AN ITERATIVE METHOD FOR SOLVING ELLIPTIC CAUCHY PROBLEMS

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Abstract

We investigate the Cauchy problem for elliptic operators with C^∞–coefficients at a regular set $\Omega \subset \mathbb{R}^2$, which is a classical example of an ill-posed problem. The Cauchy data are given at the subset $\Gamma \subset \partial\Omega$ and our objective is to reconstruct the trace of the $H^1(\Omega)$ solution of an elliptic equation at $\partial\Omega/\Gamma$. The method described here is a generalization of the algorithm developed by Maz'ya et al. [Ma] for the Laplace operator, who proposed a method based on solving successive well-posed mixed boundary value problems (BVP) using the given Cauchy data as part of the boundary data. We give an alternative convergence proof for the algorithm in the case we have a linear elliptic operator with C^∞–coefficients. We also present some numerical experiments for a special non linear problem and the obtained results are very promisive.

1 Introduction

1.1 Main results

The algorithm of Maz'ia et al. [Ma] is formulated here for general elliptic operators. A new convergence proof for this iterative algorithm using a functional analytical approach is given in section 2.1 (see Theorem 2.5), where we describe the iteration using powers of an affine operator T. The key of the proof is to define an alternative topology (see Lemma 2.1) for the space $H_{00}^{1/2}(\Gamma)'$ – where the iteration is considered – and to prove that the linear part of T satisfies special properties (see Theorem 2.4). The converse of Theorem 2.5 is also proved, i.e. if the iteration converges, it's limit is the solution of the Cauchy Problem.

Some properties of T_l (the linear part of T) such as positiveness, self adjointness and injectivity are verified in section 2.1 (see Theorem 2.3). In section 2.2 we prove a spectral property of T_l, that is attached to the ill-posedness of the elliptic Cauchy Problem.

In section 2.3 we analyze the convergence speed of the iteration for the special case when the spectral decomposition of T_l is known. The effectiveness of two regularization schemas based on the spectral decomposition of T_l (the linear part of T) is considered in section 2.4.

In section 3 some numerical experiments are presented, where we test the algorithm performance for linear consistent, linear inconsistent and non linear Cauchy problems.

An analysis of iterative method in the special case of a square region can be found in [JoNa]. The idea of this method is also applied to hyperbolic operators in [Bas] that uses semi-group theory in his approach.

1.2 About Cauchy problems

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and simply connected set. As an *elliptic Cauchy problem* at Ω we consider an (time independent) initial value problem for an elliptic differential operator defined over Ω, where the initial data is given at the manifold $\Gamma \subset \partial\Omega$.

The problem we analyze is to evaluate the trace of the solution of such an initial value problem at the part of the boundary where no data was prescribed, actually at $\partial\Omega\backslash\Gamma$. As a solution of our Cauchy problem we consider a $H^1(\Omega)$–distribution, which solves the weak formulation of the elliptic equation in Ω and also satisfies the Cauchy data at Γ in the sense of the trace operator.

It's well known that elliptic Cauchy problems are ill–posed. According to the definition of Hadamard an initial value problem (IVP) or a BVP is said to be well–posed, when the following three conditions are satisfied:[1] existence and unicity of solutions, and continuous dependence of the data. The next example was encountered by Hadamard himself [Had] and shows that the solution of an elliptic Cauchy problem may not depend continuously of the initial data. One analyzes the family of problems:

$$\left\{ \begin{array}{rcl} \Delta u_k &=& 0\,, \quad (x,y) \in \Omega = (0,1) \times (0,1) \\ u_k(x,0) &=& 0\,, \quad x \in (0,1) \\ \frac{\partial}{\partial y} u_k(x,0) &=& \varphi_k\,, \quad x \in (0,1) \end{array} \right.$$

where $\varphi_k(x) = (\pi k)^{-1} sin(\pi k x)$. The respective solutions

$$u_k(x,y) \;=\; (\pi k)^{-2} sinh(\pi k y)\; sin(\pi k x)$$

[1] More details in [Bau] or [Lo].

do exist for every $k \in \mathbb{N}$ and they are unique. The sequence $\{\varphi_k\}$ converges uniformly to zero. Taking the limit $k \to \infty$ we have a Cauchy problem with homogeneous data, which admits only the trivial solution. But for every fixed $y > 0$ the solutions u_k oscillate stronger and stronger and become unbounded as $k \to \infty$. Consequently the sequence u_k does not converge to zero in any reasonable topology.

If in this example one takes for Cauchy data the C^∞–functions (f, g) instead of $(0, \varphi_k)$, it is possible to show (see [GiTr]) that if $f \equiv 0$, then g must be analytical. This means that a *classical solution* may not exist, even if one uses smooth functions as Cauchy data.

The unique well-posedness condition that is satisfied for this problem is the second one. With adequate arguments it is possible to extend the Cauchy–Kowalewsky and Holmgren Theorem to the H^1–context in order to guarantee uniqueness of solutions also in weak sense (see Theorem D.3).

