A New Non-Iterative Reconstruction Method for a Class of Inverse Problems

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Outline

Motivation

$$\inf_{\Omega \in \mathcal{E}} \psi(\Omega)$$

- $\psi(\Omega)$: shape functional
- Ω: geometrical domain
- $\mathcal{E}$: set of admissible domains

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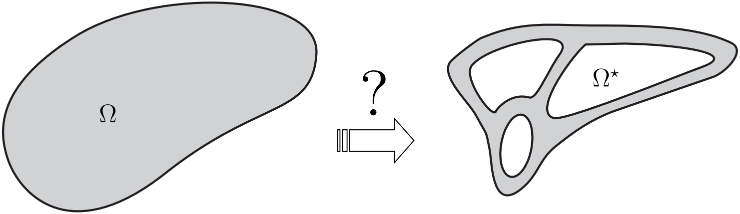
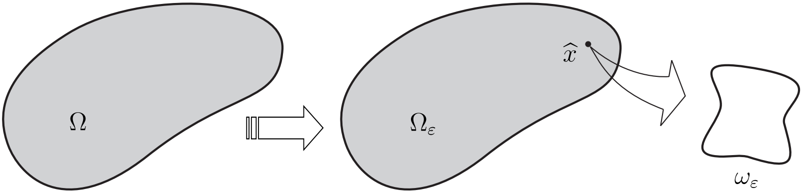
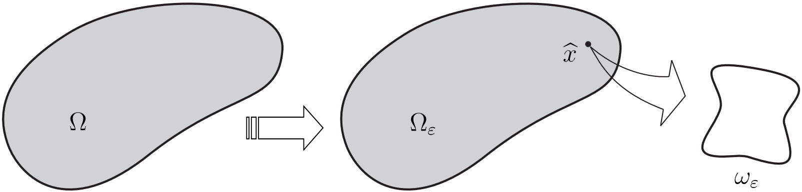


Topological Derivative Concept



Sokolowski & Zochowski, 1999

Topological Derivative Concept

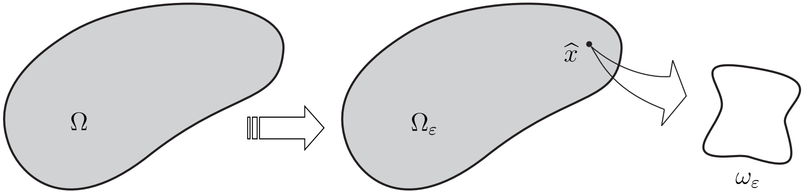


Sokolowski & Zochowski, 1999

$$\psi(\Omega_\varepsilon(\widehat{x})) = \psi(\Omega) + f(\varepsilon)\mathcal{T}(\widehat{x}) + o(f(\varepsilon)) \ ,$$

where $\Omega_\varepsilon(\widehat{x}) = \Omega \setminus \overline{\omega_\varepsilon(\widehat{x})}$ and $f(\varepsilon) \to 0$, when $\varepsilon \to 0$.

Topological Derivative Concept



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$$\mathcal{T}(\widehat{x}) = \lim_{\varepsilon \to 0} \frac{\psi(\Omega_\varepsilon(\widehat{x})) - \psi(\Omega)}{f(\varepsilon)} \ .$$

In general, $f(\varepsilon) = |\omega_\varepsilon|$. It depends on the boundary condition on $\partial\omega_\varepsilon$.

Topological Derivative Concept

$$\boxed{\psi(\Omega_\varepsilon(\widehat{x})) = \psi(\Omega) + f(\varepsilon)\mathcal{T}(\widehat{x}) + o(f(\varepsilon))}$$

The topological sensitivity analysis gives the topological asymptotic expansion of a shape functional with respect to a singular domain perturbation, like the insertion of holes, inclusions or cracks.

