New Algorithms for Parallel MRI

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Abstract. Magnetic Resonance Imaging with parallel data acquisition requires algorithms for reconstructing the patient's image from a small number of measured lines of the Fourier domain (k-space).

In contrast to well-known algorithms like SENSE and GRAPPA and its flavors we consider the problem as a non-linear inverse problem. However, in order to avoid cost intensive derivatives we will use Landweber-Kaczmarz iteration and in order to improve the overall results some additional sparsity constraints.

1. Introduction

The *Magnetic Resonance Imaging*, also known as MR imaging or simply MRI, is one of the most popular and powerful medical imaging techniques.

Using strong magnetic fields the atoms (mostly hydrogen) in a specified slice of the body are turned into small radio stations emitting signals with a specific frequency. These signals can be detected simultaneously with multiple antenna coils placed around the body (therefore "Parallel MRI").

As a result we get the 2D Fourier transform of the water distribution of this particular slice. In one pass one can read one line of this 2D Fourier transform; the acquiring time of this line is due to medical and technical constraints already at the lower possible limit. Hence a speed up of the method can just be achieved by measuring less data, ie. less lines in the Fourier domain. As all antenna coils have different spatial characteristics (i.e. they illuminate different parts of the image) even now from the theoretical point of view all necessary information to get a high resolution image are hidden in the data. We want to retrieve these information from the parallel measurements by appropriate mathematical algorithms. These have to cope with two problems, on the one hand side they have to keep the noise at a reasonable level and on the other hand need to keep undersampling (i.e. in the spatial domain overfolding) artifacts as low as possible. An additional complication is posed by the problem that the sensitivity characteristics of the antenna coils is not very well known and may change from patient to patient.

This problem has been largely investigated and a number of one pass methods have been developed, e.g. SMASH [1], SENSE [2], GRAPPA [3] and SPACE-RIP [4].

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Recently we have investigated [5] an alternative algorithm which considers this problem as non-linear inverse problem using IRGNM [6, 7] for regularization. As we have to solve several times a very high dimensional linear equation for this method speed is still an issue and therefore we seek for other less demanding iterative methods.

This article is outlined as follows. In Section 2 the description of a mathematical model for Magnetic Resonance Imaging is presented, and the inverse problem for MRI is introduced. Then we will perform a two step approach: In the first step (Section 3) we will generate a reasonable (i.e. very smooth) guess for the receiver sensitivities accepting that the resulting guess for the image will be too smooth as well. Using a particular assumption on the sensitivity kernels, we are able to derive convergence and stability results for our iterative methods. In a second step (Section 4) we will freeze the receiver sensitivities and use adapted image processing tools (mainly sparsity constraints) to regain a reasonable sharp image.

2. Direct and inverse problems

A discrete mathematical model The measurement $\mathcal{M}_j \in Y$ of one of the N antenna coils (N typically ranging from 12 to 36) can be written in the following way

$$\mathcal{M}_j = F_j(\mathcal{P}, \mathcal{S}_j) := \mathbf{P}[\mathcal{F}(\mathcal{P} \times \mathcal{S}_j)] \tag{1}$$

where we have the following notation. $\mathcal{P} \in \tilde{X} = \mathbb{R}^K$ shall denote the $2D$ image which we would like to reconstruct. Typical sizes of $\mathcal{P}$ found in applications are either $K = 256 \times 256$ or $K = 512 \times 512$ pixels.

$\mathcal{S}_j \in \mathbb{C}^K$ is the sensitivity profile of this specific antenna coil. Normally one knows (or can rapidly estimate) a rough approximation of this quantity using the fact that from the engineering side one tries that it roughly holds the (pointwise) equation

$$\sum_{j=1}^{N} \mathcal{S}_j^2 = 1. \tag{2}$$

Additionally it is known that all $\mathcal{S}_j$ are very smooth quantities.

$\times$ denotes the point-wise product, i.e. we do not have a convolution as in similar problems like blind deconvolution.