1.3 Description of the algorithm

Let Ω be an open set in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, which is divided in two open and connected components: Γ_1 and Γ_2, such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega$. Let P be the second order elliptic differential operator defined by:

$$P(u) \ := \ - \sum_{i,j=1}^{2} D_i(a_{i,j} D_j u) \tag{1}$$

where the real functions $a_{i,j}$ satisfy

$$\begin{cases} -\ a_{i,j} \in L^\infty(\Omega); \\ -\ \text{the matrix } A(x) := (a_{i,j})_{i,j=1}^2 \text{ satisfies: } \xi^t A(x)\, \xi > \alpha ||\xi||^2, \\ \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^2 \text{ where } \alpha > 0 \text{ is given (independent of } x). \end{cases} \tag{2}$$

Given the Cauchy data $(f, g) \in H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_1)'$, we search for a H^1–solution of the problem[2]

$$(CP) \qquad \begin{cases} Pu = 0 & , \text{ in } \Omega \\ u = f & , \text{ at } \Gamma_1 \\ u_{\nu_A} = g & , \text{ at } \Gamma_1 \end{cases} .$$

Our objective is to reconstruct the trace of the solution u and it's conormal derivative at Γ_2.[3] Given the approximation $\varphi_0 \in H_{00}^{1/2}(\Gamma_2)'$ for $u_{\nu_A|\Gamma_2}$, we define the sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ using the following iteration rule:

[2]Details about the notation can be found in Appendix A.

[3]Note that if we knew the conormal derivative at Γ_2, u could be evaluated as the solution of a mixed BVP.

$$
(IT) \qquad \left\{ \begin{array}{l} w \in H^1(\Omega) \text{ solve: } \quad Pw = 0; \quad w_{|\Gamma_1} = f; \quad w_{\nu_A|\Gamma_2} = \varphi_k; \\ \psi_k := w_{|\Gamma_2}; \\ \\ v \in H^1(\Omega) \text{ solve: } \quad Pv = 0; \quad v_{\nu_A|\Gamma_1} = g; \quad v_{|\Gamma_2} = \psi_k; \\ \varphi_{k+1} := v_{\nu_A|\Gamma_2}. \end{array} \right.
$$

In (IT) two differential equations are solved and two trace operators are applied. Actually we generate two sequences: the first one of Dirichlet traces and the second one of Neumann traces, both defined at Γ_2. As the functions w and v are both in $H^1(\Omega; P)$, one concludes from Theorems A.2 and A.4 respectively that $\{\varphi_k\} \subset H_{00}^{1/2}(\Gamma_2)'$ and $\{\psi_k\} \subset H^{1/2}(\Gamma_2)$.

Remark 1.1 *If the Neumann data g of (CP) is a $H^{-1/2}(\Gamma_2)$–distribution, one proves using the Theorems of Appendix C that the sequence $\{\varphi_k\}$ can be defined on the Sobolev space $H^{-1/2}(\Gamma_2)$.*

Remark 1.2 *If one supposes $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and wants to analyze a Cauchy problem with data given at Γ_1 plus a further boundary condition (Neumann, Dirichlet, ...) at Γ_3, it is possible to adapt the iteration by adding this boundary condition at Γ_3 to both BVP in (IT). This over–determination of boundary data does not affect the analysis of the algorithm.*

1.4 Functional–analytical approach

The main objective in this section is to represent the iteration (IT) using an operator $T : H_{00}^{1/2}(\Gamma_2)' \to H_{00}^{1/2}(\Gamma_2)'$. We define the operators $L_n : H_{00}^{1/2}(\Gamma_2)' \to H^1(\Omega)$ and $L_d : H^{1/2}(\Gamma_2) \to H^1(\Omega)$ by:

$$
L_n(\varphi) := w \in H^1(\Omega) \qquad \textit{and} \qquad L_d(\psi) := v \in H^1(\Omega),
$$

where the functions w and v are respectively solutions of the BVP's

$$
Pw = 0 \ \text{ in } \Omega; \quad w_{|\Gamma_1} = f; \quad w_{\nu_A|\Gamma_2} = \varphi
$$

and

$$
Pv = 0 \ \text{ in } \Omega; \quad v_{\nu_A|\Gamma_1} = g; \quad v_{|\Gamma_2} = \psi
$$

With the aid of the Neumann trace operator $\gamma_n : H^1(\Omega, P) \to H_{00}^{1/2}(\Gamma_2)'$, $\gamma_n(u) := u_{\nu_A|\Gamma_2}$ and the Dirichlet trace operator $\gamma_d : H^1(\Omega) \to H^{1/2}(\Gamma_2)$, $\gamma_d(u) := u_{|\Gamma_2}$ one can rewrite (IT) as

$$
\left\{ \begin{array}{lllllll} w & = & L_n(\varphi_k); & \psi_k & = & \gamma_d(w) \\ v & = & L_d(\psi_k); & \varphi_{k+1} & = & \gamma_n(v) \end{array} \right. \tag{3}
$$

