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Applications of the Backus–Gilbert method to linear and some nonlinear equations

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Abstract. We investigate the use of a functional analytical version of the Backus–Gilbert method as a reconstruction strategy to obtain specific information on the solution of linear and slightly nonlinear systems with Frechét derivable operators. Some *a priori* error estimates are shown and tested for two classes of problems: a nonlinear *moment problem* and a linear elliptic *Cauchy problem*. For this second class of problems a special version of the Green formula is developed in order to analyse the involved adjoint equations.

1. Introduction

1.1. Main results

The functional analytical approach of the Backus–Gilbert method in section 1.3 has already been used by other authors (see [Ch], [Ki] or [LM1, 2]). In this paper we use the differentiability of the involved nonlinear operator in order to develop the error estimates (9) and (13) for this reconstruction scheme. If the operator is linear, we obtain the estimate (17).

In order to test this reconstruction strategy, we choose the same nonlinear operator in section 3.1 as Louis does in [Lo]. The numerical tests in section 4.1 show that one can get good results even for noisy data.

For the second test in section 3.2, we choose a linear operator, which is highly ill-posed. The results are again satisfactory provided one uses appropriate *sentinels* to define the reconstruction strategy (see [Ch] or [Le]).

The results presented in this paper constitute part of the author's PhD research and they can be found in more detail in [Le].

1.2. Historical overview

This reconstruction method was first proposed in 1967 by Backus and Gilbert [BG1, 02, 3]. They were interested in the pointwise reconstruction of a function $f \in X = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is supposed to be open and bounded. The root of their problem was geophysical and the mathematical problem involved in the model is known in the literature as the *moment problem*. It can be formulated as follows: find a function $f \in X$ such that

$$\int_\Omega K_i(x) f(x)\,\mathrm{d}x = g_i \qquad i = 1, \ldots, N \tag{1}$$

where the kernels K_i are known real functions, which are well defined at Ω and the right-hand side $g = \{g_i\}_{i=1}^N \in Y = \mathbb{R}^N$ corresponds to the measured data of the physical problem. In order to determine the value of the solution f at some point $x_0 \in \Omega$, they suggest a linear reconstruction scheme, which is defined by a functional of the right hand-side of the linear system (1). One defines the linear functional $R_N \in Y'$ by

$$R_N(g) := \langle \varphi, g \rangle_Y = \int_\Omega \underbrace{\left(\sum_{i=1}^N \varphi_i K_i(x) \right)}_{\phi_N(x)} f(x)\,\mathrm{d}x = \langle \phi_N, f \rangle_X \tag{2}$$

where $\varphi \in Y'$ and $\phi_N \in X$. It is easy to observe that $f_N(x_0) := R_N(g)$ will be a good approximation for $f(x_0)$ if the condition $\phi_N(\cdot) \simeq \delta(x_0 - \cdot)$ is satisfied. The Backus–Gilbert idea is to force this condition by defining the quadratic functional

$$J(\phi) := \int_\Omega |x_0 - x|^2 \phi^2(x)\,\mathrm{d}x \tag{3}$$

on X and choosing ϕ_N such that

$$J(\phi_N) = \min_{\phi \in \mathrm{Span}\{K_i\}} J(\phi). \tag{4}$$

The linear constraint

$$\int_\Omega \phi(x)\,\mathrm{d}x = 1$$

is imposed in order to avoid the trivial solution in (4). Once one has evaluated the function $\phi_N(x) = \sum \varphi_i K_i(x)$, the approximation $f_N(x_0)$ is determined by the inner product

$$f_N(x_0) = \langle \phi_N, f \rangle_X = \langle \varphi, g \rangle_Y \tag{5}$$

where $\varphi = \{\varphi_i\}_{i=1}^N$. One great advantage of using the Backus–Gilbert method which can be recognized in (4) is that evaluation of the reconstruction operator $R_N(\cdot)$ does not depend on the system data. For different sets of data g it is possible to reconstruct the value of the respective $f(x_0)$ by only evaluating an inner product in Y.

1.3. Functional analytical formulation

Let $V \hookrightarrow X \hookrightarrow V'$ be a Hilbert triple, Y a Hilbert space, $y \in Y$ and $A : X \to Y$ a bounded linear operator. We analyse the problem of finding the value $\langle \mu, x^* \rangle$ for $\mu \in V'$, where x^* is the generalized solution obtained by the Moore–Penrose inverse of

$$Ax = y. \tag{6}$$

It is obvious that the expression $\langle \mu, x^* \rangle$ does not need to be well defined, if we do not make any further regularity assumptions about x^*. Depending on the physical situation involved, it is possible to guarantee that the expression $\langle \mu, x^* \rangle$ is well defined for some μ or even that $x^* \in V$. As we suppose y is obtained by measurement, it is to be expected that only y_ε with $\|y - y_\varepsilon\|_Y \leqslant \varepsilon$ is available, with $\varepsilon > 0$ small.