Why is making holes a good idea

$$\psi(\Omega) = \frac{1}{2}\int_{B_1} (u - z_d)^2 \, , \quad \left\{ \begin{array}{l} \text{Find } u\text{, such that} \\ \begin{array}{rcll} -\Delta u & = & b & \text{in} \quad \Omega = B_1 \subset \mathbb{R}^2 \, , \\ u & = & 0 & \text{on} \quad \partial B_1 \, . \end{array} \end{array} \right.$$

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$$z_d = u_{|_{\Omega^\star}} \,, \quad \text{where} \quad \Omega^\star = B_1 \setminus \overline{B_\rho}, \quad \text{with} \quad \rho = 1/4.$$

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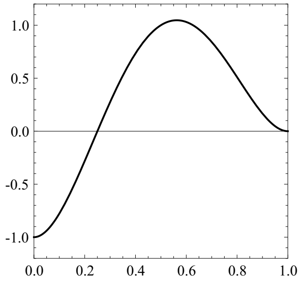


Figure: solution u

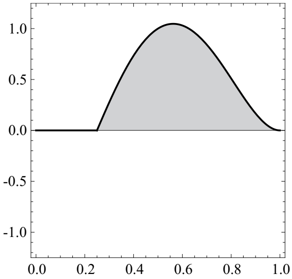


Figure: target z_d

Why is making holes a good idea

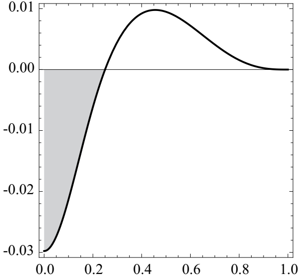
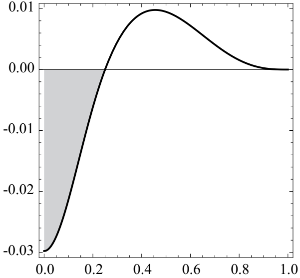


Figure: topological derivative $\mathcal{T}$

Why is making holes a good idea



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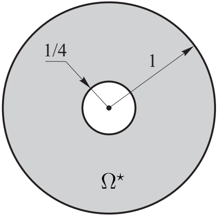

Figure: optimal domain $\Omega^\star$

Applications of the Topological Derivative

$$\mathcal{T}(\widehat{x}) = \lim_{\varepsilon \to 0} \frac{\psi(\Omega_\varepsilon(\widehat{x})) - \psi(\Omega)}{f(\varepsilon)}$$

The topological derivative $\mathcal{T}(\widehat{x})$ is now of common use for resolution of several problems, such as:

- Topology Design: Amstutz, Canelas, Leugering, Zochowski ...
- Inverse Problems: Ammari, Capdeboscq, Hintermüller, Kang, Laurain, Prakash ...
- Multi-Scale Material Design: Giusti, Souza Neto, Toader ...
- Image Processing: Auroux, Belaid, Drogoul, Masmoudi ...
- Fracture and Damage Modeling: Allaire, Jouve, Van Goethem, Xavier ...
- Theory Development: Amstutz, Nazarov, Sokolowski ...

Second Order Topological Derivative

$$\psi(\Omega_\varepsilon(\widehat{x})) = \psi(\Omega) + f(\varepsilon)\mathcal{T}(\widehat{x}) + f_2(\varepsilon)\mathcal{T}^2(\widehat{x}) + \mathcal{R}(f_2(\varepsilon)) \ ,$$

where $f(\varepsilon) \to 0$ and $f_2(\varepsilon) \to 0$ with $\varepsilon \to 0$, and

$$\lim_{\varepsilon\to 0} \frac{f_2(\varepsilon)}{f(\varepsilon)} = 0 \ , \qquad \lim_{\varepsilon\to 0} \frac{\mathcal{R}(f_2(\varepsilon))}{f_2(\varepsilon)} = 0 \ .$$

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$$\mathcal{T}(\widehat{x}) := \lim_{\varepsilon\to 0} \frac{\psi(\Omega_\varepsilon(\widehat{x})) - \psi(\Omega)}{f(\varepsilon)} \ .$$

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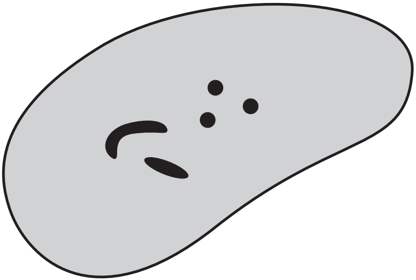
(first order) topological derivative

$$\mathcal{T}(\widehat{x}) := \lim_{\varepsilon\to 0} \frac{\psi(\Omega_\varepsilon(\widehat{x})) - \psi(\Omega)}{f(\varepsilon)} \ .$$

