega \\ \frac{\partial \delta\phi_k^i}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

4. Update the level set functions

$$\phi_{k+1}^a = \phi_k^a + \delta\phi_k^a \quad \text{and} \quad \phi_{k+1}^c = \phi_k^c + \delta\phi_k^c$$

Numerical Examples



$$a^*(x) = \begin{cases} 10\,, & \text{inside blue inclusion} \\ 1\,, & \text{elsewhere} \end{cases} \quad , \quad c^*(x) = \begin{cases} 10\,, & \text{inside red inclusion} \\ 1\,, & \text{elsewhere.} \end{cases}$$

Numerical Examples

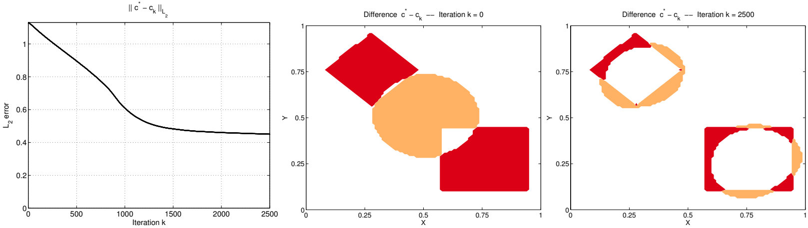
- Four ($M = 4$) distinct functions g_m, each one supported at each side of Γ. For instance,

$$g_1(x) = \begin{cases} 1\,, & if\, x \in (\frac{1}{4}, \frac{3}{4}) \times \{0\} \\ 0\,, & \text{else} \end{cases}$$

- In order to avoid inverse crimes, the direct problem was solved using FEM in an uniform grid with 100 nodes at each boundary side. Alternatively, in the iterative process, all boundary value problems were solved on a uniform grid with 50 nodes at each boundary side.
- In all cases the initial level set function ϕ_0^j was a paraboloid but with different minima.

Identification of the absorption coefficient $c(x)$

a^* is assumed to be exactly known



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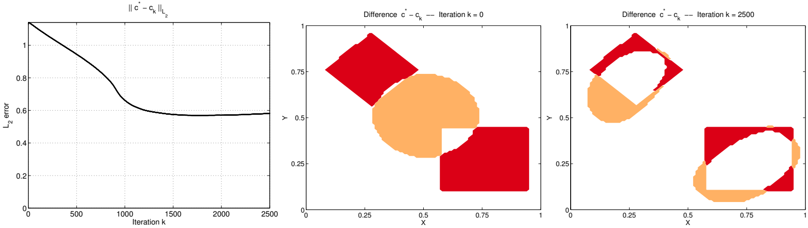
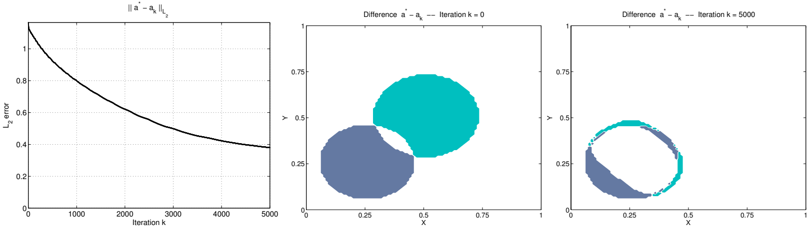


Identification of the diffusion coefficient $a(x)$

c^* is assumed to be exactly known



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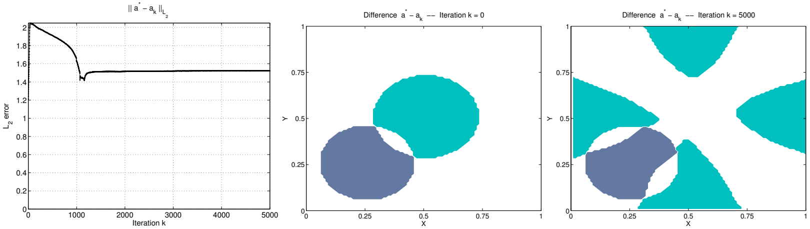