We use the Backus–Gilbert strategy and try to reconstruct the value $f := \langle \mu, x^* \rangle_X$ using a linear functional evaluated in y_ε. For $\varphi \in Y'$ we define $f_{\varepsilon,\varphi} := \langle \varphi, y_\varepsilon \rangle_Y$ and estimate the error $|f - f_{\varepsilon,\varphi}|$ by

$$\begin{aligned} |f - f_{\varepsilon,\varphi}| &= |\langle \mu, x^* \rangle_X - \langle \varphi, y_\varepsilon \rangle_Y| \\ &\leqslant |\langle \varphi, y - y_\varepsilon \rangle_Y| + |\langle \mu, x^* \rangle_X - \langle \varphi, Ax^* \rangle_Y| \\ &\leqslant \varepsilon \|\varphi\|_{Y'} + |\langle \mu - A^*\varphi, x^* \rangle_X| \end{aligned} \tag{7}$$

where $A^* : Y' \to X'$ is the adjoint operator of A. If we succeed in finding a solution $\varphi \in Y'$ for the equation $A^*\varphi = \mu$ we can write

$$f = \langle A^*\varphi, x^*\rangle_X = \langle \varphi, y\rangle_Y \simeq \langle \varphi, y_\varepsilon\rangle_Y = f_{\varepsilon,\varphi} \tag{8}$$

and the error $|f - f_{\varepsilon,\varphi}|$ behaves like $O(\varepsilon)$. Another consequence is that the approximation $f_{\varepsilon,\varphi}$ is exact if there are no errors in the measurements ($y_\varepsilon = y$).

In the special case of X and Y being spaces of functions defined over a region Ω, the Backus–Gilbert strategy suggests a pointwise reconstruction of x^*. In order to reconstruct the value of $f(\cdot)$ at the point $t \in \Omega$ we should take $\mu(\cdot) = \delta(t - \cdot)$ in (8) and solve the adjoint equation $A^*\varphi = \delta$†.

We may have difficulties if $\delta \notin \overline{Rg(A^*)}$. In this case we can use the projection of δ over $\mathrm{Ker}(A^*)^\perp$ instead of δ itself. This is equivalent to minimizing the error $\|A^*\varphi - \delta\|^2_{V'}$ or to finding a solution $\varphi \in Y'$ of the normal equation

$$(A\,A^*)\varphi = A\delta.$$

Louis and Maass propose a similar approach in [LM2] and use the projection of δ over special Sobolev spaces of negative index. In [LM1] (see also [Lo]) the equation $(A\,A^*)\varphi = Ae_h$ is considered where e_h is a *mollifier*, i.e. a smooth approximation for the Dirac distribution δ.

An alternative for the case $\delta \notin Rg(A^*)$ was proposed by Chavent in [Ch]. He tried to regularize the normal equations using the Tikhonov strategy: φ is chosen as the minimum over Y' of the functional $(\|A^*\varphi - \delta\|^2_{V'} + \alpha\|\varphi\|^2_{Y'})$, where $\alpha > 0$ is a small regularization parameter.

2. Analysis of the method

We are interested in applying the Backus–Gilbert strategy for operators of the form $A = A_0 + \gamma A_1$, where $A_0 \in \mathcal{L}(X, Y)$, $A_1 : X \mapsto Y$ is continuously differentiable‡ in X and $\gamma > 0$ is a small number. Let $\mu \in V'$ and $y_\varepsilon \in Y$ as before§.

Lemma 1. *If $x^0 \in V$ is an approximation to a solution x^* of (6), the expression $f_{\varepsilon,\varphi} = \langle\varphi, y_\varepsilon\rangle_Y$ gives an approximation for $f := \langle\mu, x^*\rangle_X$ and the error $|f - f_{\varepsilon,\varphi}|$ is estimated by*

$$\begin{aligned}|f - f_{\varepsilon,\varphi}| &\leqslant |\langle\varphi, y - y_\varepsilon\rangle_Y| + |\langle\varphi, Ax^* - Ax^0 - \mathrm{d}A(x^0)(x^* - x^0)\rangle_Y| \\ &\quad + |\langle\varphi, Ax^0 - \mathrm{d}A(x^0)x^0\rangle_Y| + |\langle \mathrm{d}A(x^0)^*\varphi - \mu, x^*\rangle_X|.\end{aligned} \tag{9}$$

Proof. Estimate (9) follows promptly from the following equality

$$|f - f_{\varepsilon,\varphi}| = |\langle\mu, x^*\rangle_X - \langle\varphi, y_\varepsilon\rangle_Y \pm \langle\varphi, y\rangle \pm \langle\varphi, Ax^0 - \mathrm{d}A(x^0)(x^* - x^0)\rangle|. \qquad \square$$

Before analysing the right-hand side of (9), let us discretize the spaces involved. We define the finite-dimensional space $Y_h = \mathrm{Span}\{y_j\}_{j=1}^N$ by

$$Y_h \subset \{\varphi \in Y / \langle\varphi, Ax^0 - \mathrm{d}A(x^0)x^0\rangle_Y = 0\}. \tag{10}$$

Further, we let $P_h : Y \mapsto Y_h$ be the orthogonal projector over Y_h and choose the finite-dimensional space $X_h = \mathrm{Span}\{x_j\}_{j=1}^N \subset D(A) \cap V$ such that the property

$$\det(\langle \mathrm{d}A(v)^* P_h^* y_i, x_j\rangle)_{1\leqslant i,j\leqslant N} \neq 0 \tag{11}$$

is satisfied.

† The Hilbert space V must be chosen, so that δ belongs to V'.
‡ The Fréchet derivative of A_1 will be denoted by $\mathrm{d}A_1$.
§ For convenience we will identify the spaces X with X' and Y with Y'.

