Chapter 2

On Iterated Tikhonov Kaczmarz Type Methods for Solving Systems of Linear Ill-posed Operator Equations

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2.1 Introduction

In this chapter we propose a nonstationary *iterated Tikhonov Kaczmarz* (iTK) type method for obtaining stable approximations to systems of linear ill-posed operator equations. The iTK methods are Kaczmarz type methods [13], where the steps are defined using the same heuristic as in the iterated Tikhonov (iT) method [3, Section 1.2].

The novelty of our approach consists in defining the Lagrange multipliers using a strategy inspired by [2]. That is the multipliers are chosen as to guarantee the residual of the next iterate to be in a *range* (use a different range as the one proposed in [2]).

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The inverse problems we are interested in consists of determining an unknown quantity $x \in X$ from the set of data $(y_0, \dots, y_{N-1}) \in Y^N$, where X and Y are Hilbert spaces, and $N \geq 1$. In practical situations, one does not know the data exactly. Instead, only approximate measured data $y_i^\delta \in Y$ satisfying

$$\|y_i^\delta - y_i\| \ \leq \ \delta_i \, , \ i = 0, \dots, N-1 \, , \tag{2.1}$$

are available, where $\delta_i > 0$ are the (known) noise levels. The available data y_i^δ are obtained by indirect measurements of the parameter x, this process being described by the system of ill-posed operator equations

$$A_i x \ = \ y_i \, , \ i = 0, \dots, N-1 \, . \tag{2.2}$$

where $A_i : X \to Y$ are bounded linear operators, whose inverses $A_i^{-1} : R(A_i) \to X$ either do not exist, or are not continuous. Consequently, approximate solutions are extremely sensitive to noise in the data.

There is a vast literature on iterative methods for the stable solution of (2.2). We refer the reader to the text books [7, 12, 1, 16, 14, 5] and the references therein.

What concerns systems of linear ill-posed equations and Kaczmarz type methods, we refer the reader to [17]. In this article Nashed considered Kaczmarz (and Cimmino) methods for solving a problem related to the inverse Radon Transform, which is written in the form of abstract equations (see (1.1) to (1.4) in [17]). It is proven that the Kaczmarz method converges to a weighted least-squares solution of (2.2) [17, Section 2], which is defined by means of the *oblique generalized inverse* defined in [17, pg.167]. In this article Nashed also discusses the generalization of Kaczmarz (and Cimmino) method to operator equations in function spaces (this is the framework presented above in (2.2), (2.1)). The final paragraph in [17] reads:

> "It is hoped that the semicontinuous (or semidiscrete, depending on your viewpoint) analogues of the methods of Kaczmarz and Cimmino developed in this paper ... will be further adapted to the context of reconstruction problems."

It is our hope that the present manuscript is able to give a small step in the direction pointed by professor Nashed back in 1980.

Iterated Tikhonov Type Methods

Standard iterated Tikhonov (iT) type methods for solving the ill-posed problem (2.1), (2.2) are defined, after rewritting (2.2) as a single equation $\mathbf{A}x = \mathbf{y}$, where $\mathbf{A} = (A_0, \dots, A_{N-1}) : X \to Y^N$ and $\mathbf{y}^\delta = (y_0^\delta, \dots, y_{N-1}^\delta)$, by the iteration formula

$$x_{k+1}^\delta \ = \ \arg\ \min_{x \in X} \big\{ \lambda_k \|\mathbf{A}x - \mathbf{y}^\delta\|^2 + \|x - x_k^\delta\|^2 \big\} \tag{2.3}$$

or, equivalentely, by

$$\begin{aligned} x_{k+1}^\delta &= x_k^\delta - \lambda_k \big(I + \lambda_k \mathbf{A}^*\mathbf{A}\big)^{-1} \mathbf{A}^* \big(\mathbf{A}x_k^\delta - \mathbf{y}^\delta\big) \\ &= \big(\lambda_k^{-1} I + \mathbf{A}^*\mathbf{A}\big)^{-1} \big[\lambda_k^{-1} x_k^\delta + \mathbf{A}^*\mathbf{y}^\delta\big], \end{aligned} \tag{2.4}$$

where $\mathbf{A}^* : Y^N \to X$ is the adjoint operator to $\mathbf{A}$. The parameter $\lambda_k > 0$ can be viewed as the Lagrange multiplier of the problem of projecting x_k^δ onto a levelset of $\|\mathbf{A}x - \mathbf{y}^\delta\|^2$. If the sequence $\{\lambda_k = \lambda\}$ is constant, iteration (2.4) is called *stationary* iT [16, 7, 15], otherwise it is denominated *nonstationary* iT [6, 10, 3].

In the nonstationary iT methods, each λ_k is chosen either *a priori* (e.g., the geometrical choice $\lambda_k = q^k$, $q > 1$) or *a posteriori* [4, 2]. In this manuscript we focus on the *a posteriori* strategy investigated in [2], where the authors propose a choice for the Lagrange multipliers, which requires the residual at the next iterate to assume a prescribed value dependent on the current residual and also on the noise level. We extend the strategy used in [2], by defining a different range. This allow us to give convergence proof different to the one in [2], which boils down to a particular instance of [3, Theorem 1.4].

