Tomographic Terahertz Imaging Using Sequential Subspace Optimization

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Overview

Introduction

Sequential subspace optimization (SESOP)

- Iterative reconstruction technique for inverse problems
- here: SESOP for nonlinear inverse problems in Hilbert spaces
- Extension (acceleration) of Landweber's method
- Regularization technique (RESESOP)
- reduction of the number of iterations

Terahertz tomography

- novel imaging technique in nondestructive testing for plastics and ceramics
- approach via scattering problems: nonlinear inverse problem
- solution of the inverse problem of THz tomography with an adapted RESESOP method

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Notation and definitions

Let X, Y be Hilbert spaces and $M_{F(x)=y} := \{x \in X : F(x) = y\}$ the solution set for a nonlinear operator $F : \mathcal{D}(F) \subseteq X \to Y$.

We assume to have noisy data y^δ with noise level $\delta \geq \|y - y^\delta\|$.

Hyperplanes, halfspaces and stripes

Let $u \in X \setminus \{0\}$ and $\alpha, \xi \in \mathbb{R}$, $\xi \geq 0$. We define the (affine) hyperplane

$$H(u, \alpha) := \{x \in X \ : \ \langle u, x \rangle = \alpha\},$$

the halfspace

$$H_{\leq}(u, \alpha) := \{x \in X \ : \ \langle u, x \rangle \leq \alpha\}$$

and the stripe

$$H(u, \alpha, \xi) := \{x \in X \ : \ |\langle u, x \rangle - \alpha| \leq \xi\}.$$

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SESOP iteration for linear problems $Ax = y$ in Hilbert spaces

$$x_{n+1} := x_n - \sum_{i \in I_n} t_{n,i} A^* w_{n,i},$$

I_n a finite index set, $w_{n,i} \in Y$ for all $i \in I_n$, and the parameters $t_n = (t_{n,i})_{i \in I_n}$ minimize

$$h_n(t) := \frac{1}{2} \left\| x_n - \sum_{i \in I_n} t_i A^* w_{n,i} \right\|^2 + \sum_{i \in I_n} t_i \left\langle w_{n,i}, y \right\rangle .$$

Lemma [Schöpfer, S., Louis (2008)]

The minimization of $h_n(t)$ is equivalent to computing the metric projection

$$x_{n+1} = P_{H_n}(x_n), \qquad H_n := \bigcap_{i \in I_n} H_{n,i},$$

onto the intersection of hyperplanes

$$H_{n,i} := \left\{ x \in X \;:\; \left\langle A^* w_{n,i}, x \right\rangle = \left\langle w_{n,i}, y \right\rangle \right\} \supseteq M_{Ax=y}.$$

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RESESOP for linear operators and noisy data

Project sequentially onto intersections of stripes $H(u, \alpha, \xi)$, where

$$\xi = \xi(\delta), \quad M_{Ax=y} \subseteq H(u, \alpha, \xi).$$

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SESOP for nonlinear inverse problems $F(x) = y$

We consider nonlinear inverse problems

$$F(x) = y, \quad F : \mathcal{D}(F) \subseteq X \to Y.$$

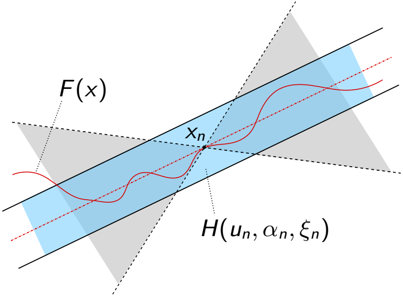
Assume that F satisfies the tangential cone condition

$$\left\|F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x})\right\| \leq c_{\mathrm{tc}} \left\|F(x) - F(\tilde{x})\right\|$$

for $x, \tilde{x} \in B_\rho(x_0)$ and $0 < c_{\mathrm{tc}} < 1$. Furthermore,

$$\sup_{x \in B_\rho(x_0)} \|F'(x)\| \leq c_{\mathrm{F}}.$$

- Use stripes also in case of exact data y.
- Take into account the local character as well as the nonlinearity of operators.