If we define $T := \gamma_n \circ L_d \circ \gamma_d \circ L_n$, we conclude immediately that T is an affine operator on $H_{00}^{1/2}(\Gamma_r)'$, which satisfies

$$
\varphi_{k+1} \; = \; T(\varphi_k) \; = \; T^{k+1}(\varphi_0).
$$

That means we are able to describe the iteration (IT) with powers of the operator T. As L_n and L_d are both affine, we can write

$$L_n(\cdot) \;=\; L_n^l(\cdot) \;+\; w_f \qquad \text{and} \qquad L_d(\cdot) \;=\; L_d^l(\cdot) \;+\; v_g,$$

where the $H^1(\Omega, P)$–functions w_f and v_g depend only of f and g respectively. With these definitions we have

$$\begin{aligned} \varphi_{k+1} \;=\; T(\varphi_k) \;&=\; \underbrace{\gamma_n \circ L_d^l \circ \gamma_d \circ L_n^l(\varphi_k)}_{T_l(\varphi_k)} \;+\; \underbrace{\gamma_n \circ L_d^l \circ \gamma_d(w_f) + \gamma_n(v_g)}_{z_{f,g}} \qquad (4) \\ &=\; T_l^{k+1}(\varphi_0) \;+\; \sum_{j=0}^{k} T_l^j(z_{f,g}). \end{aligned}$$

Remark 1.3 *If we set $\overline{\varphi} = \gamma_n u$, where u is the solution of (CP), it follows from (IT) that $T\,\overline{\varphi} = \overline{\varphi}$. Conversely, if $\overline{\varphi}$ is a fixed point of the operator T, the functions w and v in (IT) have the same traces at Γ_2. From the uniqueness Theorem D.3 follows $w = v$ and they are both solutions of (CP).*

2 Analysis of the method

2.1 Convergence proof

In order to study the iterative method proposed in section 1.3, we begin with equipping the space $H_{00}^{1/2}(\Gamma_2)'$ with a new topology.

Lemma 2.1 *Let the coefficients of P satisfy the conditions in (2). The functional*

$$||\varphi||_* := \left(\int_\Omega (\nabla L_n^l(\varphi))^t A\, (\nabla L_n^l(\varphi))\, dx \right)^{1/2}$$

defines on $H_{00}^{1/2}(\Gamma_2)'$ a norm, that is equivalent to the usual Sobolev norm of this space.

Proof. Given $\varphi \in H_{00}^{1/2}(\Gamma_2)'$, the function $u = L_n^l(\varphi)$ solves following BVP:

$$\left\{ \begin{array}{rcll} P\,u &=& 0\,, & \text{in } \Omega \\ u &=& 0\,, & \text{at } \Gamma_1 \\ u_{\nu_A} &=& \varphi\,, & \text{at } \Gamma_2 \end{array} \right.$$

From Theorem C.2 one concludes $L_n^l(\varphi)$ is the unique solution in $H_0^1(\Omega \cup \Gamma_2)$. In the same theorem the continuous dependence of the data is proved, and from this follows

$$||\varphi||_* \;\leq\; c_1\; ||L_n^l(\varphi)||_{H^1(\Omega)} \;\leq\; c_2\; ||\varphi||_{H_{00}^{1/2}(\Gamma_2)'}\,,$$

where the first inequality follows from the norm equivalence between $||\cdot||_{H^1(\Omega)}$ and $\langle A\nabla\cdot, \nabla\cdot\rangle^{1/2}_{L^2(\Omega)}$ at $H^1_0(\Omega \cup \Gamma_2)$.

The opposite inequality follows from the continuity of the Neumann trace operator in Theorem A.4 and the norm equivalence used just above. $\square$

Remark 2.2 *Actually one can prove that the norm $||\varphi||_*$ is defined by an inner product and the space $H^{1/2}_{00}(\Gamma_2)'$ is a Hilbert space with the inner product*

$$\langle \varphi, \psi \rangle_* \;=\; \int_\Omega (\nabla L^l_n(\varphi))^t A\, (\nabla L^l_n(\psi))\; dx\,.$$

In the next theorem we investigate some properties of the operator T_l, defined in section 1.4, when we equip the space $H^{1/2}_{00}(\Gamma_2)'$ with the Hilbert space structure defined by $\langle\cdot,\cdot\rangle_*$.

Theorem 2.3 *Let $T_l \in \mathcal{L}(H^{1/2}_{00}(\Gamma_2)')$ be the operator defined in (4). The following assertions hold:*

i) T_l is positive;

ii) 1 is not an eigenvalue of T_l;

iii) T_l is self adjoint;

iv) T_l is injective.