Iterated Tikhonov Kaczmarz Type Methods

The method proposed and analyzed in this manuscript for solving the the system of ill-posed problems (2.1), (2.2) is a Kaczmarz type method, where each step is defined as in the iT method (2.4) and the choice of Lagrange multipliers proposed in [2] is adopted. This iterative method is defined by

$$x_{k+1}^\delta = x_k^\delta + h_k, \tag{2.5}$$

where

$$h_k = \begin{cases} \lambda_k (I + \lambda_k A_{[k]}^* A_{[k]})^{-1} A_{[k]}^* \big(y_{[k]}^\delta - A_{[k]} x_k^\delta\big) & \text{, if } \|A_{[k]} x_k^\delta - y_{[k]}^\delta\| > \tau\delta_{[k]} \\ 0 & \text{, otherwise} \end{cases} \tag{2.6}$$

and

$$\lambda_k = \begin{cases} \text{chosen as in Algorithm 2.2.1} & \text{, if } \|A_{[k]} x_k^\delta - y_{[k]}^\delta\| > \tau\delta_{[k]} \\ 0 & \text{, otherwise.} \end{cases} \tag{2.7}$$

We use the notation $[k] = (k \bmod N) \in \{0, 1, \ldots, N-1\}$. Here $x_0^\delta \in X$ is an initial guess and $\tau > 1$ is a fixed constant.

Notice that, if $\|A_{[k]} x_k^\delta - y_{[k]}^\delta\| > \tau\delta_{[k]}$ for some k, then $h_k \in X$ is a tipical step of the iterated Tikhonov method for the $[k]$-th equation $A_{[k]}x = y_{[k]}$ of system (2.2). Otherwise, the computation of (λ_k, h_k) is avoided. We set $\lambda_k = 0$, $h_k = 0$ and $x_{k+1}^\delta = x_k^\delta$.

Following [2] we refer to this method as *range-relaxed iterated Tikhonov Kaczmarz* (rriTK) method. Essentialy, it consists in incorporating the Kaczmarz strategy into the iterated Tikhonov method investigated in [2]. This procedure is analog to the one introduced in [9, 8] regarding the Landweber Kaczmarz (LWK) iteration.

As usual in Kaczmarz type algorithms, a group of N subsequent steps (starting at some multiple of N) is called a cycle. In the noisy data case, the iTK iteration should be terminated when, for the first time, all x_k^δ are equal within a cycle. That is, the iteration is stopped at step $k_* = k_*\left(\{\delta_i\}_i, \{y_i^\delta\}_i\right)$ such that

$$k_* \ := \ \min\left\{lN \,:\, l \in N \text{ and } x_{lN}^\delta = x_{lN+1}^\delta = \cdots = x_{lN+N}^\delta\right\}. \tag{2.8}$$

In other words, $k_* \in \mathbb{N}$ is the smallest multiple of N such that $x_{k_*}^\delta = x_{k_*+1}^\delta = \cdots = x_{k_*+N}^\delta$ or, equivalently, such that $\lambda_{k_*} = \lambda_{k_*+1} = \cdots = \lambda_{k_*+N} = 0$.

Outline of the Manuscript

The article is organized as follows: In Section 2.2 we introduce the rriTK method, on which we focus on this manuscript. A detailed formulation of this method is given and some preliminary results are obtained, including an estimate for the *"gain"* (Proposition 2.2.4), as well as estimates for the Lagrange multipliers λ_k (Corollary 2.2.6) and for the stoping index k_* (Corollary 2.2.8). In Section 2.3 a convergence result for the rriTK method is presented. Section 2.4 is devoted to numerical experiments. A benchmark system of linear ill-posed equations (derived from the Hilbert matrix in $\mathbb{R}^{100,100}$) is considered. The performance of the rriTK method is compared against two other nonstationary iT type methods: the well established geometric iterated Tikhonov method (giT) with $\lambda_k = 2^k$ and the iT method in [2] (rriT). Section 2.5 is dedicated to final remarks and conclusions.

2.2 A Range-relaxed Iterated Tikhonov Kaczmarz Method

In the sequel we introduce the *range-relaxed iterated Tikhonov Kaczmarz* (rriTK) method for solving the ill-posed linear system (2.1), (2.2). Subsection 2.2.1 is devoted to main assumptions needed in the analysis. The new method is presented in Subsection 2.2.2 and a corresponding algorithm is discussed. In Subsection 2.2.3 we derive some basic properties of the proposed method, and prove preliminary results and estimates.

The implementable method proposed here happens to be a nonstationary iTK type method where, in each iteration, the set of feasible choices for the Lagrange multipliers is an interval, instead of a single real number. For this reason, this method is called a (nonstationary) range-relaxed iterated Tikhonov method.

2.2.1 Main Assumptions

For the remaining of this chapter we suppose that the following assumptions hold true:

(A1) There exists $x^\star \in X$ such that $A_i x^\star = y_i$, where $y_i \in R(A_i)$, $i = 0, \dots, N-1$, are the exact data.

(A2) The operators $A_i : X \to Y$ is linear, bounded and ill-posed, i.e., even if the operator $A_i^{-1} : R(A_i) \to X$ (the left inverse of A_i) exists, it is not continuous.

From (A2) it follows the existence of $C > 0$ with $C := \max_i \|A_i\|$. Moreover, we write $\delta := \max_i \delta_i > 0$. Thus, $\delta = 0$ in the exact data case, and $\delta > 0$ in the noisy data case.

2.2.2 Description of the Method

As already discussed in the introduction, the iterative step of the rriTK method is analog to the one proposed in [2]. This step is discussed in the sequel.

