On self-regularization of ill-posed problems in Banach spaces by projection methods

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1. The problem

2. Projection methods
   - Convergence with a priori choice of $n = n(\delta)$
   - Choice of $n = n(\delta)$ by the discrepancy principle
   - Modification of the discrepancy principle

3. Applications: Collocation method for Volterra integral equations
   - Cordial integral equations
   - Polynomial collocation method for cordial integral equations
   - Spline-collocation for Volterra integral equation

The problem

Let E, F be Banach spaces, $A \in \mathcal{L}(E, F)$. Consider the equation

$$Au = f, \tag{1}$$

where $f \in \mathcal{R}(\mathcal{A})$ is unknown, only noisy data f^δ with $\|f^\delta - f\|_F \leq \delta$ are available.

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Typically the problem is formulated in infinite-dimensional space. If the problem is ill-posed, then usually it is first regularized and then discretized. Often the discretization works as regularization as well: if data are noisy with known noise level δ, then by proper choice of $n = n(\delta)$ the solutions of discretized equations with noisy data converge to the solution of the original problem with exact data.

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Most results about self-regularization have been obtained in Hilbert spaces. But for integral equations the most natural space is C, especially for collocation or quadrature methods. Often also L^1 is useful to recover sparse solution.

Projection methods

Assume that (1) is uniquely solvable. Let u_* be the solution for exact f.

Let $E_n \subseteq E$, $Z_n \subseteq F^*$ be finite dimensional subspaces, $\dim E_n = \dim Z_n$.
General linear projection method: find $u_n \in E_n$ such that

$$\forall z_n \in Z_n \quad \langle z_n, Au_n \rangle_{F^*,F} = \langle z_n, f^\delta \rangle_{F^*,F}. \tag{2}$$

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Define $Q_n \in L(F, Z_n^*)$: $\forall g \in F, z_n \in Z_n$: $\langle Q_n g, z_n \rangle_{Z_n^*, Z_n} = \langle z_n, g \rangle_{F^*,F}$.
Then (2) $\iff Q_n A u_n = Q_n f^\delta$.

$$\|Q_n\| = \sup_{\substack{g \in F, \|g\|_F = 1 \\ z_n \in Z_n, \|z_n\|_{F^*} = 1}} \langle Q_n g, z_n \rangle_{Z_n^*, Z_n} = \sup_{\substack{g \in F, \|g\|_F = 1 \\ z_n \in Z_n, \|z_n\|_{F^*} = 1}} \langle z_n, g \rangle_{F^*,F} = 1.$$

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Example. Collocation method for integral equations.
$F = C[a, b]$, $Z_n = \mathrm{span}\{\delta(t - t_i), i = 1, \ldots, n\}$, $t_i \in [a, b]$.
(2) $\iff$ $Au_n(t_i) = f^\delta(t_i), i = 1, \ldots, n$.

$$\tilde{\kappa}_n = \sup_{w_n \in E_n} \frac{\|w_n\|_E}{\|Q_n A w_n\|_{Z_n^*}} = \sup_{w_n \in E_n} \frac{\|w_n\|_E}{\sup_{z_n \in Z_n, \|z_n\|_{F^*}=1} \langle z_n, A w_n \rangle_{F^*,F}}$$

Lemma 1 (Uniqueness of the discrete solution)

Let $\dim(E_n) = \dim(Z_n)$ and $\mathcal{N}(Q_n A) \cap E_n = \{0\}$. Then (2) is uniquely solvable for each $f^\delta \in F$.

Denote $A_n := Q_n A|_{E_n} : E_n \to Z_n^*$. Then the lemma means that A_n has an inverse and $\|A_n^{-1}\| = \tilde{\kappa}_n$.

$$\kappa_n = \sup_{v_n \in E_n} \frac{\|v_n\|_E}{\|Av_n\|_F}, \quad \breve{\kappa}_n = \|A_n^{-1} Q_n\|, \quad \tilde{\kappa}_n = \|A_n^{-1}\| = \sup_{v_n \in E_n} \frac{\|v_n\|_E}{\|Q_n A v_n\|_F},$$

$$\tau_n = \sup_{v_n \in E_n, v_n \neq 0} \frac{\|Av_n\|_F}{\|Q_n A v_n\|_{Z_n^*}}.$$