Proof. *i)* We define the operator $W : H^{1/2}_{00}(\Gamma_2)' \to H^1(\Omega)$ by $W(\varphi) := L^l_d \circ \gamma_d \circ L^l_n(\varphi)$, where the operators L^l_d, L^l_n and γ_d are the same as in section 1.4. From Theorems B.3 and B.4 follows for $\varphi, \psi \in H^{1/2}_{00}(\Gamma_2)'$

$$\begin{aligned}\int_\Omega \left(\nabla L^l_n T_l(\varphi) - \nabla W(\varphi)\right)^t A\,(\nabla L^l_n(\psi))\, dx \;&= \\ &= \int_\Omega P\left(L^l_n T_l(\varphi) - W(\varphi)\right) L^l_n(\psi)\, dx \\ &\quad + \int_{\Gamma_1\cup\Gamma_2} \left(L^l_n T_l(\varphi) - W(\varphi)\right)_{\nu_A} L^l_n(\psi) d\Gamma \;=\; 0.\end{aligned} \tag{5}$$

From an analogous argument we have

$$\int_\Omega \left(\nabla W(\varphi) - \nabla L^l_n(\varphi)\right)^t A(\nabla W(\psi))\, dx \;=\; 0. \tag{6}$$

If we denote by $\langle\cdot,\cdot\rangle$ the inner product on $L^2(\Omega)$, it follows from (5) and (6)

$$\begin{aligned}\langle T_l\,\varphi, \varphi\rangle_* \;&=\; \langle A\,\nabla L^l_n T_l(\varphi), \nabla L^l_n(\varphi)\rangle \\ &\overset{(5)}{=}\; \langle A\,\nabla W(\varphi), \nabla L^l_n(\varphi)\rangle \\ &\overset{(6)}{=}\; \langle A\,\nabla W(\varphi), \nabla W(\varphi)\rangle \\ &\geq\; c\,||W(\varphi)||^2_{H^1(\Omega)},\end{aligned}$$

for every $\varphi \in H_{00}^{1/2}(\Gamma_2)'$.

ii) Let us suppose there exists a $\varphi \in H_{00}^{1/2}(\Gamma_2)'$, such that $T_l\,\varphi = \varphi$. Define $w := L_n^l(\varphi)$ and $v := L_d \circ \gamma_d \circ L_n^l(\varphi)$. For the difference $v - w$ we have:

$$(v-w)_{|\Gamma_2} \;=\; (v-w)_{\nu_A|\Gamma_2} \;=\; 0.$$

From the unicity Theorem D.3 we have $v = w$. The definition of w and v imply $0 = w_{|\Gamma_l}$ and $0 = v_{\nu_A|\Gamma_1} = w_{\nu_A|\Gamma_1}$. Theorem D.3 now implies $\varphi = 0$.

iii) analogous to (5) and (6) one proves that for $\varphi, \psi \in H_{00}^{1/2}(\Gamma_2)'$ the identities

$$\langle A\,\nabla L_n^l(\varphi), \nabla L_n^l T_l(\psi)\rangle \;=\; \langle A\,\nabla L_n^l(\varphi), \nabla W(\psi)\rangle \tag{7}$$

and

$$\langle A\,\nabla W(\varphi), \nabla W(\psi)\rangle \;=\; \langle A\,\nabla W(\varphi), \nabla L_n^l(\psi)\rangle \tag{8}$$

hold. These last equations imply

$$\langle T_l\,\varphi, \psi\rangle_* \;=\; \langle \varphi, T_l\,\psi\rangle_*,\ \forall\ \varphi, \psi \in H_{00}^{1/2}(\Gamma_2)'.$$

iv) Take φ_1, φ_2 in $H_{00}^{1/2}(\Gamma_2)'$ with $T_l\,\varphi_1 = T_l\,\varphi_2$. Define now $w := L_n^l(\varphi_1 - \varphi_2)$ and $v := L_d^l \circ \gamma_d \circ L_n^l(\varphi_1 - \varphi_2)$. We clearly have $v_{\nu_A|\Gamma_1} = 0$ and the hypothesis $T_l(\varphi_1 - \varphi_2) = 0$ implies $v_{\nu_A|\Gamma_2} = 0$. Since v satisfies $P\,v = 0$, we conclude that v is constant in Ω. From $v_{|\Gamma_2} = w_{|\Gamma_2} \in H_{00}^{1/2}(\Gamma_2)$ follows $v \equiv 0$.[4] Then we have $w \equiv 0$ in Ω and the equality $\varphi_1 = \varphi_2$ follows. □

In the next theorem we verify two properties of T_l, that are needed in the convergence proof of the iterative method described in section 1.3.

Theorem 2.4 *Let $T_l \in \mathcal{L}(H_{00}^{1/2}(\Gamma_2)')$ be the operator defined in (4). The following assertions are valid:*

i) T_l is regular asymptotic in $H_{00}^{1/2}(\Gamma_2)'$, i.e. $\lim_{k\to\infty} ||T_l^{k+1}(\varphi) - T_l^k(\varphi)||_* = 0,\ \forall \varphi \in H_{00}^{1/2}(\Gamma_2)'$;

ii) The operator T_l is non expansive, i.e. $||T_l||_{\mathcal{L}(H_{00}^{1/2}(\Gamma_2)')} \le 1$.