For $i = 0, \dots, N-1$ and $\mu > 0$ define the levelsets $\Omega_\mu^i := \{x \in X; \|A_i x - y_i^\delta\| \le \mu\}$ of the residual w.r.t. the i$^{\text{th}}$-equation of system (2.2). Given $k \in \mathbb{N}$, set $i = [k]$. If x_k^δ does not belong to $\Omega_{\delta_i}^i$, the next iterate x_{k+1}^δ is computed by solving the *range-relaxed projection problem*

$$\begin{cases} \min_x \ \|x - x_k^\delta\|^2 \\ \text{s.t. } \|A_i x - y_i^\delta\|^2 \le \mu^2, \quad \bar{\Phi}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i) \le \mu \le \bar{\bar{\Phi}}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i) \end{cases} \tag{2.9}$$

for $(x, \mu) \in X \times \mathbb{R}$. Here

$$\bar{\Phi}(u, v) = (\bar{p}u + (1-\bar{p})v)^{\frac{1}{2}} \quad \text{and} \quad \bar{\bar{\Phi}}(u, v) = (\bar{\bar{p}}u + (1-\bar{\bar{p}})v), \quad \forall\, u, v \in \mathbb{R},$$

with $0 < \bar{p} < \bar{\bar{p}} < 1$.

If (x', μ') is a solution of (2.9), we define $x_{k+1}^\delta = x'$ and $\|A_i x_{k+1}^\delta - y_i^\delta\| = \mu'$ (see Lemma 2.2.1 below). As observed in [2], x_{k+1}^δ is generated from x_k^δ by projecting it onto any one of the range of convex sets $(\Omega_\mu^i)_{\bar{\Phi} \le \mu \le \bar{\bar{\Phi}}}$.

Since the solution of (2.9) is not unique, there are several possible choices for x_{k+1}^δ. The next lemma addresses this issue. For a proof we refer the reader to [2, Lemma 2.3].

Lemma 2.2.1 *Suppose $\|A_i x_k^\delta - y_i^\delta\| > \delta_i$. The following assertions are equivalent*

1. $x' = \Pi_{\Omega_\mu}(x_k^\delta)$ *and* $\bar{\Phi}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i) \le \mu' \le \bar{\bar{\Phi}}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i)$;

2. *$(x', \mu') \in X \times \mathbb{R}$ is a solution of the range-relaxed projection problem (2.9);*

3. *$x' = x_k^\delta - \lambda (I + \lambda A_i^* A_i)^{-1} A_i^* (A_i x_k^\delta - y_i^\delta)$, for some $\lambda > 0$,*

$$\bar{\Phi}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i) \le \|A_i x' - y_i^\delta\| \le \bar{\bar{\Phi}}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i),$$

and $\mu' = \|A_i x' - y_i^\delta\|$;

(here $\Pi_\Omega(x)$ represents the orthogonal projection of x onto the convex set Ω).

It follows from Lemma 2.2.1 that solving the range-relaxed projection problem in (2.9) sums up to solving the inequalities $\bar{\Phi}(\|A_i x_k^\delta - y_i^\delta|, \delta_i) \le \|A_i x' - y_i^\delta\| \le \bar{\bar{\Phi}}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i)$, with $x' = x_k^\delta - \lambda (I + \lambda A_i^* A_i)^{-1} A_i^* (A_i x_k^\delta - y_i^\delta)$ and $\mu' = \|A_i x' - y_i^\delta\|$. This is the quintessential ingredient to define an implementable version of the range-relaxed iterated Tikhonov Kaczmarz (rriTK) method as follows:

Algorithm 2.2.1 Range-relaxed iterated Tikhonov Kaczmarz method (rriTK)

[1] *choose an initial guess $x_0 \in X$; set $k = 0$;*
[2] *choose $0 < \bar{p} < \bar{\bar{p}} < 1$ (with $\bar{p} > \delta \bar{\bar{p}}$) and $\tau > 1$;*
[3] **repeat**
  [3.1] $i = [k]$;
  [3.2] **if** $\left[\|A_i x_k^\delta - y_i^\delta\| > \tau \delta_i \right]$ **then**
    compute $(\lambda_k, h_k) \in \mathbb{R} \times X$ such that
    $$\begin{cases} h_k = -\lambda_k \left(I + \lambda_k A_i^* A_i\right)^{-1} A_i^* \left(A_i x_k^\delta - y_i^\delta\right) \\ \bar{\Phi}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i) \le \|A_i(x_k^\delta + h_k) - y_i^\delta\| \le \bar{\bar{\Phi}}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i) \end{cases}$$
  else
    $\lambda_k = 0$; $h_k = 0$;
  [3.3] $x_{k+1}^\delta = x_k^\delta + h_k$;
  [3.4] $k = k + 1$;
  until $\left[([k] = 0) \text{ and } (\lambda_{k-1} = \lambda_{k-2} = \cdots = \lambda_{k-N} = 0) \right]$;
[4] $k_* = k - N$;

An immediate consequence of Lemma 2.2.1 is the fact that Step [3.2] of Algorithm 2.2.1 is well defined, i.e., it is allways possible to solve the problem for (λ_k, h_k) in this step.

2.2.3 Preliminary Results

For simplicity of notation we write $b_k^\delta := y_i^\delta - A_i x_{k+1}^\delta = y_i^\delta - A_i x_k^\delta - A_i h_k$, with $i = [k]$, and $C > 0$ is defined as above. Moreover, for exact data $y = (y_0, \dots, y_{N-1})$, the iterates in (2.5) are denoted by x_k, in contrast to x_k^δ in the noisy data case (analog notation for $b_k := y_i - A_i x_{k+1}$).

Our first result concerns basic properties of the iterative step of the rriTK method. The proofs of the assertions are straightforward and will be omitted.