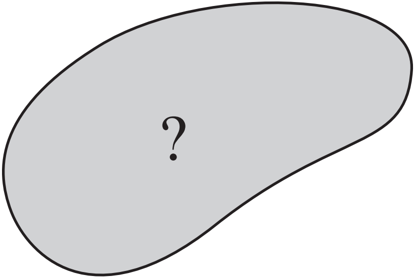
second order topological derivative

$$\mathcal{T}^2(\widehat{x}) := \lim_{\varepsilon\to 0} \frac{\psi(\Omega_\varepsilon(\widehat{x})) - \psi(\Omega) - f(\varepsilon)\mathcal{T}(\widehat{x})}{f_2(\varepsilon)} \ .$$

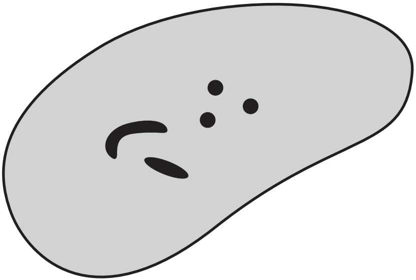
Inverse Conductivity Problem



Inverse Conductivity Problem



Inverse Conductivity Problem



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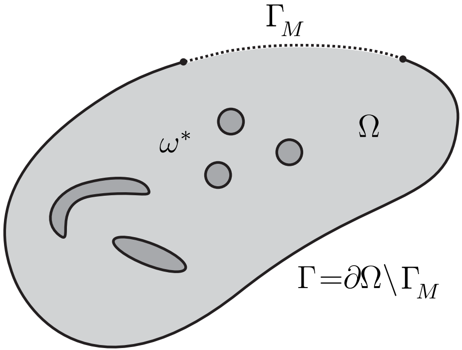

Inverse Conductivity Problem

Problem Setting

$$\left\{ \begin{array}{l} \text{Find } k_{\omega^*}, \text{ such that} \\ \begin{array}{rcll} -div(k_{\omega^*} \nabla z) & = & 0 & \text{in} \quad \Omega \\ z & = & U & \multirow{2}{*}{\text{on} \quad \Gamma_M} \\ -\partial_n z & = & Q & \end{array} \end{array} \right. ,$$

$$k_{\omega^*} = \left\{ \begin{array}{ll} 1 & \text{in } \Omega \setminus \omega^* \\ \gamma & \text{in } \omega^* \end{array} \right.$$

Inverse Conductivity Problem

Difficulties

- The problem is over determined and highly ill-posed;
- Lack of uniqueness if the contrast γ and the region ω^* are unknown simultaneously.

Inverse Conductivity Problem

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- Lack of uniqueness if the contrast γ and the region ω^* are unknown simultaneously.

$\triangleright$ We assume that the contrast γ is known

Inverse Conductivity Problem

$$\underset{\omega \subset \Omega}{\text{Minimize}} \;\; \mathcal{J}_\omega(u) = \int_{\Gamma_M} (u - z)^2$$

$$\left\{ \begin{array}{l} \text{Find } u, \text{ such that} \\ \begin{array}{rcll} -div(k_\omega \nabla u) & = & 0 & \text{in} \quad \Omega \\ -\partial_n u & = & Q & \text{on} \quad \Gamma_M \\ \displaystyle\int_{\Gamma_M} u & = & \displaystyle\int_{\Gamma_M} z & \end{array} \end{array} \right.$$

$$k_\omega = \left\{ \begin{array}{ll} 1 & \text{in } \Omega \setminus \omega \\ \gamma & \text{in } \omega \end{array} \right.$$

Inverse Conductivity Problem

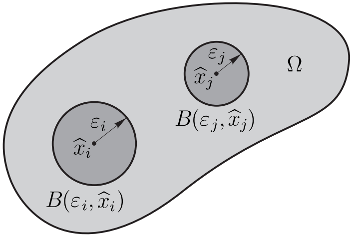
$$\underset{\omega \subset \Omega}{\text{Minimize}} \;\; \mathcal{J}_\omega(u) = \int_{\Gamma_M} (u - z)^2$$

$$\begin{cases} \text{Find } u, \text{ such that} \\ \quad -div(k_\omega \nabla u) = 0 & \text{in} \quad \Omega \\ \qquad\qquad -\partial_n u = Q & \text{on} \quad \Gamma_M \\ \qquad\quad \displaystyle\int_{\Gamma_M} u = \int_{\Gamma_M} z \end{cases}$$

$$k_\omega = \begin{cases} 1 & \text{in } \Omega \setminus \omega \\ \gamma & \text{in } \omega \end{cases} \qquad \boxed{\omega \equiv \emptyset}$$