Theorem 2. *Define* $y_h := P_h y$, $y_{\varepsilon,h} := P_h y_\varepsilon$ *and* $f_{\varepsilon,\varphi,h} := \langle \varphi, y_{\varepsilon,h}\rangle_Y$. *For every* $\varphi \in Y$ *the following estimate holds*

$$|f - f_{\varepsilon,\varphi,h}| \leqslant \varepsilon \|P_h\| \|\varphi\|_Y + \gamma \|\varphi\|_Y \|P_h\| O(\|x^* - x^0\|_X^2) \\ + |\langle \mathrm{d}A^*(x^0) P_h^* \varphi - \mu, x^*\rangle_X|. \tag{12}$$

Proof. By an argument analogous to that used in (9) we obtain that for each $\varphi \in Y$

$$|f - f_{\varepsilon,\varphi,h}| \leqslant \|\varphi\|_Y \|y_h - y_{\varepsilon,h}\|_Y + \gamma \|\varphi\|_Y \|P_h A_1 x^* - P_h A_1 x^0 - P_h\, \mathrm{d}A_1(x^0)(x^* - x^0)\|_Y \\ + |\langle \varphi, P_h A x^0 - P_h\, \mathrm{d}A(x^0) x^0\rangle_Y| + |\langle \mathrm{d}A^*(x^0) P_h^* \varphi - \mu, x^*\rangle_X|. \tag{13}$$

The first term in (13) can be estimated by

$$\|\varphi\|_Y \|y_h - y_{\varepsilon,h}\|_Y \leqslant \varepsilon \|P_h\| \|\varphi\|_Y.$$

For the second term we have

$$\gamma \|\varphi\|_Y \|P_h A_1 x^* - P_h A_1 x^0 - P_h\, \mathrm{d}A_1(x^0)(x^* - x^0)\|_Y \leqslant \gamma \|\varphi\|_Y \|P_h\| O(\|x^* - x^0\|_X^2).$$

The third term in (13) disappears because of our choice of Y_h. Putting these inequalities together we obtain (12). $\square$

The last term in (12) gives us a rule for choosing $\varphi \in Y$. This is actually

$$\langle \mathrm{d}A^*(x^0) P_h^* \varphi, x_j\rangle_X = \langle \mu, x_j\rangle_X \qquad j = 1, \ldots, N. \tag{14}$$

This means we can evaluate the coefficients of $P_h\varphi$ in Y_h by solving the N-dimensional linear system (14). Solving this system is a well defined problem, as can be seen from the determinant condition (11).

Next, we interpret the system (14) in a different way. Let us assume that the space X_h can be written as $B^* Y_h$, where B is a linear bounded operator $B : V' \mapsto Y$ with $B^* : Y \mapsto V$. We are then able to write (14) as

$$\langle \mathrm{d}A^*(x^0) P_h^* \varphi - \mu,\ B^* w\rangle_X = 0 \qquad \forall w \in Y_h$$

i.e.

$$\langle B G_h \varphi, w\rangle_Y = \langle B\mu, w\rangle_Y \qquad \forall w \in Y_h \tag{15}$$

where $G_h = \mathrm{d}A^*(x^0) P_h^*$. If $\mu = \delta$ and in the special case $\mu \in \operatorname{Ker} B$, it follows from (15) that

$$\langle B G_h \varphi, w\rangle_Y = 0 \qquad \forall w \in Y_h. \tag{16}$$

Further, if it is possible to decompose the product BG_h as a square $\mathcal{B}^2$ of a symmetric matrix $\mathcal{B}$, it follows from (16) that $\|\mathcal{B}\varphi\|_Y^2 = 0$. Instead of solving system (16), we can consider the minimization problem

$$\begin{cases} \|\mathcal{B}\varphi_h\|^2 = \min\limits_{\varphi \in Y_h} \|\mathcal{B}\varphi\|^2 \\ \text{under the linear constraint } \langle \mathrm{d}A^*(x^0)\varphi_h, 1\rangle_X = 1. \end{cases}$$

The extralinear constraint is motivated by the original Backus–Gilbert formulation in section 1.2 and introduced in order to avoid the trivial solution in the minimization problem. The constrained minimization problem above can be interpreted as an extended Backus–Gilbert method.

We proceed to develop an error estimate for the linear case when noisy data are considered.

Theorem 3. *Let A be a linear operator. Take $B^* = A^*$ and $X_h = A^*Y_h$. If we choose $\varphi_h \in Y_h$ to be the solution of (14), i.e. $\langle A^*\varphi_h - \mu, w\rangle = 0$, $\forall w \in X_h$, we obtain the error estimate*

$$|f - f_{\varepsilon,\varphi,h}| \leqslant \operatorname{dist}(\mu, A^*Y_h)_{V'}(1 + \|\mathcal{P}_h\|)\operatorname{dist}(x^*, X_h)_V + O(\varepsilon). \tag{17}$$

Proof. Using (7), for each $\varphi_h = P_h\varphi \in Y_h$ we obtain the equality

$$|f - f_{\varepsilon,\varphi,h}| = O(\varepsilon) + |\langle A^*\varphi_h - \mu, x^*\rangle_X|.$$

Define $x_h := \mathcal{P}_h x^*$, where $\mathcal{P}_h : X \mapsto X_h$ is the orthogonal projector over X_h. Now choosing $\varphi_h \in Y_h$ as the solution of (14), for every $\psi \in Y_h$ we have

$$\langle A^*\varphi_h - \mu, x^*\rangle_X = \langle A^*\varphi_h - A^*\psi, x^* - x_h\rangle_X + \langle A^*\psi - \mu, x^* - x_h\rangle_X. \tag{18}$$