Lemma 2.2.2 *Assume that (A1) and (A2) are satisfied and let x_k^δ, h_k, λ_k be defined by (2.5), (2.6) and (2.7) respectively. For all $0 \le k < k_*$ and $i = [k]$, the assertions*

a) $A_i x_{k+1}^\delta - y_i^\delta = (\lambda_k A_i A_i^* + I)^{-1}(A_i x_k^\delta - y_i^\delta)$;

b) $h_k = \lambda_k A_i^* \left(y_i^\delta - A_i x_{k+1}^\delta \right)$;

c) $\left(\bar{p} \, \|A_i x_k^\delta - y_i^\delta\| \right)^{\frac{1}{2}} \le \|A_i x_{k+1}^\delta - y_i^\delta\| \le \|A_i x_k^\delta - y_i^\delta\|$.

hold true whenever $\lambda_k > 0$.

Remark 2.2.3 *Let us consider the exact data case for a moment. From Step [3.2] of Algorithm 2.2.1 we learn that the residual w.r.t. the i^{th}-equation (with $i = [k]$) reduces from iterate x_k to the next iterate x_{k+1}, namely*

$$\|A_i x_{k+1} - y_i\| \;\le\; \bar{\bar{p}} \, \|A_i x_k - y_i\| \,. \tag{2.10}$$

In other words, we have geometrical decay of this residual.

In what follows we estimate the "gain" $\|x_{k+1}^\delta - x^\star\|^2 - \|x_k^\delta - x^\star\|^2$. This is a central result for the analysis derived in this manuscript (all subsequent corollaries in this section derive from the next proposition).

Proposition 2.2.4 *Assume that (A1) and (A2) are satisfied and let x_k^δ, h_k, λ_k be defined by (2.5), (2.6) and (2.7) respectively. For $\delta = \max_i \delta_i$ sufficiently small, it holds*

$$\|x_{k+1}^\delta - x^\star\|^2 - \|x_k^\delta - x^\star\|^2 \;\le\; -2(\bar{p} - \delta \bar{\bar{p}}) \, \lambda_k \, \|A_i x_k^\delta - y_i^\delta\| - \|x_{k+1}^\delta - x_k^\delta\|^2, \tag{2.11}$$

for $k = 0, \dots, k_* - 1$. In particular, in the exact data case ($y_i^\delta = y_i$) we have

$$\|x_{k+1} - x^\star\|^2 - \|x_k - x^\star\|^2 \;\le\; -2\bar{p} \, \lambda_k \, \|A_i x_k - y_i\| - \|x_{k+1} - x_k\|^2, \; k = 0, \dots \tag{2.12}$$

Proof. Let $i = [k]$. If $\|A_i x_k^\delta - y_i^\delta\| \le \tau\delta_i$, then $\lambda_k = 0$ and $x_{k+1}^\delta = x_k^\delta$. Thus, (2.11) is trivial. Otherwise, it follows from Lemma 2.2.2 (b) that

$$\begin{aligned}
&\|x_{k+1}^\delta - x^\star\|^2 - \|x_k^\delta - x^\star\|^2 \\
&\quad = \; 2\,\langle x_{k+1}^\delta - x_k^\delta,\, x_{k+1}^\delta - x^*\rangle - \|x_{k+1}^\delta - x_k^\delta\|^2 \\
&\quad = \; 2\lambda_k\,\langle y_i^\delta - A_i x_{k+1}^\delta,\, A_i(x_{k+1}^\delta - x^\star)\rangle - \|x_{k+1}^\delta - x_k^\delta\|^2 \\
&\quad = \; 2\lambda_k\,\langle y_i^\delta - A_i x_{k+1}^\delta,\, A_i x_{k+1}^\delta - y_i^\delta + y_i^\delta - A_i x^\star\rangle - \|x_{k+1}^\delta - x_k^\delta\|^2 \\
&\quad \le \; 2\lambda_k\left[-\|b_k^\delta\|^2 + \|b_k^\delta\|\,\delta_i\right] - \|x_{k+1}^\delta - x_k^\delta\|^2. \qquad (2.13)
\end{aligned}$$

If $\delta < 1$ then (2.11) follows from $\bar{p}\|A_i x_k^\delta - y_i^\delta\| + (1-\bar{p})\delta_i \le \|b_k^\delta\|^2$ and $\|b_k^\delta\| \le \bar{\bar{p}}\|A_i x_k^\delta - y_i^\delta\| + (1-\bar{\bar{p}})\delta_i$ (see Step [3.2] of Algorithm 2.2.1), together with $\bar{p} > \delta\bar{\bar{p}}$ (see Step [2]). To conclude the proof notice that, in the exact data case $\delta = 0$ and (2.12) follows directly from (2.11). □

Proposition 2.2.4 has several relevant consequences, namely: monotonicity of the iTK method (Corollary 2.2.5); a uniform estimate for the Lagrange multipliers (Corollary 2.2.6); the summability of important series (Corollary 2.2.7); finiteness of the stoping index k_* (Corollary 2.2.8).

Corollary 2.2.5 *Assume that (A1) and (A2) are satisfied and let x_k^δ, h_k, λ_k be defined by (2.5), (2.6) and (2.7) respectively. Then*

$$\|x_{k+1}^\delta - x^\star\|^2 \;\le\; \|x_k^\delta - x^\star\|^2,\; k = 0,\ldots,k_* - 1\,. \qquad (2.14)$$

Additionaly, in the exact data case we have $\|x_{k+1} - x^\star\|^2 \le \|x_k - x^\star\|^2$, *for* $k = 0,1,\ldots$

Corollary 2.2.6 *Assume that (A1) and (A2) are satisfied and let x_k^δ, h_k, λ_k be defined by (2.5), (2.6) and (2.7) respectively. Moreover, let b_k^δ be defined as above. Then*

$$\lambda_k \;\ge\; \frac{\left(\|A_i x_k^\delta - y_i^\delta\| - \|b_k^\delta\|\right)\|A_i x_k^\delta - y_i^\delta\|}{\|A_i^*(A_i x_k^\delta - y_i^\delta)\|^2}\,,\; k = 0,\ldots,k_* - 1\,. \qquad (2.15)$$

Moreover, if $\|A_i x_k^\delta - y_i^\delta\| > \tau\delta_i$ (for some $0 \le k < k_* - 1$) then $\lambda_k > C^{-2}(1-\bar{\bar{p}})(1-\frac{1}{\tau})$, with $C > 0$ defined as above.