$\mathcal{F}$ is the *Discrete Fourier Transform* (DFT). The numerical evaluation of the DFT requires $O(p_{num}^2)$ arithmetical operations. If p_{num} is a power of two, the DFT can be replaced by the *Fast Fourier Transform* (FFT), which can be computed by the Cooley-Tukey algorithm[3] and requires only $O(p_{num} \log(p_{num}))$ operations.

$\mathbf{P}$ is a projection operator which in the typical setting selects the center lines of the Fourier domain and in the high frequent parts uses every k^{th} line.

Hence combining the above information $F_i : X \to Y$ is a bilinear operator.

Formulation of the inverse problem Next we use the discrete model above as starting point to formulate an inverse problem for MRI.

Due to the experimental nature of the data acquisition process, we shall assume that the data $\mathcal{M}_i$ in (1) is not exactly known. Instead, we have only approximate measured data $\mathcal{M}_i^\delta \in Y$ satisfying

$$\|\mathcal{M}_i^\delta - \mathcal{M}_i\| \le \delta_i\,, \tag{3}$$

[3] The FFT algorithm was published independently by J.W. Cooley and J.W. Tukey in 1965. However, this algorithm was already known to C.F. Gauß around 1805.

with $\delta_i > 0$ (noise level). Therefore, the inverse problem for MRI can be written in the form of the following system of nonlinear equations

$$F_i(\mathcal{P}, \mathcal{S}_i) \ = \ \mathcal{M}_i^\delta \, , \ i = 0, \ldots, N-1 \, . \tag{4}$$

It is worth noticing that the nonlinear operators F_i's are continuously Fréchet differentiable, and the derivatives are locally Lipschitz continuous:

$$F_i[\overline{\mathcal{P}}, \overline{\mathcal{S}}_i]'(\mathcal{P}, \mathcal{S}_i) = \mathbf{P}[\mathcal{F}(\mathcal{P} \times \overline{\mathcal{S}}_j + \overline{\mathcal{P}} \times \mathcal{S}_j)]. \tag{5}$$

3. Iterative regularization

In this section we analyze efficient iterative methods for obtaining stable solutions of the inverse problem in (4).

3.1. An image identification problem

Our first goal is to consider a simplified version of problem (4). We assume that the sensitivity kernels $\mathcal{S}_j$ are known, and we have to deal with the problem of determining only the image $\mathcal{P}$. This assumption can be justified by the fact that, in the praxis, one has very good approximations for the sensitivity kernels, while the image $\mathcal{P}$ is completely unknown.

In this particular case, the inverse problem reduces to

$$\tilde{F}_i(\mathcal{P}) \ = \ \mathcal{M}_i^\delta \, , \ i = 0, \ldots, N-1 \, , \tag{6}$$

where $\tilde{F}_i(\mathcal{P}) = F_i(\mathcal{P}, \mathcal{S}_i)$. This is a much simpler problem, since $\tilde{F}_i : \tilde{X} \to Y$ are linear and bounded operators.

We follow the approaches in [8, 9] and derive two iterative regularization methods of Kaczmarz type for problem (6). Both iterations can be written in the form

$$\mathcal{P}_{k+1}^\delta = \mathcal{P}_k^\delta - \omega_k \alpha_k s_k \, , \tag{7}$$

where

$$s_k := \tilde{F}_{[k]}(\mathcal{P}_k^\delta)^* (\tilde{F}_{[k]}(\mathcal{P}_k^\delta) - \mathcal{M}_i^\delta) \, , \tag{8}$$

$$\omega_k := \begin{cases} 1 & \|\tilde{F}_{[k]}(\mathcal{P}_k^\delta) - \mathcal{M}_i^\delta\| > \tau \delta_{[k]} \\ 0 & \text{otherwise} \end{cases} . \tag{9}$$