Inverse Conductivity Problem

Topological Asymptotic Expansion



$$k_\varepsilon = \begin{cases} 1 & \text{in } \Omega \setminus B_\varepsilon(\xi) \\ \gamma & \text{in } B_\varepsilon(\xi) \end{cases}$$

$$B_\varepsilon(\xi) = \bigcup_{i=1}^{m} B(x_i, \varepsilon_i)$$

$$\varepsilon := \{\varepsilon_1, \varepsilon_2, \ldots \varepsilon_m\} \quad \text{and} \quad \xi := \{x_1, x_2, \ldots, x_m\}$$

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \int_{\Gamma_M} (u_\varepsilon - z)^2$$

Inverse Conductivity Problem

Theorem

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_0(u_0) - \alpha \cdot d(\xi) + \frac{1}{2} H(\xi)\alpha \cdot \alpha + o(|\alpha|^2) \ ,$$

where the vector $\alpha = (\alpha_1, \cdots, \alpha_m)$, *with* $\alpha_i = |B(x_i, \varepsilon_i)|$.

$$\left\{ \begin{array}{rcll} -div(k_\varepsilon \nabla u_\varepsilon) & = & 0 & \text{in} \quad \Omega \\ -\partial_n u_\varepsilon & = & Q & \text{on} \quad \Gamma_M \\ \int_{\Gamma_M} u_\varepsilon & = & \int_{\Gamma_M} z & \end{array} \right. \quad \left\{ \begin{array}{rcll} -\Delta u_0 & = & 0 & \text{in} \quad \Omega \\ -\partial_n u_0 & = & Q & \text{on} \quad \Gamma_M \\ \int_{\Gamma_M} u_0 & = & \int_{\Gamma_M} z & \end{array} \right.$$

Joint work with A.D. Ferreira, A. Laurain & M. Hintermüller

Inverse Conductivity Problem

The vector $d \in \mathbb{R}^m$ and the matrix $H \in \mathbb{R}^m \times \mathbb{R}^m$ are defined as

$$d_i := 2 \int_{\Gamma_M} \rho(u_0 - z)(g_i + \tilde{u}_i),$$

$$H_{ii} := 4 \int_{\Gamma_M} (u_0 - z)(\rho h_i + \rho \tilde{g}_i + \tilde{\tilde{u}}_i) + 2 \int_{\Gamma_M} (\rho g_i + \tilde{u}_i)^2,$$

$$H_{ij} := 2 \int_{\Gamma_M} (u_0 - z)(\rho \theta_i^j + \rho \theta_j^i + u_i^j + u_j^i)$$
$$+ 2 \int_{\Gamma_M} (\rho g_i + \tilde{u}_i)(\rho g_j + \tilde{u}_j), \qquad j \neq i.$$

with

$$\rho = \frac{1-\gamma}{1+\gamma},$$

Inverse Conductivity Problem

$$\begin{cases} -\Delta \tilde{u}_i = 0 & \text{in} \quad \Omega \\ \partial_n \tilde{u}_i = -\rho \partial_n g_i & \text{on} \quad \partial\Omega \\ \int_{\Gamma_M} \tilde{u}_i = -\rho \int_{\Gamma_M} g_i & \end{cases}$$

$$\begin{cases} -\Delta \tilde{\tilde{u}}_i = 0 & \text{in} \quad \Omega \\ \partial_n \tilde{\tilde{u}}_i = -\rho \partial_n (h_i + \tilde{g}_i) & \text{on} \quad \partial\Omega \\ \int_{\Gamma_M} \tilde{\tilde{u}}_i = -\rho \int_{\Gamma_M} h_i + \tilde{g}_i & \end{cases}$$

$$\begin{cases} -\Delta u_i^j = 0 & \text{in} \quad \Omega \\ \partial_n u_i^j = -\rho \partial_n \theta_i^j & \text{on} \quad \partial\Omega \\ \int_{\Gamma_M} u_i^j = -\rho \int_{\Gamma_M} \theta_i^j & \end{cases}$$

Inverse Conductivity Problem

$$
\begin{aligned}
g_i(x) &= \frac{1}{\|x - x_i\|^2} \nabla u_0(x_i) \cdot (x - x_i), \\
h_i(x) &= \frac{1}{2} \frac{1}{\|x - x_i\|^4} \nabla^2 u_0(x_i)(x - x_i)^2, \\
\tilde{g}_i(x) &= \frac{1}{\|x - x_i\|^2} \nabla \tilde{u}_i(x_i) \cdot (x - x_i), \\
\theta_i^j(x) &= \frac{1}{\|x - x_j\|^2} A(x_j) \nabla u_0(x_i) \cdot (x - x_j).
\end{aligned}
$$

$$
A(x) = \frac{1}{\|x - x_i\|^2} \left[I - 2 \frac{(x - x_i) \otimes (x - x_i)}{\|x - x_i\|^2} \right].
$$