Additionaly, in the exact data case we have $\lambda_k \ge C^{-2}(1-\bar{\bar{p}})$, for $k = 0,1,\ldots$

Proof. Let $0 \le k < k_*$. If $\|A_i x_k^\delta - y_i^\delta\| \le \tau\delta_i$ then $\lambda_k = 0$ and $x_{k+1}^\delta = x_k^\delta$. Thus, (2.15) is trivial. On the other hand, if $\|A_i x_k^\delta - y_i^\delta\| > \tau\delta_i$, the proof of (2.15) follows the lines of [2, Corollary 2.5].

To prove the second assertion, notice that Step [3.2] of Algorithm 2.2.1 guarantees $\|b_k^\delta\| \le \bar{\bar{p}}\|A_i x_k^\delta - y_i^\delta\| + (1-\bar{\bar{p}})\delta_i$. Consequently, $\|A_i x_k^\delta - y_i^\delta\| - \|b_k^\delta\| \ge$

$(1-\bar{\bar{p}})(\|A_i x_k^\delta - y_i^\delta\| - \delta_i)$. Thus, we obtain from (2.15)

$$\lambda_k \ \geq \ \frac{\|A_i x_k^\delta - y_i^\delta\| - \|b_k^\delta\|}{C^2\|A_i x_k^\delta - y_i^\delta\|} \ \geq \ \frac{1}{C^2}(1-\bar{\bar{p}})\left(1-\delta_i/\|A_i x_k^\delta - y_i^\delta\|\right) \tag{2.16}$$

and the second assertion follows from the aditional assumption $\|A_i x_k^\delta - y_i^\delta\| > \tau\delta_i$.

Finally, in the case of exact data, $\lambda_k \geq C^{-2}(1-\bar{\bar{p}})$ follows from the first inequality in (2.16) together with $\|b_k\| \leq \bar{\bar{p}}\|A_i x_k - y_i\|$. □

Corollary 2.2.7 *Assume that (A1) and (A2) are satisfied and let x_k, h_k, λ_k be defined by (2.5), (2.6) and (2.7) in the exact data case (i.e., $y_j^\delta = y_j$, $j = 0, \ldots N-1$). Then the series*

$$\sum_{k=0}^{\infty}\|x_{k+1}-x_k\|^2, \ \sum_{k=0}^{\infty}\lambda_k\|A_{[k]}x_k - y_{[k]}\|, \ \sum_{k=0}^{\infty}\lambda_k\|b_k\|, \ \sum_{k=0}^{\infty}\lambda_k^2\|b_k\|^2, \ \sum_{k=0}^{\infty}\|A_{[k]}x_k - y_{[k]}\|$$

are all summable.

Proof. The first two assertions follow from (2.12), using a telescopic series argument. The next two assertions follow from a comparison test and Lemma 2.2.2 (c). The last assertion follows from the second one and Corollary 2.2.6. □

Corollary 2.2.8 *Assume that (A1) and (A2) are satisfied and let x_k^δ, h_k, λ_k be defined by (2.5), (2.6) and (2.7). Then the stopping index k_* defined in (2.8) is finite and*

$$k_* \ \leq \ N\|x_0 - x^\star\|^2\left[\frac{2(\bar{p}-\delta\bar{\bar{p}})(1-\bar{\bar{p}})(\tau-1)}{C^2}\,\delta_{\min}\right]^{-1}. \tag{2.17}$$

Proof. Assume by contradiction that k_* is not finite, i.e., in each cycle $\{lN, \ldots, lN+N-1\}$, $l \in \mathbb{N}$, of the rriTK method, there exists at least one index $j(l) \in \{0, \ldots, N-1\}$ such that $\|A_{j(l)}x_{lN+j(l)} - y_{j(l)}^\delta\| \geq \tau\delta_{j(l)}$.
From Proposition 2.2.4 follows that (2.11) holds for $k \in \mathbb{N}$. Summing over k and using the fact that either $\|A_{[k]}x_k^\delta - y_{[k]}^\delta\| \geq \tau\delta_{[k]}$ or $\lambda_k = 0$, we obtain (with the notation $i = [k]$)

$$\begin{aligned}
\|x_0 - x^*\|^2 &\geq 2(\bar{p}-\delta\bar{\bar{p}})\sum_{k=0}^{lN}\lambda_k\|A_i x_k^\delta - y_i^\delta\| \\
&\geq 2(\bar{p}-\delta\bar{\bar{p}})\sum_{s=0}^{l}\lambda_{sN+j(s)}\|A_{j(s)}x_{sN+j(s)}^\delta - y_{j(s)}^\delta\| \\
&\geq 2(\bar{p}-\delta\bar{\bar{p}})\sum_{s=0}^{l}\lambda_{sN+j(s)}\tau\delta_{j(s)} \ \geq \ l\,\tfrac{2(\bar{p}-\delta\bar{\bar{p}})(1-\bar{\bar{p}})(\tau-1)}{C^2}\,\delta_{\min}\,, \ l \in \mathbb{N}
\end{aligned} \tag{2.18}$$

(the last inequality follows from Corollary 2.2.6). Since the right hand side of (2.18) becomes unbounded as $l \to \infty$ a contradiction is established, and the finiteness of k_* follows. Estimate (2.17) follows now substituting $k_* = lN$ in (2.18). □

2.3 A Convergence Result for Exact Data

Our main goal in this section is to prove convergence of the rriTK method in the case $\delta_i = 0$, $i = 0, \ldots, N-1$. Notice that, in this exact data case, $\lambda_k > 0$ and $h_k = x_{k+1} - x_k = 0$ if and only if $\|A_i x_k - y_i\| = 0$ (see Step [3.2] of Algorithm 2.2.1).