Lemma 2

Let $\dim(E_n) = \dim(Z_n)$ *and* $\mathcal{N}(Q_n A) \cap E_n = \{0\}$ *hold. Then*

$$\kappa_n \leq \breve{\kappa}_n \leq \tilde{\kappa}_n \leq \tau_n \kappa_n.$$

If there exists $\tau < \infty$ *such that* $\tau_n \leq \tau$ *for all* $n \in \mathbb{N}$ *then also* $\tilde{\kappa}_n \leq \tau \kappa_n$, *i.e. the quantities* κ_n, $\breve{\kappa}_n$ *and* $\tilde{\kappa}_n$ *are all equivalent as* $n \to \infty$.

Convergence with a priori choice of $n = n(\delta)$

Theorem 3

Let (1) and (2) be uniquely solvable for each $n \geq n_0$ and u_*, u_n be their solutions. Then the following error estimate holds:

$$\begin{aligned} \|u_n - u_*\|_E &\leq \min_{v_n \in E_n} [\|u_* - v_n\|_E + \|A_n^{-1} Q_n A(u_* - v_n)\|_E] + \breve{\kappa}_n \delta \\ &\leq (1 + \|A_n^{-1} Q_n A\|) \, dist(u_*, E_n) + \breve{\kappa}_n \delta. \end{aligned}$$

In case of exact data ($\delta = 0$) the convergence $\|u_n - u_*\|_E \to 0$ as $n \to \infty$ holds if and only if there exists a $(\hat{u}_n)_{n \in \mathbb{N}}$, $\hat{u}_n \in E_n$, satisfying the convergence conditions $\|u_* - \hat{u}_n\|_E \to 0$ as $n \to \infty$ and

$$\|A_n^{-1} Q_n A(u_* - \hat{u}_n)\| \to 0 \text{ as } n \to \infty.$$

If these conditions hold and the data are noisy, then choosing $n = n(\delta)$ according to a priori rule $n(\delta) \to \infty$ and $\breve{\kappa}_{n(\delta)} \delta \to 0$ as $\delta \to 0$ we have convergence $\|u_{n(\delta)} - u_*\|_E \to 0$ as $\delta \to 0$.

According to the previous theorem the convergence may hold due to sufficient smoothness of the solution.
The next theorem gives conditions for convergence for every $f \in \mathcal{R}(A)$ (i.e. for every $u_* \in E$ without additional smoothness requirements).

Theorem 4 (Convergence for every f)

Let (1) and (2) be uniquely solvable for $n \geq n_0$. Then in case of exact data ($\delta = 0$) the convergence $\|u_n - u_*\|_E \to 0$ as $n \to \infty$ holds for every $f \in \mathcal{R}(A)$ if and only if the subspaces E_n satisfy condition $\inf_{v_n \in E_n} \|v_n - v\| \to 0 \ \forall v \in E$ as $n \to \infty$, and the projectors $A_n^{-1} Q_n A : E \to E_n$ are uniformly bounded, i.e.,

$$\|A_n^{-1} Q_n A\| \leq M$$

for all $n \geq n_0$ and some constant M.
The last two conditions are necessary and sufficient for existence of relations $n = n(\delta)$ for convergence $\|u_{n(\delta)} - u_*\|_E \to 0$ as $\delta \to 0$ for every $f \in \mathcal{R}(A)$ given approximately as arbitrary f^δ with $\|f^\delta - f\| \leq \delta$.

For the convergence analysis in case of exact data we can choose different image spaces, particularly such that the equation becomes well-posed. But for noisy data the image space is determined by the data.
The following theorem shows that convergence for the "main part" of the equation implies convergence for the whole equation.

Theorem 5

Let (1) and (2) be uniquely solvable for $n \geq n_0$. Assume that there exists a $(\hat{u}_n)_{n\in\mathbb{N}}$, $\hat{u}_n \in E_n$, satisfying $\|u_* - \hat{u}_n\|_E \to 0$ as $n \to \infty$. Let the operator $A : E \to F$ have the form $A = S + K$, where $S : E \to W \subset F$ is invertible, W is a Banach space with continuous imbedding and $K : E \to W$ is compact. Let the operator $S_n := Q_n S|_{E_n} : E_n \to Z_n^*$ be invertible and $\|S_n^{-1} Q_n S\| \leq M$ for some constant M. Then the projection equation $Q_n A u_n = Q_n f$ has for n large enough a unique solution $u_n \in E_n$, and $u_n \to u_*$ as $n \to \infty$.