Proof. *i)* Because of the identity $(T_l^{k+1}(\varphi) - T_l^k(\varphi)) = T_l^k(T_l - I)(\varphi)$, it is enough to prove that $T_l^k(\varphi_0) \to 0$ for every $\varphi_0 \in \mathrm{Rg}(T_l - I)$. Take $\psi \in H_{00}^{1/2}(\Gamma_2)'$ with $(T_l - I)\psi = \varphi$. Note that we can describe the iteration $T_l^k(\psi)$ using the functions

$$\left\{ \begin{array}{rcll} w_k(\psi) & = & L_n^l(\gamma_n(v_{k-1}(\psi))), & k \ge 1 \\ v_k(\psi) & = & L_d^l(\gamma_d(w_k(\psi))), & k \ge 0 \end{array} \right. \tag{9}$$

[4]The unique constant function in $H_{00}^{1/2}(\Gamma_l)$ is the null function.

and $w_0(\psi) = L_n^l(\psi)$. From (9) follows $w_k = (v_k)_{\nu_A} = 0$ at Γ_1, $(w_k)_{\nu_A} = (v_{k-1})_{\nu_A}$ and $v_k = w_k$ at Γ_2. These identities and Theorem B.3 give us

$$\begin{aligned}
\int_\Omega (\nabla v_k)^t A\,(\nabla v_k)\,dx &= \int_{\Gamma_2} v_k (v_k)_{\nu_A}\,d\Gamma,\\
\int_\Omega (\nabla w_k)^t A\,(\nabla w_k)\,dx &= \int_{\Gamma_2} w_k (w_k)_{\nu_A}\,d\Gamma,\\
\int_\Omega (\nabla v_k)^t A\,(\nabla w_k)\,dx &= \int_{\Gamma_2} w_k (v_k)_{\nu_A}\,d\Gamma,\\
\int_\Omega (\nabla v_{k-1})^t A\,(\nabla w_k)\,dx &= \int_{\Gamma_2} w_k (v_{k-1})_{\nu_A}\,d\Gamma.
\end{aligned}$$

From these identities we obtain

$$\begin{aligned}
&\int_\Omega \nabla(w_k - v_{k-1})^t A\,\nabla(w_k - v_{k-1})\,dx \;= \\
&= \int_\Omega \Big[(\nabla v_{k-1})^t A\,(\nabla v_{k-1}) \;-\; (\nabla w_k)^t A\,(\nabla w_k)\Big]\,dx
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\int_\Omega \nabla(v_k - w_k)^t A\,\nabla(v_k - w_k)\,dx \;= \\
&= \int_\Omega \Big[(\nabla w_k)^t A\,(\nabla w_k) \;-\; (\nabla v_k)^t A\,(\nabla v_k)\Big]\,dx.
\end{aligned} \tag{11}$$

Note that the definition of φ and ψ imply $w_k(\varphi) = w_{k+1}(\psi) - w_k(\psi)$. Equations (10) and (11) now imply

$$\begin{aligned}
&\int_\Omega (\nabla w_k(\varphi))^t A\,(\nabla w_k(\varphi))\,dx \;\le \\
&\qquad \le\; 2\int_\Omega \Big[(\nabla w_k(\psi))^t A\,(\nabla w_k(\psi)) \;-\; (\nabla w_{k+1}(\psi))^t A\,(\nabla w_{k+1}(\psi))\Big]\,dx.
\end{aligned}$$

From this last equation we obtain

$$||T_l^k(\varphi)||_*^2 \;\le\; 2\left(||T_l^k(\psi)||_*^2 \;-\; ||T_l^{k+1}(\psi)||_*^2\right). \tag{12}$$

Another consequence of (10) and (11) is the inequality

$$\int_\Omega (\nabla w_k)^t A(\nabla w_k)\,dx \;-\; \int_\Omega (\nabla w_{k+1})^t A(\nabla w_{k+1})\,dx \;\ge\; 0,$$

for every k, i.e. the sequence $\{||T_l^k(\psi)||_*\}$ does not increase. Now from (12) follows

$$\lim_{k\to\infty} ||T_l^k(\varphi)||_* \;=\; 0.$$

ii) For $\varphi \in H_{00}^{1/2}(\Gamma_r)'$ define $w := L_n^l(\varphi)$ and $v := L_d^l \circ \gamma_d \circ L_n^l(\varphi)$. We claim that the inequality

$$\langle T_l(\varphi), T_l(\varphi)\rangle_* \;\le\; \int_\Omega (\nabla v)^t A\,(\nabla v)\,dx \tag{13}$$

holds. Indeed, as $T_l(\varphi) = \gamma_n v$ we have

$$\begin{aligned}
\langle T_l(\varphi), T_l(\varphi)\rangle_* &= \int_\Omega (\nabla L_n^l(\gamma_n v))^t A\, (\nabla L_n^l(\gamma_n v))\, dx \\
&= \int_{\Gamma_1\cup\Gamma_2} (L_n^l(\gamma_n v))_{\nu_A} L_n^l(\gamma_n v)\, d\Gamma \\
&\quad + \int_\Omega L_n^l(\gamma_n v)\, P(L_n^l(\gamma_n v))\, dx \\
&= \int_{\Gamma_1\cup\Gamma_2} v_{\nu_A}\, L_n^l(\gamma_n v)\, d\Gamma \ + \int_\Omega P(v)\, L_n^l(\gamma_n v)\, dx \\
&= \int_\Omega (\nabla v)^t A\, (\nabla L_n^l(\gamma_n v))\, dx \\
&\le \left(\int_\Omega (\nabla v)^t A\, (\nabla v)\, dx\right)^{1/2} \langle T_l(\varphi), T_l(\varphi)\rangle_*^{1/2},
\end{aligned}$$