The first term on the right-hand side of (18) disappears by the definition of x_h. For the second term we have

$$|\langle A^*\varphi_h - \mu, x^* - x_h\rangle_X| \leqslant \|A^*\psi_h - \mu\|_{V'}\|x^* - x_h\|_V.$$

In order to estimate $\|x^* - x_h\|_V$ we define $\tilde{x} \in X_h$ as the solution of the minimization problem

$$\|x^* - \tilde{x}\|_V^2 = \min_{x \in X_h} \|x^* - x\|_V^2.$$

From this definition follows

$$\begin{aligned}\|x^* - x_h\|_V &\leqslant \|x^* - \tilde{x}\|_V + \|\tilde{x} - \mathcal{P}_h x^*\|_V \\ &\leqslant \operatorname{dist}(X_h, x^*)_V + \|\mathcal{P}_h(\tilde{x} - x^*)\|_V \\ &\leqslant (1 + \|\mathcal{P}_h\|)\operatorname{dist}(X_h, x^*)_V\end{aligned}$$

and the theorem is proven. $\square$

It is easy to conclude from (17) that the error in the approximation $f_{\varepsilon,\varphi,h}$ will converge to zero with h and ε only when we have $\mu \in \overline{Rg A^*}$.

3. Applications

3.1. A nonlinear moment problem

We start this discussion with the special class of nonlinear moment problems. Quadratic moment problems were also analysed by Louis in [Lo]. Let $X = Y = L^2(0, 1)$ and $A : X \mapsto Y$ the operator defined by

$$(Ax)(t) = \int_0^t x^0(t-s)x(s)\,\mathrm{d}s + \nu \int_0^t x(t-s)x(s)\,\mathrm{d}s \qquad t \in [0, 1] \tag{19}$$

where the kernel x^0 of the linear component of A is a $L^2(0, 1)$ function and $\nu > 0$ is a small parameter, that controls the nonlinear component of A. Just as in section 2 we will analyse the system $Ax = (A_0 + \nu A_1)x = y$.

The right-hand side of this system consists of measured data, so we assume we know only a finite number of $y_i = y(t_i)$, $t_i \in (0, 1)$. If we define the projection operator $P_h : Y \mapsto Y_h = \mathbb{R}^N$, it is possible to define a discrete version of A in (19) by setting

$$(P_h A)(x) := [Ax(t_i)]^t = \left[\int_0^{t_i} x^0(t_i - s)x(s)\,\mathrm{d}s + \nu \int_0^{t_i} x(t_i - s)x(s)\,\mathrm{d}s\right]^t.$$

If we further assume that our measurements are inexact, then we actually have $y_{h,\varepsilon} \in Y_h$ with $\|P_h y - y_{h,\varepsilon}\| \leqslant \varepsilon$, where $\varepsilon > 0$ is small. We will be interested in finding the solution x^* of the discrete nonlinear system

$$(P_h A)x^* = y_{h,\varepsilon}.$$

We saw in section 2 that an approximation for the solution x^* is needed. For this propose we choose the kernel x^0 in (19). We also need the operator $P_h\,\mathrm{d}A$ and its adjoint $(P_h\,\mathrm{d}A)^* : Y_h \mapsto X$. One can easily see that for $f \in L^2(0,1)$ and $w \in \mathbb{R}^N$ the equalities

$$(P_h\,\mathrm{d}A(x))(f) = \left[\int_0^{t_i} x^0(t_i - s) f(s)\,\mathrm{d}s + 2\nu \int_0^{t_i} x(t_i - s) f(s)\,\mathrm{d}s\right]^t_{1\leqslant i\leqslant N}$$

and

$$(P_h\,\mathrm{d}A(x))^*(w) = \sum_{i=1}^{N} w_i[x^0(t_i - s) + 2\nu x(t_i - s)]\chi_{[0,t_i]}(s)$$

are valid.

Now we have to choose the space $X_h = \mathrm{Span}\{x_i\}_{i=1}^N$. This choice must reflect the expected regularity of the solution x^* and should be such that the system in (14) has nice properties. We choose a cubic B-spline basis for X_h for the numerical experiments. Given $\mu \in X'$ we will have to solve the system

$$[\langle (P_h\,\mathrm{d}A(x^0))^* e_i, x_j\rangle_X]_{i,j=1}^N [\varphi_j]^t = [\langle \mu, x_j\rangle_X]^t \qquad (20)$$

where the matrix of (20) will have almost upper triangular form if x_j are B-splines. We assume the points t_j are uniformly placed on the interval $[0,1]$ and define for $j = 0, \ldots, N-1$ the cubic B-splines

$$S_j(t) = \frac{1}{4h^3}\begin{cases} (t - t_{j-2})^3 & t \in [t_{j-2}, t_{j-1}] \\ h^3 + 3h^2(t - t_{j-1}) + 3h(t - t_{j-1})^2 - 3(t - t_{j-1})^3 & t \in [t_{j-1}, t_j] \\ h^3 + 3h^2(t_{j+1} - t) + 3h(t_{j+1} - t)^2 - 3(t_{j+1} - t)^3 & t \in [t_j, t_{j+1}] \\ (t_{j+2} - t)^3 & t \in [t_{j+1}, t_{j+2}]. \end{cases}$$

In the formulation of our strategy we assume the space Y_h satisfies the condition in (10). In order to rescue our choice of Y_h, we add to the system (20) the following linear restriction of φ

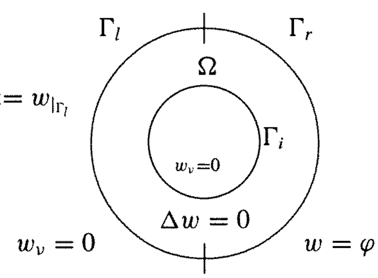
$$\langle \varphi, P_h A_1(x^0) - \mathrm{d}P_h A_1(x^0)x^0\rangle_Y = 0. \qquad (21)$$

Joining equations (20) and (21), we have an overdetermined system with $(N+1)$ equations to solve, in order to determine the N-coefficients of φ. We observe that the matrix coefficients $a_{i,j}$ of this system vanish for $i > j+2$ and $i \neq N+1$.