For considering the influence of the noisy data, the behaviour of the quantities $\breve{\varkappa}_n$ is essential. For estimating these quantities we introduce operators $\Pi_n : Z_n^* \to F$ such that the equality $Q_n \Pi_n Q_n = Q_n$ holds. Then the operator $\Pi_n Q_n$ is a projector in F.
Let $F_n = \mathcal{R}(\Pi_n)$. We assume that $F_n \subset W$ and let $W_n = F_n$, equipped with the norm of W. Let I_n be the identity operator, considered as acting from F_n to W_n.

Theorem 6 (Estimate of $\breve{\varkappa}_n$)

Let (1) and (2) be uniquely solvable for $n \geq n_0$. Let the operator $A : E \to W$ be invertible. Then

$$\breve{\varkappa}_n \leq C \| I_n \|_{F_n \to W_n}, \quad n \geq n_0.$$

Choice of $n = n(\delta)$ by the discrepancy principle

Theorem 7

Let (1) and (2) be uniquely solvable for $n \geq n_0$. Let the convergence

$$\breve{\kappa}_{n+1} dist(f, AE_n) \to 0 \text{ as } n \to \infty$$

hold. We also assume that there exists a sequence of approximations $(\hat{u}_n)_{n\in\mathbb{N}}$, $\hat{u}_n \in E_n$, satisfying $\|u_* - \hat{u}_n\|_E \to 0$ as $n \to \infty$ and

$$\|A_n^{-1} Q_n A(u_* - \hat{u}_n)\| \to 0 \text{ as } n \to \infty.$$

Assume there exists $\tau < \infty$ such that $\tau_n \leq \tau$ for all $n \in \mathbb{N}$. Let $b > \tau + 1$ be fixed and for $\delta > 0$, let $n = n_{DP}(\delta)$ be the first index such that $\|Au_n - f^\delta\|_F \leq b\delta$.
Then $n_{DP}(\delta)$ is finite and $\|u_{n_{DP}(\delta)} - u_*\|_E \to 0$ as $\delta \to 0$.

Modification of the discrepancy principle

In Theorem 7 the assumption $\tau_n \leq \tau$ for all $n \in \mathbb{N}$ was required. For collocation methods this is the uniform boundedness of the interpolation projector onto the subspace $AE_n \subset F$. If $F = C^m$, this assumption does not hold in general.

Theorem 8

Let the assumptions of Theorem 7 be satisfied without uniform boundedness of τ_n. Let the sequence $b_n > (1+\tau_n)(1+\varepsilon)$ be fixed with some fixed $\varepsilon > 0$ and $n = n_{DP}(\delta)$ be chosen as the first index such that $\|Au_n - f^\delta\|_F \leq b_n\delta$.
Then $n_{DP}(\delta)$ is finite and $\|u_{n_{DP}(\delta)} - u_*\|_E \to 0$ as $\delta \to 0$.

Cordial integral equations

Consider cordial integral equations of first kind

$$\int_0^t \frac{1}{t} a(t,s)\varphi(\frac{s}{t})u(s)ds = f(t), \quad 0 \le t \le T, \tag{3}$$

where $\varphi \in L^1(0,1)$ is called the core of the cordial integral operator, a, f are given smooth enough functions.

Define the cordial integral operators

$$(V_\varphi u)(t) = \int_0^t \frac{1}{t}\varphi(\frac{s}{t})u(s)ds,$$
$$(V_{\varphi,a} u)(t) = \int_0^t \frac{1}{t} a(t,s)\varphi(\frac{s}{t})u(s)ds.$$

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Make a change of variables $s = tx$, denote $b(t,x) = a(t,tx)$ and define the corresponding integral operator

$$(\widetilde{V}_{\varphi,b} u)(t) = \int_0^1 \varphi(x)b(t,x)u(tx)dx.$$

Cordial integral operators in spaces C^m

Denote $\Delta_T = \{(s,t) : \ t \in [0,T], s \in [0,t]\}$.