proving (13). Now from the definition of v and w follows

$$\begin{aligned}
\int_\Omega (\nabla v)^t A\, (\nabla v)\, dx &= \int_{\Gamma_2} v_{\nu_A}\, v\, d\Gamma = \int_{\Gamma_1\cup\Gamma_2} v_{\nu_A}\, w\, d\Gamma = \int_\Omega (\nabla v)^t A\, (\nabla w)\, dx \\
&\le \left(\int_\Omega (\nabla v)^t A\, (\nabla v)\, dx\right)^{1/2} \left(\int_\Omega (\nabla w)^t A\, (\nabla w)\, dx\right)^{1/2}.
\end{aligned}$$

Putting all together we have

$$||T_l(\varphi)||_* \ \le\ \left(\int_\Omega (\nabla w)^t A\, (\nabla w)\, dx\right)^{1/2} \ =\ \langle \varphi, \varphi\rangle_*^{1/2} \ =\ ||\varphi||_*,$$

proving *(ii)*. □

The next theorem guarantees the convergence of the iterative algorithm.

Theorem 2.5 *Let T and T_l be the operators defined in section 1.4. If we have consistent Cauchy–data (f, g), then the sequence $\varphi_k = T^k \varphi_0$ converges to the Neumann–trace at Γ_2 of the solution of (CP) for every $\varphi_0 \in H_{00}^{1/2}(\Gamma_2)'$.*

Proof. It is enough to prove that $T^k\varphi_0$ converges to the fixed point $\overline{\varphi}$ of T. If we define $\varepsilon_k = \varphi_k - \overline{\varphi}$ we have

$$\begin{aligned}
\varepsilon_{k+1} &= \varphi_{k+1} - \overline{\varphi} \\
&= T(\varphi_k) - T(\overline{\varphi}) \\
&= T_l(\varphi_k) + z_{f,g} - T_l(\overline{\varphi}) - z_{f,g} \\
&= T_l(\varepsilon_k)\,.
\end{aligned}$$

Theorems 2.3 and 2.4 imply $\varepsilon_k \to 0$. □

The converse of Theorem 2.5 is valid, i.e. when the sequence $\varphi_k = T^k\varphi_0$ converges, the associated Cauchy problem is consistent and $\overline{\varphi} := \lim \varphi_k$ is the Neumann–trace of the problem's solution. This result can also be understood as an existence criterion for Cauchy problems.

Theorem 2.6 *Given the Cauchy data* $(f,g) \in H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_1)'$, *we denote by* $\{\varphi_k\}$ *the sequence generated by the iteration (IT). If* $\{\varphi_k\}$ *converges in* $H_{00}^{1/2}(\Gamma_2)'$, *the Cauchy problem (CP) has a solution* u *in* $H^1(\Omega; P)$ *and* $u_{\nu_A|\Gamma_2} = \lim_k \varphi_k$.

Proof. If we define $\overline{\varphi} = \lim_k \varphi_k \in H_{00}^{1/2}(\Gamma_2)'$, we have

$$T\,\overline{\varphi} \;=\; T(\lim_{k\to\infty} \varphi_k) \;=\; \lim_{k\to\infty} \varphi_{k+1} \;=\; \overline{\varphi}\,.$$

Therefore $\overline{\varphi}$ is a fixed point of T. The argument of Remark 1.3 implies the existence of a solution for (CP) and the theorem follows. □

2.2 A spectral property of T_l

Before going any further with the analysis of the iterative algorithm, we discuss an important spectral property of T_l. We have already proved in Theorem 2.3 that T_l is positive, self adjoint and it's spectrum belongs to $[0,1]$. Now we verify that 1 belongs to $\sigma(T_l)$.

Theorem 2.7 *Let* T_l *be the operator defined in section 1.4. If there exists* $(f,g) \in H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_1)'$ *such that the Cauchy problem (CP) is inconsistent for the data* (f,g), *then 1 belongs to the continuous spectrum* $\sigma_c(T_l)$ *of* T_l.

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family for T_l and denote by I be the identity operator in $H_{00}^{1/2}(\Gamma_2)'$. From Theorem 2.3 we conclude that $E_\lambda = I$ for $\lambda \geq 1$. It's enough to prove that given $\delta \in (0,1)$ there exists an eigenvalue λ_0 of T_l in the interval $(\delta, 1)$.

If this condition were not satisfied the point spectrum $\sigma_p(T_l)$ would be a subset of $[0,\delta]$ and T_l would be contractive with norm $||T_l|| \leq \delta$. An immediate consequence of this is the convergence of the sequence $\varphi_k = T^k\varphi_0$, where $T = T_l + z_{f,g}$. Now Theorem 2.6 would imply the existence of a solution for the Cauchy problem (CP) with data (f,g), contradicting the hypothesis of (f,g) being inconsistent Cauchy data. □

Corollary 2.8 *From Theorems 2.3 and 2.7 follows* $||T_l|| = 1$.