Theorem 2.3.1 ((Convergence for exact data)) *Assume that (A1) and (A2) are satisfied and let x_k, h_k, λ_k be defined by (2.5), (2.6) and (2.7) in the exact data case (i.e., $y_i^\delta = y_i$, $i = 0, \ldots\ N-1$). Then x_k converges to a solution of (2.2) as $k \to \infty$.*

Proof. We define $e_k := x^\star - x_k$. From Corolary 2.2.5 follows that $\|e_k\|$ is monotone non-increasing. Thus, $\|e_k\|$ converges to some $\varepsilon \geq 0$. In what follows we show that e_k is in fact a Cauchy sequence.

In order to prove that e_k is indeed a Cauchy sequence, it suffices to prove $|\langle e_n - e_k, e_n\rangle| \to 0$, $|\langle e_n - e_l, e_n\rangle| \to 0$ as $k, l \to \infty$ with $k \leq l$ for some $k \leq n \leq l$ [11, Theorem 2.3]. Let $k \leq l$ be arbitrary and write $k = k_0 N + k_1$, $l = l_0 N + l_1$, with $k_1, l_1 \in \{0, \ldots, N-1\}$. Now let $n_0 \in \{k_0, \ldots, l_0\}$ be such that

$$\sum_{s=0}^{N-1} \lambda_{n_0 N+s} \|A_s x_{n_0 N+s} - y_s\| \leq \sum_{s=0}^{N-1} \lambda_{i_0 N+s} \|A_s x_{i_0 N+s} - y_s\|, \text{ for all } i_0 \in \{k_0, \ldots, l_0\}, \tag{2.19}$$

and set $n = n_0 N + N - 1$. Therefore

$$\begin{aligned} |\langle e_n - e_k, e_n\rangle| &= \left|\textstyle\sum_{i=k}^{n-1} \langle (x_{i+1} - x_i), (x_n - x^\star)\rangle\right| \\ &= \left|\textstyle\sum_{i=k}^{n-1} \lambda_i \langle A_{[i]} x_{i+1} - y_{[i]}, A_{[i]} x_n - A_{[i]} x^\star\rangle\right| \\ &\leq \textstyle\sum_{i_0=k_0}^{n_0} \sum_{i_1=0}^{N-1} \lambda_i \|A_{i_1} x_{i+1} - y_{i_1}\| \, \|A_{i_1} x_n - y_{i_1}\| \\ &\leq \textstyle\sum_{i_0=k_0}^{n_0} \sum_{i_1=0}^{N-1} \lambda_i \|b_i\| \, \|A_{i_1} x_n - y_{i_1}\| \end{aligned} \tag{2.20}$$

(we use the notation $i = i_0 N + i_1$). The last term on the right hand side of (2.20) can be estimated by

$$
\begin{aligned}
\|A_{i_1} x_n - y_{i_1}\| &= \|A_{i_1} x_{n_0N+N-1} - y_{i_1}\| \\
&\le \|A_{i_1} x_{n_0N+i_1+1} - y_{i_1}\| + \textstyle\sum_{s=i_1+1}^{N-2} \|A_{i_1} x_{n_0N+s+1} - A_{i_1} x_{n_0N+s}\| \\
&\le \|A_{i_1} x_{n_0N+i_1+1} - y_{i_1}\| + \textstyle\sum_{s=i_1+1}^{N-2} C\|x_{n_0N+s+1} - x_{n_0N+s}\| \\
&= \|A_{i_1} x_{n_0N+i_1+1} - y_{i_1}\| + \textstyle\sum_{s=i_1+1}^{N-2} C\lambda_{n_0N+s}\|A_s^*(A_s x_{n_0N+s+1} - y_s)\| \\
&\le \|A_{i_1} x_{n_0N+i_1+1} - y_{i_1}\| + \textstyle\sum_{s=0}^{N-1} C^2\lambda_{n_0N+s}\|A_s x_{n_0N+s+1} - y_s\| \\
&\le (\tfrac{1}{\lambda_{\min}} + C^2)\textstyle\sum_{s=0}^{N-1} \lambda_{n_0N+s}\|A_s x_{n_0N+s+1} - y_s\|
\end{aligned}
$$