Theorem 9

Let $\varphi \in L^1(0,1)$*,* $a \in C^m(\Delta_T)$*. Then* $V_{\varphi,a} \in \mathcal{L}(C^m[0,T])$ *and* $\|V_{\varphi,a}\|_{C^m[0,T]} \leq C\|\varphi\|_{L^1(0,1)}\|a\|_{C^m(\Delta_T)}$.

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Theorem 10

Let $\varphi \in L^1(0,1)$ *and let* $\lambda \in \mathbf{C}$ *with* $\mathrm{Re}\,\lambda > 0$. *Then* t^λ *is an eigenfunction of* V_φ *in* $C[0,T]$, *and the corresponding eigenvalue is* $\hat{\varphi}(\lambda) = \int_0^1 \varphi(x)x^\lambda dx$. *If* $\mathrm{Re}\,\lambda > m$, *then the eigenfunction belongs to* $C^m[0,T]$.

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Theorem 10

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Theorem 11

Let $\varphi \in L^1(0,1)$, $a \in C^m(\Delta_T)$. *Then the spectrum of* $V_{\varphi,a}$ *in* $C^m[0,T]$ *is given by*
$\sigma_m(V_{\varphi,a}) = \{0\} \cup \{a(0,0)\hat{\varphi}(k),\ k = 0,\dots,m\} \cup \{a(0,0)\hat{\varphi}(\lambda),\ Re\,\lambda > m\}$.

Existence and uniqueness of solution

$$\int_0^t \frac{1}{t} a(t,s)\varphi(\frac{s}{t})u(s)ds = f(t), \quad 0 \le t \le T \tag{3}$$

Theorem 12

Let $\varphi \in L^1(0,1)$, $x(1-x)\varphi'(x) \in L^1(0,1)$, $\int_0^1 \varphi(x)dx > 0$ *and there exists* $\beta < 1$ *such that* $(x^\beta \varphi(x))' \ge 0$ *for* $x \in (0,1)$. *Assume also that* $a \in C^{m+1}(\Delta_T)$ *and* $a(t,t) \ne 0$. *Then* $V_{\varphi,a}$ *is injective in* $C[0,T]$, $C^{m+1}[0,T] \subset V_{\varphi,a}(C^m[0,T]) \subset C^m[0,T]$, *and* $V_{\varphi,a}^{-1} \in \mathcal{L}(C^{m+1}[0,T], C^m[0,T])$.

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Corollary 13

Let the assumptions of Theorem 12 be satisfied and let $f \in C^{m+1}[0,T]$ *be given. Then the equation is uniquely solvable in* $C[0,T]$ *and its solution is in* $C^m[0,T]$.

Polynomial collocation method for cordial integral equations

Look for solutions in the form $u_n(t) = \sum_{k=0}^{N} c_k t^k$.

Collocation method:
choose the collocation points t_k, $k = 0, \ldots, n$,
solve the collocation equations

$$\sum_{j=0}^{N} c_j \int_0^T \frac{1}{t_k} a(t_k, s) \varphi(\frac{s}{t_k}) s^j \, ds = f(t_k), \quad k = 0, \ldots n. \tag{4}$$

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Need to calculate exactly or "well enough" the integrals

$$\int_0^T \frac{1}{t_k} a(t_k, s)\varphi(\frac{s}{t_k}) s^j \, ds$$

In the following $E = F = C[0, T]$, E_n is the space of polynomials of order up to n and Z_n is the linear span of δ-functions with supports t_k, $k = 0, \ldots, n$.
Let $a(t, s) \equiv 1$. Then $V_\varphi : E_n \to E_n$ and τ_n is simply the norm of the interpolation projector from C to C with the interpolation nodes t_k, $k = 0, \ldots, n$. If t_k are the Chebyshev nodes, then $\tau_n = \frac{2}{\pi} \ln(n + 1) + 1$.

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We chose certain noise levels and the noise was generated by random numbers with uniform distribution at the collocation nodes. We also found the optimal number n_{opt} and the corresponding error

$$e_{opt} = \min_{n \in \mathbb{N}} \|u_n - u_*\|_E = \|u_{n_{opt}} - u_*\|_E.$$

We used the modified discrepancy principle to find the first $n = n_{DP}$ satisfying the inequality $\|Au_n - f^\delta\|_F \leq b_n \delta$ with $b_n = 1.001(1 + \tau_n)$. We denote the corresponding error by $e_{DP} = \|u_{n_{DP}} - u_*\|$.