From SESOP for linear problems to SESOP for nonlinear problems

Stripes in the linear case

$$H_{n,i}^{\delta} := \left\{ x \in X : \left| \left\langle A^* w_{n,i}^{\delta}, x \right\rangle - \left\langle w_{n,i}^{\delta}, y^{\delta} \right\rangle \right| \leq \delta \| w_{n,i}^{\delta} \| \right\}$$

Stripes in the nonlinear case

$$H_{n,i}^{\delta} := \left\{ x \in X : \left| \left\langle F'(x_i^{\delta})^* w_{n,i}^{\delta}, x_i^{\delta} - x \right\rangle - \left\langle w_{n,i}^{\delta}, F(x_i^{\delta}) - y^{\delta} \right\rangle \right| \right.$$
$$\left. \leq \| w_{n,i}^{\delta} \| \left(c_{\mathrm{tc}} \left(\| R_i^{\delta} \| + \delta \right) + \delta \right) \right\} \supseteq M_{F(x)=y}$$

RESESOP algorithm

As long as the residual $R_n^{\delta} := F(x_n^{\delta}) - y^{\delta}$ fulfills $\| R_n^{\delta} \| > \tau \delta$, compute

$$x_{n+1}^{\delta} = x_n^{\delta} - \sum_{i \in I_n^{\delta}} t_{n,i}^{\delta} F'(x_i^{\delta})^* w_{n,i}^{\delta} \quad \in \bigcap_{i \in I_n^{\delta}} H(u_{n,i}^{\delta}, \alpha_{n,i}^{\delta}, \xi_{n,i}^{\delta})$$

and $\| z - x_{n+1}^{\delta} \|^2 \leq \| z - x_n^{\delta} \|^2 - C\left(\| R_n^{\delta} \|, \delta, c_{\mathrm{tc}}, c_{\mathrm{F}} \right)$ for $z \in M_{F(x)=y}$.

Convergence of the SESOP algorithm

Let

- $n \in I_n$ for each $n \in \mathbb{N}$,
- $I_n \subseteq \{n - N + 1, ..., n\} \cap \mathbb{N}$, $N \geq 1$ fixed
- $w_{n,i} := R_i = F(x_i) - y$ for each $i \in I_n$,
- $x^+ \in B_{\rho/2}(x_0)$ be the unique solution in $B_\rho(x_0)$.

Theorem [Wald, S. (2016)]

The sequence of iterates $\{x_n\}_{n\in\mathbb{N}}$, generated by the SESOP algorithm, converges strongly to x^+, if the optimization parameters $t_{n,i}$ are bounded,

$$|t_{n,i}| \leq t$$

for some $t > 0$ for all $i \in I_n$ and $n \in \mathbb{N}$.

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Convergence and regularization of the RESESOP algorithm

Theorem [Wald, S. (2016)]

- The RESESOP algorithm, together with the discrepancy principle, where the tolerance parameter satisfies

$$\tau > \frac{1 + c_{\mathrm{tc}}}{1 - c_{\mathrm{tc}}} > 1,$$

yields a finite stopping index $n_*(\delta)$.

- The iterates x_n^δ depend, for a fixed $n \in \mathbb{N}$, continuously on the data y^δ and we have

$$x_n^\delta \to x_n \quad \text{for } \delta \to 0.$$

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- The algorithm yields a regularized solution $x_{n_*(\delta)}^\delta$ of the nonlinear problem $F(x) = y$, if noisy data y^δ are given, i.e.,

$$x_{n_*(\delta)}^\delta \to x^+ \quad \text{for } \delta \to 0,$$

  if there is only one solution $x^+ \in B_\rho(x_0)$ and if $\lim_{\delta \to 0} \left| t_{n,i}^\delta \right| < t$ holds for all $n \in \mathbb{N}$ and $i \in I_n^\delta$ for some $t > 0$.

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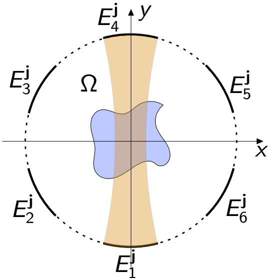
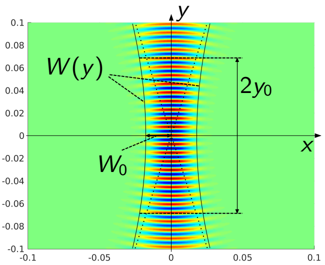
THz tomography - an introduction

- novel imaging technique in nondestructive testing (plastics, ceramics)
- non-ionizing electromagnetic radiation, frequency range 0.1 - 10 THz
- THz radiation: Gaussian beams with wave number $k_0 = {}^{2\pi f}/_{c}$





Inverse problem of THz tomography

Reconstruct the complex refractive index $\tilde{n} = n + i\kappa$ to detect defects, inclusions or the moisture content from measurements of the resulting electric field.