2.3 Error estimation

For simplicity we investigate in this section the iteration (IT) for the operator $P = -\Delta$ in two special domains. Analog results can be obtained for general operators of the form (1) every time the spectral decomposition of the operator is known.

In the first problem we take $\Omega = (-\pi, \pi) \times (-\pi, \pi)$, $\Gamma_1 = \{(x, 0);\ x \in (-\pi, \pi)\}$, $\Gamma_2 = \{(x, \pi);\ x \in (-\pi, \pi)\}$ and want to solve the problem

$$(CP\ 1) \qquad \left\{ \begin{array}{rl} \Delta u = 0, & \text{in } \Omega \\ u = f, & \text{at } \Gamma_1 \\ u_\nu = g, & \text{at } \Gamma_1 \\ u(x, \pm\pi) = 0, & x \in (-\pi, \pi) \end{array} \right.$$

In the second problem Ω is the ring centered at the origin with inner and outer radius respectively r_0 and 1, $\Gamma_1 = \{(1, \theta);\ \theta \in (-\pi, \pi)\}$, $\Gamma_2 = \{(r_0, \theta);\ \theta \in (-\pi, \pi)\}$. The problem to be solved is

$$(CP\ 2) \qquad \left\{ \begin{array}{rl} \Delta u = 0, & \text{in } \Omega \\ u = f, & \text{at } \Gamma_1 \\ u_\nu = g, & \text{at } \Gamma_1 \end{array} \right.$$

As we are working on special domains, it is possible to describe the action of the operator T_l explicitly. If $\varphi_0 = \sum \varphi_{0,j} \sin(jy)$ is given in the Sobolev space of periodic functions[5] $H^{-1/2}_{\text{per}}(\Gamma_2)$, we have for (CP 1)

$$(T_l^k\, \varphi_0)(x) \;=\; \sum_{j=1}^{\infty} \lambda_j^{2k}\, \varphi_{0,j}\, \sin(jx), \tag{14}$$

where $\lambda_j = sinh(2j\pi)/cosh(2j\pi)$, $j \in \mathbb{N}$. As we intend to measure how fast the error $\varepsilon_k := \varphi_k - \overline{\varphi}$ converges to zero, we deduce from (14) and from the equality $\varepsilon_{k+1} = T_l\, \varepsilon_k$ the estimate

$$||\varepsilon_k||^2_{H^{-1/2}_{\text{per}}(\Gamma_2)} \;\le\; \sum_{j \ge 1} j^{-1} \left(\lambda_j^k\, \varepsilon_{0,j} \right)^2 . \tag{15}$$

If the initial error ε_0 has the nice property of consisting only of the lower frequencies $j \le J$, equation (15) simplifies to

$$||\varepsilon_k||^2_{H^{-1/2}_{\text{per}}(\Gamma_2)} \;\le\; \lambda_j^{2k}\, ||\varepsilon_0||^2_{H^{-1/2}_{\text{per}}(\Gamma_2)} .$$

In the very special case $J = 1$ and $\varepsilon_0 = \varepsilon_{0,j} \sin(x)$ one calculates for $k = 10^5$ the power of the first eigenvalue $\lambda_1^{2k} = 0.061$. Therefore we must evaluate 10^5 iteration steps to reduce the error to 6% of the initial error.

Next we analyze a more realistic situation, in which the initial error $\varepsilon_0 = \varphi_0 - \overline{\varphi}$ has more regularity than a $H^{-1/2}_{\text{per}}(\Gamma_2)$ distribution. We assume that there exists a monotone sequence of positive real numbers $\{c_j\}$ such that

$$\lim_{j \to \infty} c_j \;=\; \infty \quad \text{and} \quad \sum_{j \ge 1} j^{-1}\, c_j^2\, \varepsilon_{0,j}^2 \;=\; M \;<\; \infty .$$

[5] For $s \in \mathbb{R}$ one defines $H^s_{\text{per}}((-\pi, \pi)) := \{\varphi(y) = \sum_{k \in \mathbb{Z}} c_k\, e^{iky} \mid \sum_{k \in \mathbb{Z}} (1 + k^2)^s c_k^2 \;<\; \infty\}$.