(with $\lambda_{\min} = C^{-2}(1-\bar{\bar{p}})$, cf. Corolary 2.2.6). Hence, by the minimality property (2.19) follows $\|A_{i_1} x_n - y_{i_1}\| \le (\frac{1}{\lambda_{\min}} + C^2)\sum_{s=0}^{N-1} \lambda_{i_0N+s}\|A_s x_{i_0N+s+1} - y_s\|$, for $i_0 \in \{k_0, \ldots, l_0\}$. Inserting this last inequality into (2.20) we obtain

$$
\begin{aligned}
|\langle e_n - e_k, e_n\rangle| &\le (\tfrac{2-\bar{\bar{p}}}{1-\bar{\bar{p}}})C^2\textstyle\sum_{i_0=k_0}^{n_0} \sum_{i_1=0}^{N-1}\lambda_i\|b_i\| \left[\sum_{s=0}^{N-1} \lambda_{i_0N+s}\|A_s x_{i_0N+s+1} - y_s\|\right] \\
&= (\tfrac{2-\bar{\bar{p}}}{1-\bar{\bar{p}}})C^2\textstyle\sum_{i_0=k_0}^{n_0} \left[\sum_{i_1=0}^{N-1}\lambda_i\|b_i\|\right]^2 \\
&\le (\tfrac{2-\bar{\bar{p}}}{1-\bar{\bar{p}}})C^2 N\textstyle\sum_{i_0=k_0}^{n_0} \sum_{i_1=0}^{N-1}\lambda_i^2\|b_i\|^2 \\
&= (\tfrac{2-\bar{\bar{p}}}{1-\bar{\bar{p}}})C^2 N\textstyle\sum_{i=k_0}^{n}\lambda_i^2\|b_i\|^2.
\end{aligned}
\tag{2.21}
$$

Hence by Corolary 2.2.7 the right hand side of (2.21) go to zero as $k,\, l \to \infty$. Analogously one shows that $|\langle e_n - e_l, e_n\rangle| \to 0$ as $k,\, l \to \infty$.

Thus, e_k is a Cauchy sequence and x_k converges to some $x^+ \in X$. Since the residuals $\|A_{[k]} x_k - y_{[k]}\|$ converge to zero as $k \to \infty$ (see Corolary 2.2.7), this x^+ is a solution of (2.2). □

2.4 Numerical Experiments

In this section the rriTK method (see Algorithm 2.2.1) is implemented for solving a benchmark problem, which happens to be a well known system of linear ill-posed equations.

The setup is inspired in [17, Introduction]. Let the operator $\mathbf{A} = \big[a_{i,j} = 1/(i+j-1)\big]_{i,j=1}^{24}$ be a Hilbert matrix. The Hilbert matrix is scaled such that each line a_i of $\mathbf{A}$ satisfy $\|a_i\| = 1$.

Set $N = 8$ and define $A_i \in \mathbb{R}^{3,24}$ to be the block of $\mathbf{A}$ formed by lines $a_{3i}, \ldots, a_{3i+2}$, $i = 0, \ldots, N-1$. In this setup we have $X = \mathbb{R}^{24}$ and $y_i \in Y = \mathbb{R}^3$.

The performance of the rriTK method is compared against three concurrent Kaczmarz type methods: (i) the Landweber Kaczmarz (LWK) method; (ii) the stationary iTK (siTK) method with constant $\lambda_k = 2$; (iii) the geometric iTK method (giTK) with $\lambda_k = 2^k$.

In our numerical experiments we set $x^\star = (1,\dots,1) \in X$ and compute the corresponding exact data $y_i = A_i x^\star$. The noise levels are $\delta_i = \delta = 0.1\%$; noisy data y_i^δ satisfying (2.1) is generated by adding to y_i randomly generated noise. The discrepancy principle constant is $\tau = 4.0$. Moreover, in Step [3.2] of Algorithm 2.2.1 we use the constants $\bar{p} = 0.1$ and $\bar{\bar{p}} = 0.8$

The first two plots in Figure 2.1 (TOP and CENTER) show the evolution of iteration error and residual (respectively) for the Kaczmarz type methods implemented in this section. The LWK (gray) reached the discrepancy principle after 3050 cycles; the siTK (black) after 1528 cycles, the giTK (blue) after 23 cycles, and the rriTK (red) after 10 cycles. In these plots the x-axis shows the number of computed cycles.

In the last plot of Figure 2.1 (BOTTOM) the values of $\|A_i x_{k+1}^\delta - y_i^\delta\|$, $i = [k]$, for the rriTK method are shown (black). Moreover, in order to verify the two inequalities in Step [3.2] of Algorithm 2.2.1, the upper bound $\bar{\bar{\Phi}}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i)$ (green) and the lower bound $\bar{\Phi}(\|A_i x_k^\delta - y_i^\delta\|, \delta_i)$ (magenta) are shown. In this plot the x-axis shows the number of effectively computed iterates x_k^δ. Although 10 cycles were computed (i.e., a total of 80 steps), only 44 iterates x_k were actually computed. For the remaining 36 steps, the residual of the current equation $\|A_i x_k^\delta - y_i^\delta\|$ was below the discrepancy $\tau\delta_i$; consequently, $x_{k+1}^\delta = x_k^\delta$ in Step [3.2] avoiding the task of solving problem (2.9).

A careful reader will notice that the comparison of methods presented in Figure 2.1 is far from being fair. Indeed, in Kaczmarz type methods, the computation of an iterative step is avoided (i.e., $x_{k+1}^\delta = x_k^\delta$) whenever the residual satisfies $\|A_i x_k^\delta - y_i^\delta\| \le \tau\delta_i$. Consequently, the numerical burden of computing a cycle differs from method to method (as well as from cycle to cycle of the same method). Therefore, plotting iteration errors (or residuals) after each cycle does not give a proper comparison of the efficiency of these methods.

In Figure 2.2 the evolution of iteration error and residual (for the same Kaczmarz type methods as before) are plotted as functions of effectively computed iterates x_k. This allows a fair comparison between these methods, since the number of computed iterates is proportional to the total computational burden of a iTK type method.

It is worth noticing that, in Kaczmarz type methods we have monotonicity of the iteration error $\|x_k^\delta - x^\star\|$, see Corollary 2.2.5. However monotonicity of residual $\|\mathbf{A}x_k^\delta - \mathbf{y}^\delta\|$ cannot be guaranteed. In this regard, the best result available is Lemma 2.2.2 (c). These two facts can be observed in both Figures 2.1 and 2.2.