Here $\tau > 2$ is an appropriately chosen constant, $[k] := (k \mod N) \in \{0, \ldots, N-1\}$ (a group of N subsequent steps, starting at some multiple k of N, is called a *cycle*), $\mathcal{P}_0^\delta = \mathcal{P}_0 \in \tilde{X}$ is an initial guess, possibly incorporating some *a priori* knowledge about the exact image, and $\alpha_k \geq 0$ are relaxation parameters. Distinct choices for the relaxation parameters α_k lead to the definition of the two iterative methods:

1) If α_k is defined by

$$\alpha_k := \begin{cases} \|s_k\|^2 / \|\tilde{F}_{[k]}(\mathcal{P}_k^\delta) s_k\|^2 & \omega_k = 1 \\ 1 & \omega_k = 0 \end{cases} , \tag{10}$$

the iteration (7) corresponds to the *loping Steepest-Descent-Kaczmarz method* (lSDK) [9].

2) If $\alpha_k \equiv 1$, (7) corresponds to the *loping Landweber-Kaczmarz method* (lLK) [8].

The iterations should be terminated at k_*^δ when, for the first time, all $\mathcal{P}_k$ are equal within a cycle, i.e. all ω_k were 0 in the cycle.

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Convergence analysis of the lSDK method From (1) follows that the operators $\tilde{F}_i$ are linear and bounded. Therefore, there exist $M > 0$ such that

$$\|\tilde{F}_i\| \leq M\,,\ i = 0, \dots, N-1\,. \tag{11}$$

Since the operators $\tilde{F}_i$ are linear, the *local tangential cone condition* is trivially satisfied (see (20) below). Thus, the constant τ in (9) can be chosen such that $\tau > 2$. Moreover, we assume the existence of

$$\mathcal{P}^* \in B_{\rho/2}(\mathcal{P}_0) \ \text{ such that } \ \tilde{F}_i(\mathcal{P}^*) = \mathcal{M}_i\,,\ i = 0, \dots, N-1\,, \tag{12}$$

where $\rho > 0$ and $(\mathcal{M}_i)_{i=0}^{N-1} \in Y^N$ denotes to exact data satisfying (3).

In the sequel we summarize several properties of the lSDK iteration. For a complete proof of the results we refer the reader to [9, Section 2].

Lemma 3.1 *Let the coefficients α_k be defined as in* (10)*, the assumption* (12) *be satisfied for some $\mathcal{P}^* \in \tilde{X}$, and the stopping index k_*^δ be defined as above. Then the following assertions hold true:*

1) *There exists a constant $\underline{\alpha} > 0$ such that $\alpha_k > \underline{\alpha}$, for $k = 0, \dots, k_*^\delta$.*
2) *Let $\delta_i > 0$ in* (3)*. Then the stopping index k_*^δ is finite.*
3) *$\mathcal{P}_k^\delta \in B_{\rho/2}(\mathcal{P}_0)$ for all $k \leq k_*^\delta$.*
4) *The following monotony property is satisfied:*

$$\|\mathcal{P}_{k+1}^\delta - \mathcal{P}^*\|^2 \ \leq \ \|\mathcal{P}_k^\delta - \mathcal{P}^*\|^2\,,\ k = 0, 1, \dots, k_*^\delta\,, \tag{13}$$

$$\|\mathcal{P}_{k+1}^\delta - \mathcal{P}^*\|^2 \ = \ \|\mathcal{P}_k^\delta - \mathcal{P}^*\|^2\,,\ k > k_*^\delta\,. \tag{14}$$

Moreover, in the case of noisy data (i.e. $\delta_i > 0$) we have

$$\|\tilde{F}_i(\mathcal{P}_{k_*^\delta}^\delta) - \mathcal{M}_i^\delta\| \leq \tau\delta_i\,,\ i = 0, \dots, N-1\,. \tag{15}$$

The lSDK method is a convergent regularization method in the sense of [10] which has been proven in [9]. Please note that (14) is a trivial consequence of the definition of k_*^δ and hence we can indeed stop at k_*^δ as afterwards nothing changes any more.