3.2. A linear elliptic Cauchy problem

We begin with the definition of the linear operator $A : H^{1/2}(\Gamma_r) \to H^{1/2}(\Gamma_l)$

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = \varphi & \text{at } \Gamma_r \\ w_\nu = 0 & \text{at } \Gamma_l \\ w_\nu = 0 & \text{at } \Gamma_i \end{cases} \qquad A(\varphi) := w_{|\Gamma_l}$$



where H^s are the Soblev spaces† of index $s \in \mathbb{R}$ and w is the $H^1(\Omega; \Delta)$ solution of the mixed boundary value problem on the left-hand side.

Note that solving the equation $A\varphi = f$ is equivalent to finding the trace $\varphi = w_{|\Gamma_r}$ of the $H^1(\Omega; \Delta)$ solution of the following elliptic Cauchy problem

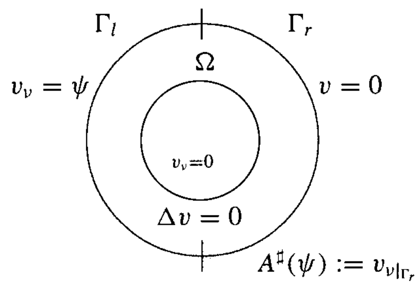
$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = f & \text{at } \Gamma_l \\ w_\nu = 0 & \text{at } \Gamma_l \\ w_\nu = 0 & \text{at } \Gamma_i. \end{cases}$$

Given a distribution $\mu \in H^{-1/2}(\Gamma_r)$ we will use the Backus–Gilbert strategy to approximate the value $\langle \mu, \varphi \rangle$ by $\langle \psi, f \rangle$, where ψ is the solution of

$$A^*\psi = \mu. \tag{22}$$

We see, using integration by parts, that the adjoint operator of the restriction of A to $H^{1/2}_{00}(\Gamma_r)$ is the operator $A^\sharp : H^{1/2}_{00}(\Gamma_l)' \mapsto H^{1/2}_{00}(\Gamma_r)'$ defined by

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = 0 & \text{at } \Gamma_r \\ v_\nu = \psi & \text{at } \Gamma_l \\ v_\nu = 0 & \text{at } \Gamma_i. \end{cases}$$



For $\psi \in H^{1/2}_{00}(\Gamma_l)'$, if $\varphi \in H^{1/2}(\Gamma_r) \backslash H^{1/2}_{00}(\Gamma_r)$, it is not true that

$$\int_{\Gamma_l} A(\varphi)\psi \, \mathrm{d}\Gamma = -\int_{\Gamma_r} \varphi A^\sharp(\psi) \, \mathrm{d}\Gamma.$$

To correct this problem we need the following theorem.

Theorem 4. *For $a, b \in \mathbb{R}$ let $\eta_{a,b} \in C^\infty(\Gamma_r)$ be a function with $\eta_{a,b}(P_1) = a$ and $\eta_{a,b}(P_2) = b$, where P_1 and P_2 are the contact points between Γ_r and Γ_l. If $V_{a,b}$ is the subspace of $H^{1/2}(\Gamma_r)$ defined by*

$$V_{a,b} := \{\varphi \in H^{1/2}(\Gamma_r) / \eta_{a,b} - \varphi \in H^{1/2}_{00}(\Gamma_r)\}$$

then for $\varphi_1, \varphi_2 \in V_{a,b}$ we have

$$\int_{\Gamma_l} A\varphi_1\psi \, \mathrm{d}\Gamma + \int_{\Gamma_r} \varphi_1 A^\sharp\psi \, \mathrm{d}\Gamma = \int_{\Gamma_l} A\varphi_2\psi \, \mathrm{d}\Gamma + \int_{\Gamma_r} \varphi_2 A^\sharp\psi \, \mathrm{d}\Gamma$$

for every ψ in $H^{-1/2}(\Gamma_l)$.

† For details see [Ad] or [DaLi].

A complete proof of this theorem can be found in [Le]. A direct consequence of theorem 4 is that for $a, b \in \mathbb{R}$ one can define over $H^{-1/2}(\Gamma_l)$ the linear functional

$$r_{a,b}(\psi) := \langle A\eta_{a,b}, \psi\rangle + \langle \eta_{a,b}, A^\sharp\psi\rangle$$

and obtain

$$\langle A\varphi, \psi\rangle = -\langle \varphi, A^\sharp\psi\rangle + r_{a,b}(\psi)$$

for every $\varphi \in V_{a,b}$ and $\psi \in H^{-1/2}(\Gamma_l)$.