Numerical examples: cordial integral equation

Example 1. Consider the cordial integral equation (here $\phi(x) = \frac{1}{\sqrt{x}}$)

$$\int_0^t \frac{u(s)ds}{\sqrt{st}} = \frac{1}{t^2+1}, \quad t \in [0, T]$$

with exact solution $u(s) = \frac{1-3s^2}{2(s^2+1)^2}$.

For this equation κ_n can be estimated using Markoff's inequality, by Cn^2. Since the right-hand side of the equation is analytic, $\mathrm{dist}(f, AE_n)$ converges to zero exponentially, hence the assumptions of Theorem 8 (modification of the discrepancy principle) are satisfied.

We took $T = 10$ and used noisy data with noise levels $\delta = 10^{-4}, 10^{-6}, \ldots, 10^{-14}$. The number of collocation nodes was 10, 15, 20, ..., 110.

The optimal errors and the errors obtained by using the discrepancy principle with $b_{n_{DP}}$ are presented in the following table.

δ	e_{opt}	n_{opt}	e_{DP}	n_{DP}	$b_{n_{DP}}$
10^{-4}	$6 \cdot 10^{-2}$	25	$8 \cdot 10^{-2}$	20	3.94
10^{-6}	$1.01 \cdot 10^{-3}$	40	$2.4 \cdot 10^{-3}$	30	4.19
10^{-8}	$1.51 \cdot 10^{-5}$	40	$1.51 \cdot 10^{-5}$	40	4.36
10^{-10}	$1.8 \cdot 10^{-7}$	50	$1.8 \cdot 10^{-7}$	50	4.56
10^{-12}	$4.69 \cdot 10^{-9}$	75	$9.58 \cdot 10^{-9}$	60	4.62
10^{-14}	$7.04 \cdot 10^{-11}$	105	$7.57 \cdot 10^{-11}$	70	4.71

Example 2. Consider the equation

$$\int_0^t \frac{u(s)ds}{\sqrt{st}} = t^{3/2}(2-t)^{5/2}, t \in [0,2].$$

The exact solution is $2t^{3/2}(2-t)^{5/2} - \frac{5}{2}t^{5/2}(2-t)^{3/2}$.
The integral operator is the same as in Example 1, hence $\kappa_n \leq Cn^2$. The distance $\mathrm{dist}(f, AE_n)$ can be estimated by Cn^{-3}, hence the assumptions of Theorem 8 are satisfied.
We used noisy data with noise levels $\delta = 10^{-3}, 10^{-4}, \dots, 10^{-7}$. The noise was generated by random numbers with uniform distribution at the collocation nodes. The number of collocation nodes was 10, 20, $30, \dots, 300$.

The optimal errors and the errors obtained by using the discrepancy principle with $b_{n_{DP}}$ are presented in the following table.

δ	e_{opt}	n_{opt}	e_{DP}	n_{DP}	$b_{n_{DP}}$
10^{-3}	$1.5 \cdot 10^{-1}$	10	$1.5 \cdot 10^{-1}$	10	3.53
10^{-4}	$5 \cdot 10^{-2}$	40	$1.1 \cdot 10^{-1}$	30	4.19
10^{-5}	$5.24 \cdot 10^{-3}$	20	$2 \cdot 10^{-2}$	50	4.5
10^{-6}	$6.13 \cdot 10^{-4}$	40	$5.16 \cdot 10^{-3}$	100	4.94
10^{-7}	$9.17 \cdot 10^{-5}$	90	$5.77 \cdot 10^{-3}$	230	5.46

Numerical examples: spline-collocation for Volterra integral equation

Consider Volterra integral equation of the first kind

$$(Au)(t) := \int_0^t K(t,s)u(s)\,ds = f(t), \;\; t \in [0,1]$$

with the operator $A \in L(L^p(0,1), C[0,1]), \;\; 1 \leq p \leq \infty$. The approximation space is $E_n = S_{k-1}^{(-1)}(I_\Delta)$, the space of discontinuous piecewise polynomials of order $k-1$ with mesh Δ. We find $u_n \in E_n$ such that

$$Au_n(t_{i,j}) = f^\delta(t_{i,j}), \quad i = 1, \ldots, n, \;\; j = 1, \ldots, k$$

where $t_{i,j} = (i-1+c_j)h \in [0,1]$, $i = 1, \ldots, n$, $j = 1, \ldots, k$ are collocation nodes and $0 < c_1 < \ldots < c_k \leq 1$ are collocation parameters whose choice is essential.