Theorem 3.2 (Convergence) *Let α_k be defined as in* (10)*, the assumption* (12) *be satisfied for some $\mathcal{P}^* \in \tilde{X}$, and the data be exact, i.e. $\mathcal{M}_i^\delta = \mathcal{M}_i$ in* (3)*. Then, the sequence $\mathcal{P}_k^\delta$ defined in* (7) *converges to a solution of* (6) *as $k \to \infty$.*

Theorem 3.3 (Stability) *Let the coefficients α_k be defined as in* (10)*, and the assumption* (12) *be satisfied for some $\mathcal{P}^* \in \tilde{X}$. Moreover, let the sequence $\{(\delta_{1,m}, \dots, \delta_{N,m})\}_{m\in\mathbb{N}}$ (or simply $\{\delta_{\mathbf{m}}\}_{m\in\mathbb{N}}$) be such that $\lim_{m\to\infty}(\max_i \delta_{i,m}) = 0$, and let $\mathcal{M}_i^{\delta_{\mathbf{m}}}$ be a corresponding sequence of noisy data satisfying* (3) *(i.e. $\|\mathcal{M}_i^{\delta_{\mathbf{m}}} - \mathcal{M}_i\| \leq \delta_{i,m}$, $i = 0, \dots, N-1$, $m \in \mathbb{N}$). For each $m \in \mathbb{N}$, let k_*^m be the stopping index defined above. Then, the lSDK iterates $\mathcal{P}_{k_*^m}^{\delta_{\mathbf{m}}}$ converge to a solution of* (6) *as $m \to \infty$.*

Convergence analysis of the lLK method The convergence analysis results for the lLK iteration are analog to the ones presented in Theorems 3.2 and 3.3 for the lSDK method. In the sequel we summarize the main results that we could extend to the lLK iteration [8].

Theorem 3.4 (Convergence Analysis) *Let* $\alpha_k \equiv 1$*, the assumption* (12) *be satisfied for some* $\mathcal{P}^* \in \tilde{X}$*, the operators* $\tilde{F}_i$ *satisfy* (11) *with* $M = 1$*, and the stopping index* k_*^δ *be defined as above. Then the following assertions hold true:*

1) *Let* $\delta_i > 0$ *in* (3)*. Then the stopping index* k_*^δ *defined is finite.*
2) $\mathcal{P}_k^\delta \in B_{\rho/2}(\mathcal{P}_0)$ *for all* $k \le k_*^\delta$*.*
3) *The monotony property in* (13)*,* (14) *is satisfied. Moreover, in the case of noisy data,* (15) *holds true.*
4) *For exact data, i.e.* $\delta_i = 0$ *in* (3)*, the sequence* $\mathcal{P}_k^\delta$ *defined in* (7) *converges to a solution of* (6) *as* $k \to \infty$*.*
5) *Let the sequence* $\{\delta_{\mathbf{m}}\}_{m\in\mathbb{N}}$*, the corresponding sequence of noisy data* $\mathcal{M}_i^{\delta_{\mathbf{m}}}$*, and the stopping indexes* k_*^m *be defined as in Theorem 3.3. Then, the lLK iterates* $\mathcal{P}_{k_*^m}^{\delta_{\mathbf{m}}}$ *converge to a solution of* (6) *as* $m \to \infty$*.*

Notice that the assumption $M = 1$ in Theorem 3.4 is nonrestrictive. Indeed, since the operators $\tilde{F}_i$ are linear, it is enough to scale the equations in (6) with appropriate multiplicative constants.

3.2. Identification of image and sensitivity

Our next goal is to consider the problem of determining both the image $\mathcal{P}$ as well as the sensitivity kernels $\mathcal{S}_j$ in (4). The lLK and lSDK iterations can be extended to the nonlinear system in a straightforward way

$$(\mathcal{P}_{k+1}^\delta, (\mathcal{S}_j)_{k+1}^\delta) = (\mathcal{P}_k^\delta, (\mathcal{S}_j)_k^\delta) - \omega_k \alpha_k s_k \,, \tag{16}$$

where

$$s_k := F_{[k]}'(\mathcal{P}_k^\delta, (\mathcal{S}_j)_k^\delta)^* (F_{[k]}(\mathcal{P}_k^\delta, (\mathcal{S}_j)_k^\delta) - \mathcal{M}_i^\delta) \,, \tag{17}$$

$$\omega_k := \begin{cases} 1 & \|F_{[k]}(\mathcal{P}_k^\delta, (\mathcal{S}_j)_k^\delta) - \mathcal{M}_i^\delta\| > \tau\delta_{[k]} \\ 0 & \text{otherwise} \end{cases} . \tag{18}$$