If we are able to find $\psi \in H^{-1/2}(\Gamma_l)$ that solves the equation

$$-A^\sharp\psi = \mu$$

we can solve our reconstruction problem as before, using

$$\begin{aligned}\langle \mu, \varphi\rangle &= -\langle A^\sharp\psi, \varphi\rangle \\ &= \langle \psi, A\varphi\rangle - r_{a,b}(\psi) \\ &= \langle \psi, f\rangle - r_{a,b}(\psi).\end{aligned}$$

We observe that, if $\varphi \in H^{1/2}_{00}(\Gamma_r)$, then $a = b = 0$ and $r_{a,b} \equiv 0$. In this case we have

$$\langle \mu, \varphi\rangle = \langle \psi, f\rangle.$$

4. Numerical results

4.1. The moment problem

In this section we study the operator $A : L^2(0, 1) \to L^2(0, 1)$ defined in (19) for $x^0(t) = t$. Let us start with the linear case, i.e. taking $\nu = 0$ in (19).

We generate different right-hand sides by solving the direct problem $y = Ax$ for three functions

$$x_a(t) = \begin{cases} t/2 & t \leqslant 1/2 \\ t - 1/4 & t \geqslant 1/2 \end{cases} \qquad x_b(t) = \begin{cases} 2t & t \leqslant 1/2 \\ 2 - 2t & t \geqslant 1/2 \end{cases}$$

and

$$x_c(t) = \begin{cases} 1 & 1/4 \leqslant t \leqslant 3/4 \\ 0 & \text{otherwise.} \end{cases}$$

Our grid is defined by $t_j = j/N$, $0 \leqslant j \leqslant N$. For the space X_h we choose the B-spline basis corresponding to this grid. Our objective is to reconstruct the values of the different solutions x_a, x_b and x_c at the grid points t_j and at the points $(t_{j+1} + t_j)/2$, $0 \leqslant j \leqslant N - 1$. In figure 1 we give the results for $N = 25$ and $N = 50$, when the exact right-hand side y is used.

In figure 2, we show the reconstruction results for the linear operator and perturbed data. The system is solved for a right-hand side y_ε generated by adding a 1% random noise to the original y, i.e. $|y_j - y_{\varepsilon,j}| \leqslant (1/100)y_j$.

Next we analyse the reconstruction error at the point $t = 1/2$ for exact data and the functions

$$x_a(t) = 2t \qquad x_b(t) = \begin{cases} 2t & t \leqslant 1/2 \\ 2 - 2t & t \geqslant 1/2. \end{cases}$$

Analysing figure 3 we observe that the reconstruction is better for even values of N. This can be explained by the existence of a B-spline centred at the point $t = 1/2$ in the

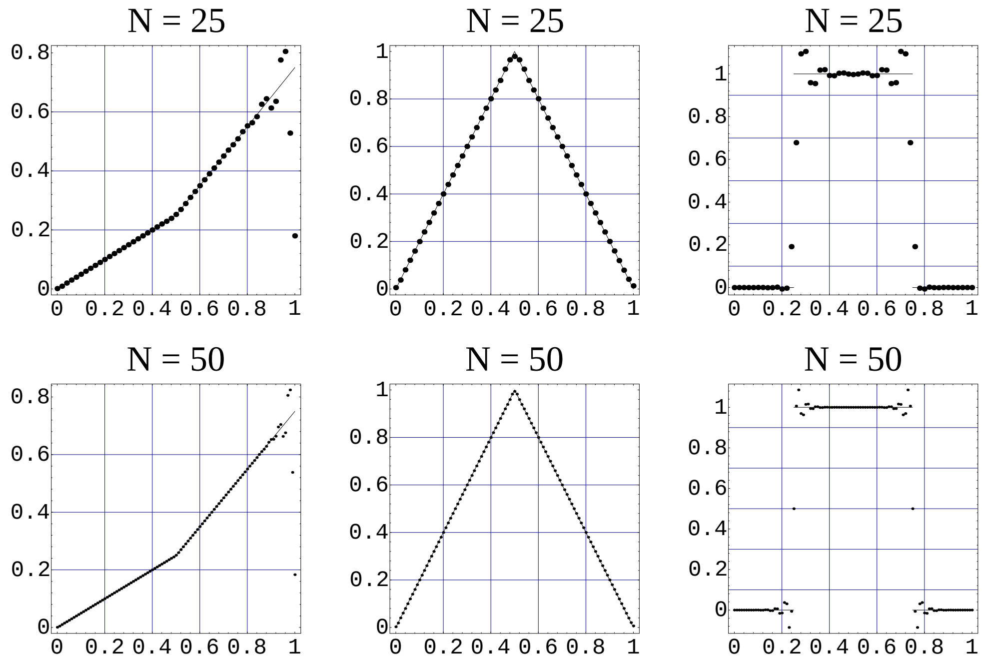


Figure 1. Reconstruction results for a linear operator and exact data.

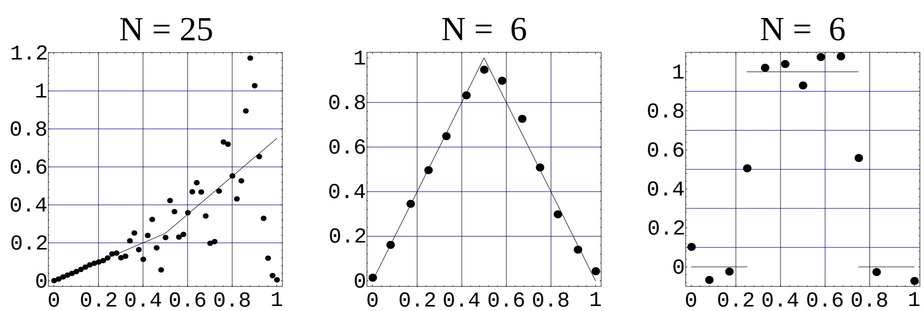


Figure 2. Reconstruction results for a linear operator and noisy data.