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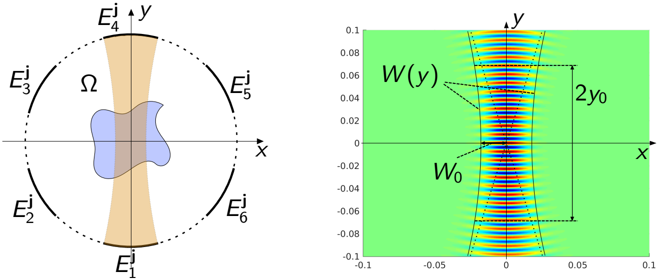

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Scattering of THz radiation

For convenience, define

$$m : L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega) \to \mathbb{C}, \;\; m(\mathbf{x}) := 1 - \tilde{n}^2(\mathbf{x}).$$

By $\Omega \subseteq \mathbb{R}^2$, we denote the domain (with C^1-boundary $\partial\Omega$) in which m is to be reconstructed.

Boundary value problem for u_{t}

Given the incident field u_{i} and m, the total field u_{t} is determined by

$$\begin{aligned} \Delta u_{\mathrm{sc}} + k_0^2(1-m)u_{\mathrm{sc}} &= k_0^2 m u_{\mathrm{i}} \quad \text{in } \Omega, \\ \frac{\partial u_{\mathrm{sc}}}{\partial \mathbf{n}} - ik_0 u_{\mathrm{sc}} &= 0 \quad \text{on } \partial\Omega, \\ u_{\mathrm{sc}} + u_{\mathrm{i}} &= u_{\mathrm{t}} \quad \text{in } \overline{\Omega}, \end{aligned}$$

where $\mathbf{n}$ is the outward normal vector of $\partial\Omega$.

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Existence and uniqueness of a weak solution

For $m \in L^\infty(\Omega)$ with $\mathrm{Im}(m) \leq 0$, consider the boundary value problem

$$\begin{aligned} \Delta u + k_0^2(1-m)u &= k_0^2 m u_\mathrm{i} \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} - ik_0 u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{BVP}$$

and its variational formulation

$$(\nabla u, \nabla v)_{L^2(\Omega)} - k_0^2 \left((1-m)u, v\right)_{L^2(\Omega)} - ik_0(u,v)_{L^2(\partial\Omega)} = -k_0^2(mu_\mathrm{i}, v)_{L^2(\Omega)}.$$

Theorem [Wald, S. (2017)]

The boundary value problem (BVP) possesses a unique weak solution $u \in H^1(\Omega)$, which satisfies

$$\|u\|_{H^1(\Omega)} \leq C_1 \|m\|_{L^\infty(\Omega)} \|u_\mathrm{i}\|_{L^2(\Omega)}$$

for a constant $C_1 > 0$.

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The scattering operator S

Definition of S

Let $u_{\rm i} \in H^1(\Omega)$ and let

$$S : \mathcal{D}(S) \to H^1(\Omega),\ m \mapsto u_{\rm t} = u_{\rm i} + u_{\rm sc},$$

where

$$\mathcal{D}(S) \subseteq \left\{ m \in L^\infty(\Omega) \cap L^2_{\rm comp}(\Omega) : \|m\|_{L^\infty(\Omega)} \leq M,\ \mathrm{Im}(m) \leq 0 \right\}$$

for a fixed $M > 0$ and $u_{\rm sc}$ is the weak solution of (BVP).

Lemma (continuity of S) [Wald, S. (2017)]

The operator S is well-defined and Lipschitz-continuous on $\mathcal{D}(S)$, more precisely

$$\|S(m_1) - S(m_2)\|_{H^1(\Omega)} \leq C_2 \|m_1 - m_2\|_{L^\infty(\Omega)} \|u_{\rm i}\|_{L^2(\Omega)},$$

where $C_2 = C_2(k_0, \Omega) > 0$.

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Properties of the scattering operator S

Theorem (Fréchet differentiability) [Wald, S. (2017)]

The operator S is Fréchet differentiable w.r.t. $m \in \mathcal{D}(S)$ with continuous Fréchet derivative

$$S'(m) : \mathcal{D}(S) \to H^1(\Omega), \; h \mapsto S'(m)h = w,$$

where $w \in H^1(\Omega)$ solves the boundary value problem

$$\Delta w + k_0^2(1-m)w = k_0^2 S(m) \cdot h \quad \text{in } \Omega,$$
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Lemma (tangential cone condition) [Wald, S. (2017)]

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The forward operator F

Tomographic measurements

The tomograph (emitter and receivers) are rotated on a circle around the object m, i.e.,

the incident field u_{i}^{j} depends on the position $j \in \{1, \ldots, J\}$ of the tomograph.

The same holds for the scattered and total field and the scattering map S^j.