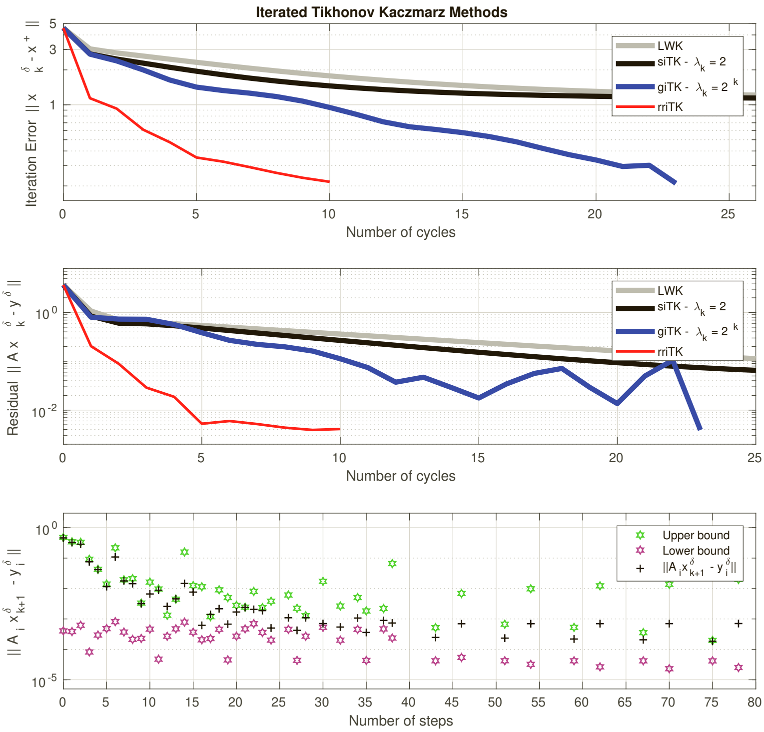

Figure 2.1: (TOP) Iteration error; (CENTER) Residual; (BOTTOM) Inequalities in Step [3.2].

2.5 Conclusions

We investigate nonstationary iTK type methods for computing stable approximate solutions to systems of linear ill-posed operator equations. The main contribution of this chapter is to extend the strategy for choosing the Lagrange multipliers in [2] (we propose a different range). This modification allow us to couple the iT method with the Kaczmarz strategy and also to give a convergence proof (Section 2.3) completely different from the one in [2].

This strategy is advantageous, since it allows each of these multipliers to belong to a non-degenerate interval. Consequently, the actual computation of Lagrange multipliers satisfying the theoretical requirements for the convergence analysis (Step [3.2]) is in much simplified.

We prove monotonicity of the proposed rriTK method (2.14) and verify geometrical decay of the residual (2.10). Moreover, we provide estimates to the "gain" $\|x^\star - x_k^\delta\|^2 - \|x^\star - x_{k+1}^\delta\|^2$ in (2.11) and (2.12), as well as a lower bound to the Lagrange multipliers (2.15), and an estimate to stopping index k_* (2.17). A convergence proof in the case of exact data is given (Theorem 2.3.1).

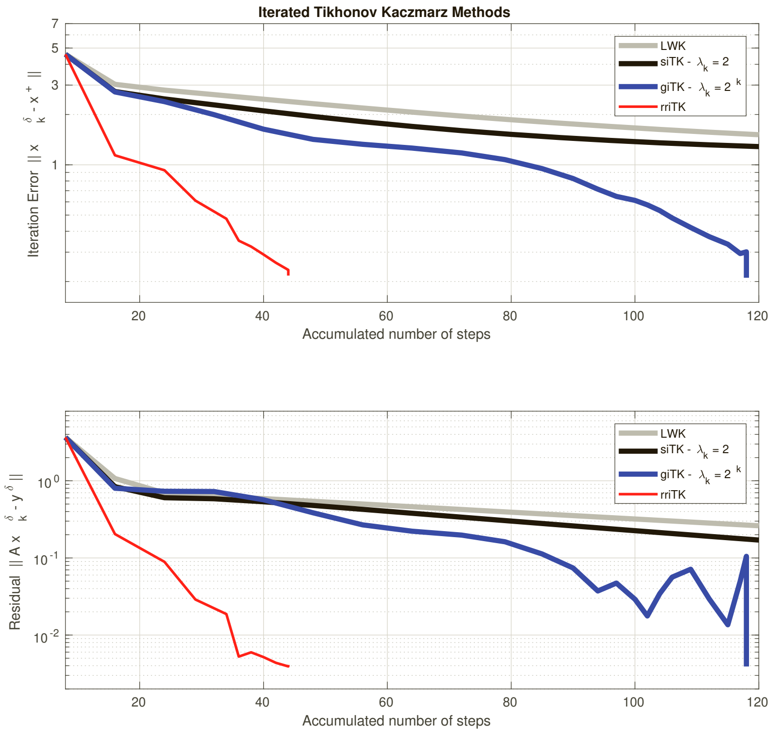

Figure 2.2: Iteration error and residual as functions of the accumulated number of steps.

An algorithmic implementation of the rriTK method is proposed (Algorithm 2.2.1). The resulting rriTK algorithm is competitive with giTK and also with other two well known Kaczmarz type methods (LWK and siTK).

The rriTK is tested for a well known benchmark problem modeled by a Hilbert matrix in the noisy data case. The obtained results validate the efficiency of our method.

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