X_h-basis. A consequence of this is that the functional $\delta(\cdot - 1/2)$ will be better approximated in X_h if N is even.

Next we analyse the operator A in (19) for small values of ν. We use the same grid as before with $N = 25$ and try to reconstruct the polynomial x^2 at the points t_j and $(t_{j+1}+t_j)/2$ using exact data. The results are shown in figure 4.

The next example in figure 5 shows a reconstruction for $\nu = 0.01$ and exact data of the functions

$$x_b(t) = \begin{cases} 2t & t \leqslant 1/2 \\ 2 - 2t & t \geqslant 1/2 \end{cases} \qquad x_c(t) = \begin{cases} 1 & 1/4 \leqslant t \leqslant 3/4 \\ 0 & \text{otherwise.} \end{cases}$$

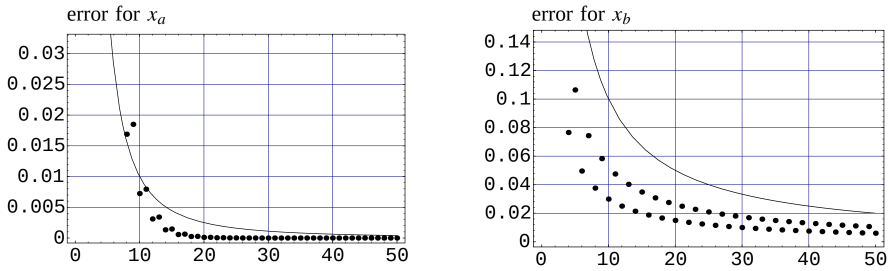


Figure 3. Reconstruction error at $t = \frac{1}{2}$ for exact data.

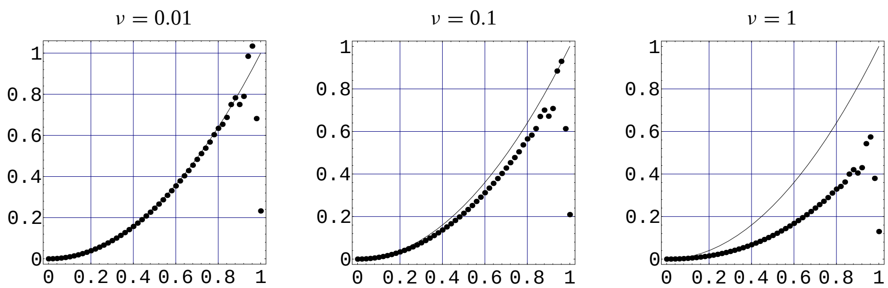


Figure 4. Reconstruction results for different parameters of nonlinearity.

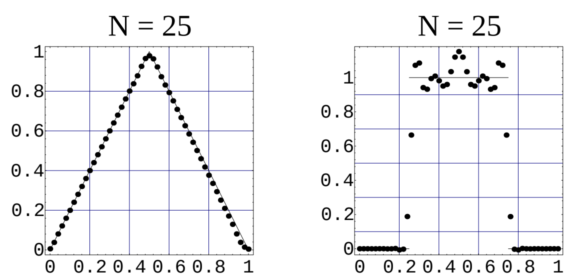


Figure 5. Reconstruction results for a nonlinear operator $\nu = 0.01$ and exact data.

4.2. The elliptic Cauchy problem

We analyse the elliptic Cauchy problem in an anulus Ω with inner radius $1/2$ and outer radius 1. Let us take the linear operator A defined in section 3.2. The problem we want to solve is, given $\mu \in H^{-1/2}(\Gamma_r)$ to reconstruct the value of $\langle \mu, \varphi \rangle$, where $\varphi \in H^{1/2}(\Gamma_r)$ is the solution of the equation $A\varphi = f$. In order to generate consistent data f, we solve the direct problems for $\varphi_1(t) = (t - \pi/2)^2$ and $\varphi_2(t) = \pi - 2|t - \pi/2|$, where $t \in [0, \pi]$.

The formulation of this elliptic Cauchy problem in Ω involves an extra difficulty: we are not able to characterize the space $Rg(A^\sharp)$. As we want to have an element $\mu \in Rg(A^\sharp)$, we first solve the direct problem $\mu = A^\sharp \psi$ for $\psi \in H^{-1/2}(\Gamma_l)$. For this proposal we choose

$\psi \equiv 1$ to solve the mixed boundary value problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = 0 & \text{at } \Gamma_r \\ v_\nu = \psi & \text{at } \Gamma_l \\ v_\nu = 0 & \text{at } \Gamma_i \end{cases}$$

and set $\mu = v_{\nu|_{\Gamma_r}} \in H^{-1/2}(\Gamma_r)$.

According to the Backus–Gilbert strategy discussed in section 3.2, the first thing to do is to solve the equation $-A^\sharp\psi = \mu$. To approximate the solution ψ, we use the iterative method described in [MaKo] (this iterative method is also extensively discussed in [Le]). The approximations ψ_k are shown in figure 6, where k represents the iteration index. The grid node 0 represents the point $(0, -1)$ and the grid node 32 the point $(0, 1)$ of Γ_r.