The (nonlinear) forward operator F^j

Let

$$F = \left(F^1, \ldots, F^J\right) \, : \, \mathcal{D}(S) \to \mathbb{C}^{N \times J},$$

where $F^j(m) := Q^j \gamma S^j(m)$ and

$$\gamma : H^1(\Omega) \to L^2(\partial\Omega), \; u \mapsto u|_{\partial\Omega}$$

denotes the *trace operator* and the linear operator $Q^j \, : \, L^2(\partial\Omega) \to \mathbb{C}^N$,

$$y_\nu^j = (Q^j v)_\nu = \int_{\partial\Omega} \chi_{E_\nu^j}(\mathbf{x}) v(\mathbf{x}) \, \mathrm{d}s_{\mathbf{x}}, \quad \nu = 1, \ldots, N$$

describes the *measuring process*.

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Evaluation of the gradient g

The inverse problem of 2D THz tomography

Reconstruct $m : \Omega \to \mathbb{C}$ from measurements $y^{j,\delta} \in \mathbb{C}^N$ of the electric field, where

$$F^j(m) = y^j, \quad \|y^{j,\delta} - y^j\| \leq \delta, \quad j = 1, ..., J.$$

(RE)SESOP and Landweber method

We use the gradient

$$g^j(m) := \big((F^j)'(m)\big)^*(F^j(m) - y^\delta)$$

of the functional

$$\Psi^j(m) := \frac{1}{2}\left\|F^j(m) - y^\delta\right\|^2$$

as search directions.

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The linearity of γ and Q^j yields

- $(F^j)'(m) = Q^j\gamma(S^j)'(m)$,
- $(F^j)'(m)^* = \left(\gamma(S^j)'(m)\right)^*(Q^j)^*$.

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(RE)SESOP and Landweber method

We use the gradient

$$g^j(m) := \big((F^j)'(m)\big)^*(F^j(m) - y^\delta)$$

of the functional

$$\Psi^j(m) := \frac{1}{2}\left\|F^j(m) - y^\delta\right\|^2$$

as search directions.

The linearity of γ and Q^j yields

- $(F^j)'(m) = Q^j\gamma(S^j)'(m)$,
- $(F^j)'(m)^* = \big(\gamma(S^j)'(m)\big)^*(Q^j)^*$.

Adjoint operators

Theorem [Wald, S. (2017)]

For $R \in L^2(\partial\Omega)^*$ there exists a $\phi \in H^1(\Omega)$ such that

$$\left(\gamma S'(m)\right)^* R = -k_0^2 \overline{S(m)} \phi$$

holds, where

$$\Delta\phi + k_0^2(1 - \overline{m})\phi = 0 \quad \text{in } \Omega,$$
$$\frac{\partial\phi}{\partial\mathbf{n}} + ik_0\phi = -R \quad \text{on } \partial\Omega.$$

For $\beta = (\beta_\nu)_{\nu=1,\ldots,N} \in \mathbb{C}^N$, we thus have

$$\left((F^j)'(m)\right)^* \beta = \left(\gamma (S^j)'(m)\right)^* (Q^j)^* \beta = -k_0^2 \overline{S^j(m)} \phi,$$

where

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Numerical experiments - some adjustments

Landweber method

$$m_{n+1}^\delta = m_n^\delta - \omega F'(m_n^\delta)^*(F(m_n^\delta) - y^\delta) = m_n^\delta - \omega g_n^\delta$$

Two adaptions

(1) Use averaged gradients

$$g_n^\delta := \frac{1}{J}\sum_{j=1}^{J} g_n^{j,\delta} = \frac{1}{J}\sum_{j=1}^{J}\left((F^j)'(m_n^\delta)\right)^*\left(F^j(m_n^\delta) - y^{j,\delta}\right).$$

(2) Adapt (RE)SESOP method to solve nonlinear inverse problems in complex Hilbert spaces.

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Landweber method

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RESESOP iteration for complex-valued problems

$$m_{n+1}^{\delta} = \left(\mathrm{Re}(m_n^{\delta}) - \sum_{l\in I_n^{\delta}} t_{n,l}^{\delta,\mathrm{r}} \cdot \mathrm{Re}(g_l^{\delta})\right) + i\cdot\left(\mathrm{Im}(m_n^{\delta}) - \sum_{l\in I_n^{\delta}} t_{n,l}^{\delta,\mathrm{i}} \cdot \mathrm{Im}(g_l^{\delta})\right)$$

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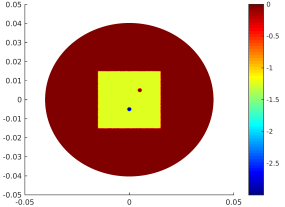
An example in the Terahertz regime