Split strategy

- Some facts to take into account:
  1. The method for identifying c^* performs well, even if a good approximation of a^* is not known.
  2. On the other hand, the method may generate a sequence a^k that does not approximate a^* if $\|c^k - c^*\|$ is large.
  3. For simultaneous identification of (a^*, c^*) we observed that the error $\|c^k - c^*\|$ decreases from the very first iteration. However, the error $\|a^k - a^*\|$ only starts improving when $\|c^k - c^*\|$ is sufficiently small.
- Split strategy:
  1. Set $a^k(x) \equiv 1$ and iterate w.r.t. c^k until the sequenece c^k stagnates ($\|c^k - c^*\|$ is small).
  2. Set $c^k(x) \equiv c^{k_1}$ and iterate w.r.t. a^k until the sequenece a^k stagnates ($\|a^k - a^*\|$ is small).
  3. Each iteration step consist in one iteration w.r.t. c^k and two iterations w.r.t a^k.

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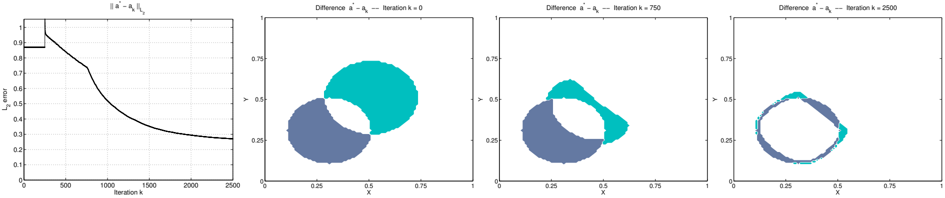
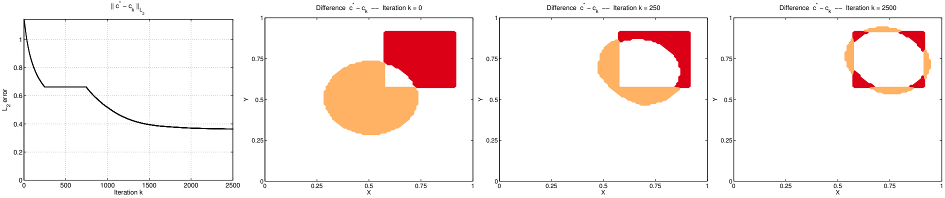
Identification of both coefficients: example 1

Diffusion coefficient $a(x)$

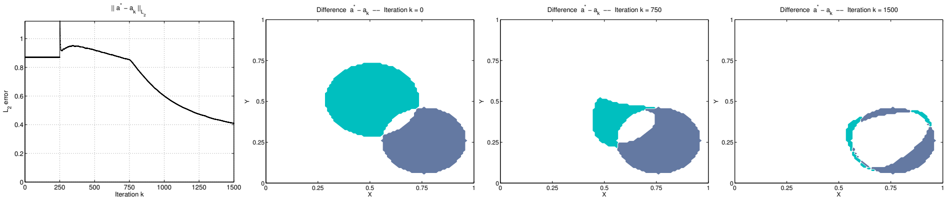


Absorption coefficient $c(x)$

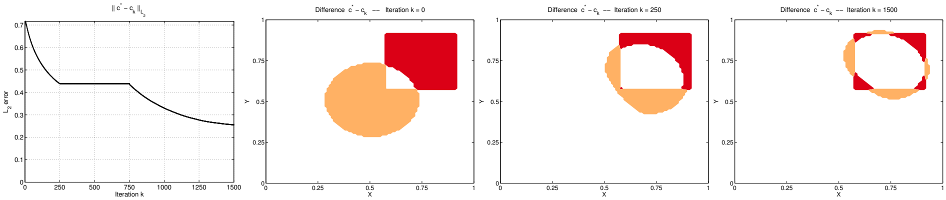


Identification of both coefficients: example 2

Diffusion coefficient $a(x)$



Absorption coefficient $c(x)$



Conclusion

- We developed a level set approach for simultaneous reconstruction of the piecewise constant coefficients (a, c) from a finite set of boundary measurements of optical tomography in the diffusive regime.
- We proved that the forward map F is continuous in the L^1-topology. Hence, by previous results, the presented level set approach is a regularization method.
- We proposed a split strategy for the simultaneous identification of the diffusion and absorption coefficient.
- This numerical strategy has not only demonstrated that gives very good results but also reduces significantly the computational time.

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Thank you for your atention !