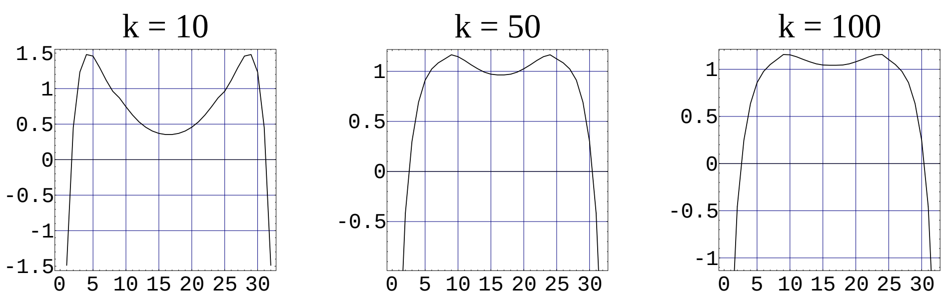


Figure 6. Approximations ψ_k obtained by alternative method to the solution of $-A^\sharp\psi = \mu$.

Next, we compare the values $\langle\mu, \varphi\rangle$ with $\langle\psi, f\rangle - r_{a,b}(\psi)$. The results are shown in table 1 (note that $r_{a,b} \equiv 0$ for $\varphi = \varphi_2$).

Table 1.

	$\langle\mu, \varphi\rangle$	$\langle\psi, f\rangle$	$\langle\psi, f\rangle - r_{a,b}(\psi)$	Relative error
$\varphi = \varphi_1$	3.995 77	5.844 96	3.891 91	0.017 76
$\varphi = \varphi_2$	0.523 09	0.499 03	0.499 03	0.045 99

Other numerical tests related to this specific Cauchy problem and to the validation of theorem 4 can be found in [Le].

5. Final remarks and conclusions

(1) Numerical experiments show that one can obtain good approximations for $\mu = \delta$ in $Rg(\mathrm{d}A^* P_h^*)$ if A is the integral operator defined in section 3.1 and the nonlinearity in A is small. In the nonlinear case we can always improve our approximation by defining a new $\tilde{x}^0$ as the B-spline interpolation of the evaluated values $x(t_j)$ and solving the new system

$$\langle \mathrm{d}A^*(\tilde{x}^0) P_h^* \varphi, x_j\rangle_X = \langle\mu, x_j\rangle_X.$$

Comparable and related results can be found in [Ch], [Hu], [Ki], [Lo], [LM1, 2], [ScBe] and [Sn].

(2) We observe an unwanted Gibb phenomenon in figures 1(a), 2(a) and 4. An explanation for this fact is that $\delta(\cdot) \in H^{-1/2-\varepsilon}$ for $\varepsilon > 0$ but $H_0^s([0,1]) \not\subseteq H^s([0,1])$ for $s > 1/2$. Thus, the inner product $\langle \delta, x \rangle_{L^2}$ will be in duality only if the boundary conditions $x(0) = x(1) = 0$ are satisfied.

The same phenomenon can also be observed in figure 5 (right), where the lack of regularity of the solution x_c is responsible for the effect.

(3) If the operator A is defined by the elliptic Cauchy problem in section 3.2, we do not know, for an arbitrary set Ω, how to characterize the space $Rg(A^\sharp)$. However, if some argument guarantees that μ_i are in $Rg(A^\sharp)$, we can proceed as in section 4.2 and solve the Cauchy problems $A^\sharp \psi_i = \mu_i$ once for each μ_i, in order to obtain the *observations*

$$\langle \mu_i, \varphi \rangle = \langle \psi_i, f \rangle$$

of φ, each time we have a different set of data f. Such μ_i are also known in the literature as *sentinels* (see [Ch]).

(4) When we analysed the Cauchy problem, we tried first to evaluate the reconstruction with $\mu = \delta$ and μ as a C^∞-mollifier. Using classical arguments (see [GiTr]) one can prove that no analytical solution exists in such cases when Ω has an analytical boundary. Our numerical results show, that in this case the equation $A^\sharp \psi = \mu$ has no solutions.

(5) It is important to point out here the ill-posed nature of the involved reconstruction problems. Fredholm operators of the first kind are typically ill-posed [Gro]. Concerning elliptic Cauchy problems, Hadamard elaborated an example with Cauchy data that converge uniformly to zero but the respective solutions become unbounded. The example follows

$$\begin{cases} \Delta u_k = 0 & (x, y) \in \Omega = (0,1) \times (0,1) \\ u_k(x, 0) = 0 & x \in (0,1) \\ (\partial/\partial y) u_k(x, 0) = \varphi_k(x) & x \in (0,1) \end{cases}$$

where $\varphi_k = (\pi k)^{-1} \sin(\pi k x)$. The respective solutions are

$$u_k(x, y) = (\pi k)^{-2} \sinh(\pi k y) \sin(\pi k x).$$

(6) Our numerical experiments were realized on a IBM RISC 6000/250 workstation. It took some seconds to generate and solve the systems in section 4.1 for $N = 50$. To evaluate the first 100 steps of the iterative method, in order to solve the Cauchy problem in section 4.2, we needed about 30 min CPU time (we used the finite element method on a grid with $\simeq$8000 nodes to solve each mixed BVP involved on the iterative method).

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