Inverse Conductivity Problem

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$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_0(u_0) - \alpha \cdot d(\xi) + \frac{1}{2} H(\xi)\alpha \cdot \alpha + o(|\alpha|^2) \ ,$$

where the vector $\alpha = (\alpha_1, \cdots, \alpha_m)$*, with* $\alpha_i = |B(x_i, \varepsilon_i)|$.

$$\boxed{\delta J(\alpha, \xi, m) := -\alpha \cdot d(\xi) + \frac{1}{2} H(\xi)\alpha \cdot \alpha}$$

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$$\Rightarrow (\xi^\star, \alpha(\xi^\star))$$

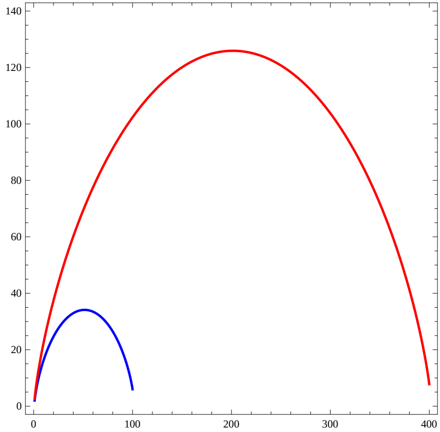
Inverse Conductivity Problem

Complexity order of the Resulting Algorithm

$$\mathcal{C}(n,m) = \left(\begin{array}{c} n \\ m \end{array} \right) m^3 = \frac{n!}{m!(n-m)!} m^3$$

Inverse Conductivity Problem

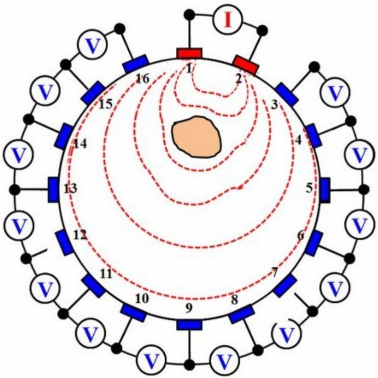
Complexity order of the Resulting Algorithm



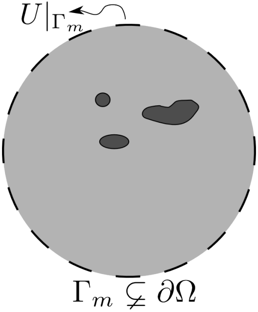
$m \times \log_{10}(\mathcal{C}(n, m))$, for $n = 100$ in blue and $n = 400$ in red