Now $\breve{\varkappa}_n$ can be estimated using Theorem 6 (Estimate of $\breve{\varkappa}_n$); it depends on how much A is smoothing.

Example 3. Consider the equation

$$Au(t) = \int_0^t u(s)ds = \frac{t^q}{q}, \quad t \in [0,1], \quad q \in \{3/2, 5/2\}$$

with operator $A : L^1(0,1) \to C[0,1]$. The exact solution is $u(s) = s^{q-1}$. We used for E_n the space of discontinuous linear splines with uniform mesh ih, $i = 0, \ldots, n$, where $h = 1/n$. The collocation points are $t_{i1} = (i - 1 + c))h$, $t_{i2} = ih$, $c \in (0,1)$. We took for F_n the space of continuous linear splines and the inverse property of splines gives $\|w_n'\| \leq Cn\|w_n\| \ \forall w_n \in F_n$, hence $\breve{\kappa}_n \leq Cn$. The distance $\operatorname{dist}(f, AE_n)$ can be estimated by Cn^{-q}. It can be shown that here

$$\tau = \begin{cases} 1 + \frac{c^2}{2(1-c)}, & \text{if } c \geq \frac{1}{2}, \\ 1 + \frac{(1-c)^2}{2c} & \text{if } c \leq \frac{1}{2}. \end{cases}$$

τ is minimal for $c = \frac{1}{2}$, then $\tau = 1.25$. In this example $\tau_n = \tau$. We used $c = \frac{1}{2}$ and took $b = 1.01 + \tau = 2.26$ for the discrepancy principle. The noisy data were generated by the formula $f^\delta(t_{i,j}) = f(t_{i,j}) + \delta\theta_{i,j}$, where $\delta = 10^{-m}, m \in \{2, \ldots, 7\}$ and $\theta_{i,j}$ are random numbers with normal distribution, normed after being generated: $\max_{i,j} |\theta_{i,j}| = 1$.

The optimal errors and the errors obtained by using the discrepancy principle with $q = 3/2$ and $q = 5/2$.

δ	e_{opt}	n_{opt}	e_{DP}	n_{DP}	e_{opt}	n_{opt}	e_{DP}	n_{DP}
10^{-1}	$2.5 \cdot 10^{-1}$	1	$2.5 \cdot 10^{-1}$	1	$2.9 \cdot 10^{-1}$	1	$2.9 \cdot 10^{-1}$	1
10^{-2}	$6.8 \cdot 10^{-2}$	2	$6.8 \cdot 10^{-2}$	2	$5.4 \cdot 10^{-2}$	2	$5.4 \cdot 10^{-2}$	2
10^{-3}	$1.3 \cdot 10^{-2}$	8	$1.8 \cdot 10^{-2}$	5	$9 \cdot 10^{-3}$	6	$1.1 \cdot 10^{-2}$	5
10^{-4}	$3.2 \cdot 10^{-3}$	24	$3.3 \cdot 10^{-3}$	20	$1.7 \cdot 10^{-3}$	15	$3 \cdot 10^{-3}$	8
10^{-5}	$7.6 \cdot 10^{-4}$	72	$8.4 \cdot 10^{-4}$	86	$3.5 \cdot 10^{-4}$	32	$6.2 \cdot 10^{-4}$	18
10^{-6}	$1.9 \cdot 10^{-4}$	128	$3.3 \cdot 10^{-4}$	512	$6.8 \cdot 10^{-5}$	72	$9.9 \cdot 10^{-5}$	46
10^{-7}	$4.5 \cdot 10^{-5}$	512	$1.2 \cdot 10^{-4}$	2048	$1.5 \cdot 10^{-5}$	128	$1.5 \cdot 10^{-5}$	128

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