In the lLK iteration we choose $\alpha_k \equiv 1$, and in the lSDK iteration we choose

$$\alpha_k := \begin{cases} \|s_k\|^2 / \|F_{[k]}'(\mathcal{P}_k^\delta, (\mathcal{S}_j)_k^\delta) s_k\|^2 & \omega_k = 1 \\ 1 & \omega_k = 0 \end{cases} . \tag{19}$$

In order to extend the convergence results in [8, 9] for these iterations, we basically have to prove two facts:

1) Assumption (14) in [8].
2) The *local tangential cone condition* [8, Eq. (15)], i.e. the existence of $(\mathcal{P}_0, (\mathcal{S}_j)_0) \in X$ and $\eta < 1/2$ such that

$$\|F_i(\mathcal{P}, \mathcal{S}_j) - F_i(\bar{\mathcal{P}}, \bar{\mathcal{S}}_j) - F_i'(\mathcal{P}, \mathcal{S}_j)[(\mathcal{P}, \mathcal{S}_j) - (\bar{\mathcal{P}}, \bar{\mathcal{S}}_j)]\|_Y \le \eta \|F_i(\mathcal{P}, \mathcal{S}_j) - F_i(\bar{\mathcal{P}}, \bar{\mathcal{S}}_j)\|_Y \,, \tag{20}$$

for all $(\mathcal{P}, \mathcal{S}_j)$, $(\bar{\mathcal{P}}, \bar{\mathcal{S}}_j) \in B_\rho(\mathcal{P}_0, (\mathcal{S}_j)_0)$, and all $i = 1, \dots, N$.

The first one represents no problem. Indeed, the Fréchet derivatives of the operators F_i are locally Lipschitz continuous. Thus, for any $(\mathcal{P}_0, (\mathcal{S}_j)_0) \in X$ and any $\rho > 0$ we have $\|F_i'(\mathcal{P}, (\mathcal{S}_j))\| \leq M = M_{\rho,\mathcal{P}_0,(\mathcal{S}_j)_0}$ for all $(\mathcal{P}, (\mathcal{S}_j))$ in the ball $B_\rho(\mathcal{P}_0, (\mathcal{S}_j)_0) \subset X$.

For the local cone condition we can rewrite using the definitions

$$\|\mathbf{P}[\mathcal{F}[(\mathcal{P} - \overline{\mathcal{P}}) \times (\mathcal{S}_j - \overline{\mathcal{S}}_j)]]\|_Y \leq \eta \|\mathbf{P}[\mathcal{F}(\mathcal{P} \times \mathcal{S}_j - \overline{\mathcal{P}} \times \overline{\mathcal{S}}_j)]\|_Y \,, \tag{21}$$

Obviously this condition does not need to hold; however the breakdown will just occur when $\mathcal{P} - \overline{\mathcal{P}}$ and $\mathcal{S}_j - \overline{\mathcal{S}}_j$ are big (which we rule out anyway) or $(\mathcal{P} - \overline{\mathcal{P}}) \times \overline{\mathcal{S}}_j \approx \mathcal{P} \times (\mathcal{S}_j - \overline{\mathcal{S}}_j)$ point-wise.

Therefore, the techniques used to prove convergence of the lLK and lSDK iterations in [8, 9] cannot be extended to the nonlinear system (4), however in practice it can be consider highly unlikely that the above case happens.