Inverse Conductivity Problem

Numerical Experiments



(a) electrodes

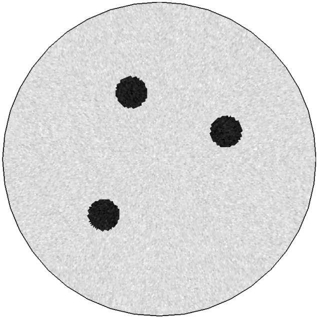


(b) measurements

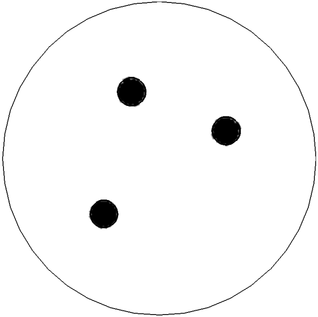
Inverse Conductivity Problem

Inverse Conductivity Problem

Numerical Results



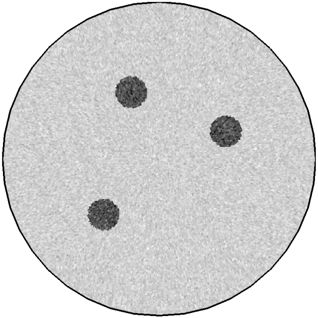
(a) WGN = 5%



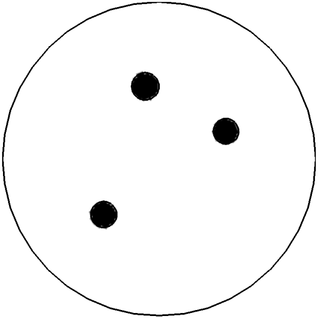
(b) M = 64

Inverse Conductivity Problem

Numerical Results



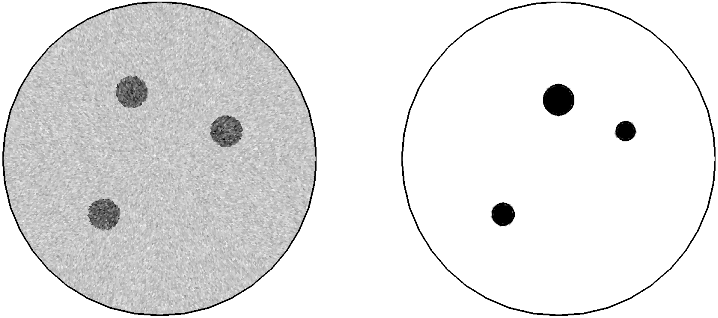
(a) WGN = 10%



(b) M = 64

Inverse Conductivity Problem

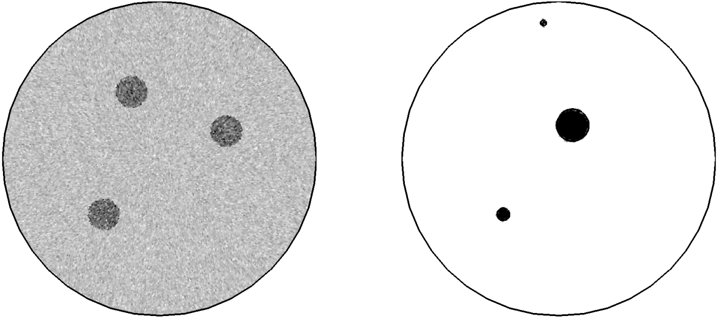
Numerical Results



(a) WGN = 15%   (b) M = 64

Inverse Conductivity Problem

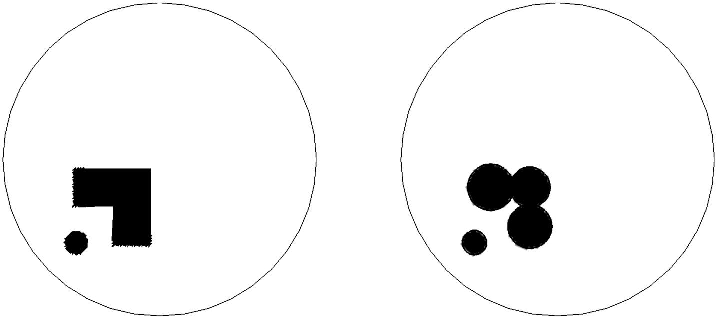
Numerical Results



(a) WGN = 20%   (b) M = 64

Inverse Conductivity Problem

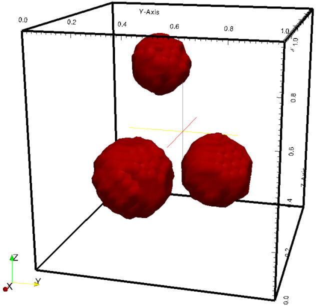
Numerical Results



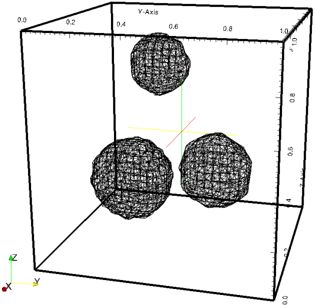
(a) target (b) result

Inverse Conductivity Problem

Numerical Results



(a) target



(b) result

Conclusions Remarks

- Non-iterative reconstruction method, robust with respect to noisy data and free of any initial guess;
- Specifically designed for a class of inverse problems where the unknown is given by a set of anomalies;
- Gravimetry, inverse potential problem, seismic, obstacle reconstruction ...
- The main drawback is the combinatorial nature of the algorithm, summarized as follows:
  1. If $m \ll n$ and m small, the complexity is treatable;
  2. If $m \sim n$, m can assume high values and the complexity remains treatable, since the number of combinations becomes small;
  3. If $m < n$ $(m \sim n/2)$ and m high, the complexity blows up and the combinatorial search becomes unfeasible;
- Actually, we are thinking about different possibilities to explore these features of the algorithm.

References

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Muito Obrigado!