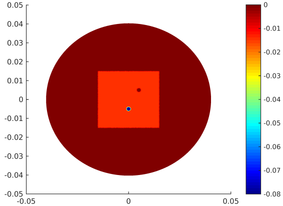
Three different complex refractive indices:

$$m_1 = -1.249975 - i \cdot 0.015, \quad m_2 = 0, \quad m_3 = -2.2396 - i \cdot 0.072$$

Algorithm: RESESOP with two search directions, $I_n^\delta = \{n-1, n\}$



(a) Real part of m



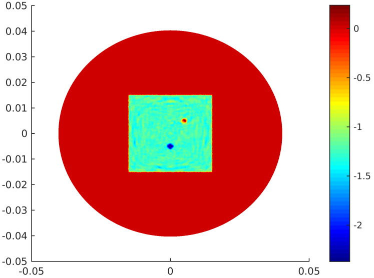
(b) Imaginary part of m

Parameters of the experiment

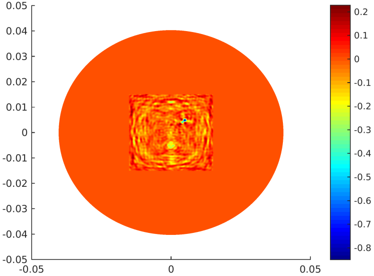
$$f = 0.1 \text{ THz}, \quad N = 40, \quad J = 180, \quad c_{\mathrm{tc}} = 0.9, \quad \tau = 20$$

An example in the Terahertz regime

- Assumption: The outer interfaces of the object are known.
- Noise level δ: 2% noise plus error from implementation



(a) Reconstruction of $\mathrm{Re}(m)$

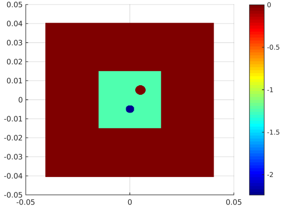


(b) Reconstruction of $\mathrm{Im}(m)$

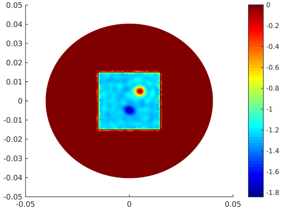
- Relative error in reconstruction of $\mathrm{Re}(m)$: 7.98 %
- Relative error in reconstruction of $\mathrm{Im}(m)$: $>$ 100 %
- $n_* = 30$ iterations in 14h 13min 51s

A comparison of Landweber and RESESOP method for $f = 2.5 \cdot 10^{10}$ Hz

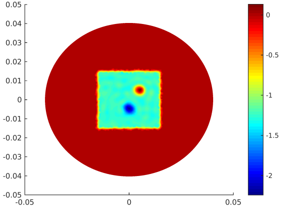
Landweber: 155 iterations in 5h 38min 49s, **RESESOP**: 20 iterations in 1h 12min 40s



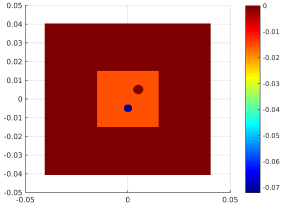
(a) $\mathrm{Re}(m)$



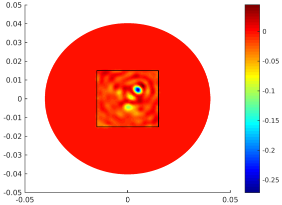
(b) Landweber: $\mathrm{Re}(m^\delta_{n_*})$



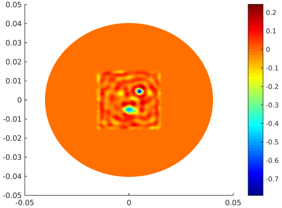
(c) RESESOP: $\mathrm{Re}(m^\delta_{n_*})$



(d) $\mathrm{Im}(m)$



(e) Landweber: $\mathrm{Im}(m^\delta_{n_*})$



(f) RESESOP: $\mathrm{Im}(m^\delta_{n_*})$

Conclusion and outlook

Summary of the results

- extension of SESOP and RESESOP techniques to nonlinear inverse problems in Hilbert spaces
- proof of convergence and regularization results
- modeling and analysis of the inverse problem of THz tomography
- application of SESOP techniques in parameter identification significantly reduces computation time

Future research

- extension of subspace optimization techniques to nonlinear inverse problems in Banach spaces
- development of a hybrid algorithm that combines the modified ART (Tepe et al.) with our scattering approach

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Thank you for your attention!