It is worth noticing that the local tangential cone condition is a standard assumption in the convergence analysis of adjoint type methods (Landweber, steepest descent, Levenberg-Marquardt, asymptotical regularization) for nonlinear inverse problems [10, 11, 12, 13, 6, 14, 15]. Thus, none of the classical convergence proofs for these iterative methods can be extended to system (4).

The general problem with the approach presented up till now is that it will though returning mathematically seen good results be not very valuable for the medical imaging part where as in every imaging application one has an emphasis on small scale high frequent structures like edges. Therefore we will use this approach to generate a good approximation to the sensitivities which we can use for the approach presented in the next section.

4. Tikhonov regularization

As mentioned earlier, the above algorithm produces reconstructions of the sensitivities $\mathcal{S}_j$ as well as of the image $\mathcal{P}$. Since the sensitivities are essentially smooth functions their approximations will be good, whereas the image we obtain would be 'oversmoothed', i.e. we would not get a good reconstruction of edges.

Therefore, as a second step we propose to reconsider the linear inverse problem (6) using only the sensitivities from the above algorithm. In order to get better results, this time we use a procedure that is suitable for jumps in the data such as edges. One way of doing this is by using bounded variation penalization, but this has the disadvantage of having a high numerical complexity, which is why we propose to use a Besov space penalty instead. In particular, we want to minimize the functional

$$J_\alpha(\mathcal{P}) = \sum_{i=0}^{N-1} \|\tilde{\mathcal{F}}_i(\mathcal{P}) - \mathcal{M}_i^\delta\|^2 + \alpha \cdot \|\mathcal{P}\|_{B^s_{p,p}}, \tag{22}$$

where $\|.\|_{B^s_{p,p}}$ denotes the norm of the Besov space $B^s_{p,p}$. If we take a sufficiently smooth wavelet basis $(\varphi_\gamma)_{\gamma\in\Gamma}$ of $L^2(\mathbb{R}^d)$ and denote the coefficients of the expansion with respect to this basis by $\mathcal{P}_\gamma$, i.e.

$$\mathcal{P}_\gamma = \langle \mathcal{P}, \varphi_\gamma \rangle, \tag{23}$$

then the norms $B^s_{p,p}$ are equivalent to the norms

$$\|\mathcal{P}\|_{s,p} := \Big(\sum_{\gamma\in\Gamma} 2^{\sigma p|\gamma|} \, |\langle \mathcal{P}, \varphi_\gamma \rangle| \Big)^{1/p} \tag{24}$$

where $\sigma = s + d(\frac{1}{2} - \frac{1}{p}) \geq 0$. In our case the dimension d is equal to 2 and with the particular choice $p = 1$ the norm is simply the ℓ^1-norm of the coefficients $(\mathcal{P}_\gamma)_\gamma$. The functional (22) then reduces to

$$J_\alpha(\mathcal{P}) = \sum_{i=0}^{N-1} \|\tilde{\mathcal{F}}_i(\mathcal{P}) - \mathcal{M}_i^\delta\|^2 + \alpha \cdot \sum_{\gamma\in\Gamma} |\mathcal{P}_\gamma|. \tag{25}$$

This functional is a slight extension to the type considered in [16] and adapting these results, the minimizer can be computed iteratively as a limit of the sequence $(\mathcal{P}^k)_k$, where

$$\mathcal{P}^{k+1} = S_{\alpha/(2N)}\Big(\mathcal{P}^k + \frac{1}{N}\sum_i \tilde{\mathcal{F}}_i^*\big(\mathcal{M}_i^\delta - \tilde{\mathcal{F}}_i(\mathcal{P})\big)\Big). \tag{26}$$

and

$$S_\mu(x) = \begin{cases} x - \mathrm{sign}(x)\cdot\mu & \text{if} \quad |x| \geq \mu \\ 0 & \text{if} \quad |x| < \mu. \end{cases} \tag{27}$$

5. Conclusions

As we have seen we have presented a method for parallel MRI which both uses the specific structure of the problem and the demands of medical image processing.

A challenging numerical task will be the incorporation of as accurate as possible a priori information in order to both reduce the noise and the undersampling (i.e. overfolding) artifacts.

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