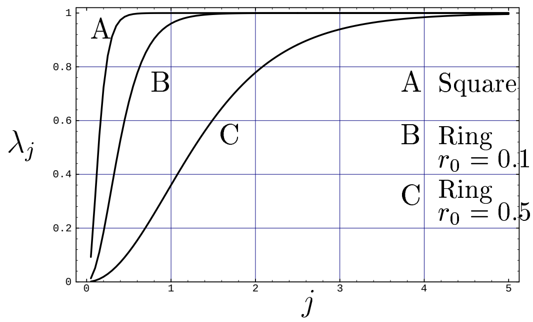


Figure 1: Eigenvalues of T_l in different domains

In this case the error at the k$^{\text{th}}$–iteration step can be estimated by

$$\begin{aligned} ||\varepsilon_k||^2_{H^{-1/2}_{\text{per}}(\Gamma_2)} &\le \lambda_J^{2k} \left(\frac{c_J}{c_1}\right)^2 \sum_{j\le J} j^{-1}\,\varepsilon_{0,j}^2 \;+\; \frac{1}{c_J^2} \sum_{j>J} j^{-1}\, c_j^2\, \lambda_j^{2k} \varepsilon_{0,j}^2 \\ &\le \lambda_J^{2k} \left(\frac{c_J}{c_1}\right)^2 ||\varepsilon_0||^2_{H^{-1/2}_{\text{per}}(\Gamma_2)} \;+\; \frac{M}{c_J^2}. \end{aligned} \tag{16}$$

For the Cauchy problem (CP 2) we have an analogous result. If the iteration is again formulated at $H^{-1/2}_{\text{per}}(\Gamma_2)$ one obtains for the operator T_l the eigenfunctions $\sin(j\theta)$, $cos(j\theta)$ with corresponding eigenvalues

$$\lambda_j \;=\; [(r_0^{-(j+1)} - r_0^{j-1})(r_0^{-j} - r_0^{j})] \;/\; [(r_0^{-(j+1)} + r_0^{j-1})(r_0^{-j} + r_0^{j})]\,.$$

In Figure 1 we show a qualitative comparison between the eigenvalues of T_l in the different domains considered in this section.

2.4 Regularization

The objective of regularizing the iteration (IT) is to choose an operator T_{reg} such that the regularized sequence $\tilde{\varphi}_k := T_{\text{reg}}^k\,\varphi_0 + \sum_{j<k} T_{\text{reg}}^j\, z_{f,g}$ converges faster than the original sequence $\varphi_k := T_l^k\,\varphi_0 + \sum_{j<k} T_l^j\, z_{f,g}$. We also have to assure that the difference $\|\lim\tilde{\varphi}_k - \lim\varphi_k\|$ remains small.

We start with the *a priori* assumption that the Cauchy data of (CP) satisfies $(f,g) \in H^r(\Gamma_1) \times H^{1/2}_{00}(\Gamma_1)'$, with $r > 1/2$.

Given the measured data $(f_\varepsilon, g_\varepsilon)$ in $L^2(\Gamma_1)\times H^{1/2}_{00}(\Gamma_1)'$, we claim that using a smoothing operator $S : L^2 \to H^{1/2}$ it is possible to generate a $\tilde{f}_\varepsilon := Sf_\varepsilon \in H^{1/2}$ satisfying $\|f - \tilde{f}_\varepsilon\|_{1/2} \le \varepsilon'$. Indeed this is a consequence of

Lemma 2.9 *Let $f \in H^r$, $r > s > 0$. There exists a smoothing operator $S : L^2 \to H^s$ and a positive function γ with $\lim_{x\downarrow 0}\gamma(x) = 0$, such that for $\varepsilon > 0$ and $f_\varepsilon \in L^2$ with $\|f - f_\varepsilon\|_{L^2} \le \varepsilon$, we have $\|f - Sf_\varepsilon\|_s \le \gamma(\varepsilon)$.*

Proof. This lemma describes a standard procedure in inverse problems. A complete proof can be found in [BaLe]. □

After smoothing the data f_ε, we obtain a corresponding $z_\varepsilon \in H^{1/2}(\Gamma_l)'$ such that $||z_{f,g} - z_\varepsilon||_{H^{1/2}(\Gamma_l)'} < \varepsilon$.

We analyze the choice of two different regularization strategies. The first one is a cut–off method, where we consider only the eigenvalues of T_l lower than $1 - 1/n$.[6] In the second method we use powers of T_l to define $T_{\rm reg}$. For $n \geq 2$ we set

$$A_n \;:=\; \int_0^{1-\frac{1}{n}} \lambda\, dE_\lambda \qquad \text{and} \qquad B_n \;:=\; \int_0^1 (\lambda - \lambda^n)\, dE_\lambda\,, \tag{17}$$

where E_λ is the spectral family of T_l. Both operators A_n and B_n are positive, self adjoint and contractive. Let $T_{\rm reg}^{(n)}$ represent one of the families defined in (17) and define $\overline{\varphi}$ and $\varphi^{(n)}$ as the fixed points of

$$\overline{\varphi} \;=\; T_l\, \overline{\varphi} + z_{f,g} \qquad \text{and} \qquad \varphi^{(n)} \;=\; T_{\rm reg}^{(n)}\, \varphi^{(n)} + z_\varepsilon$$

respectively. $\varphi^{(n)}$ will exist, as $T_{\rm reg}^{(n)}$ is contractive. We have now

$$\begin{aligned} ||\varphi^{(n)} - \overline{\varphi}|| \;&=\; ||T_l\, \overline{\varphi} + z_{f,g} - T_{\rm reg}^{(n)}\, \varphi^{(n)} - z_\varepsilon|| \\ &=\; ||T_{\rm reg}^{(n)}(\varphi^{(n)} - \overline{\varphi}) + \left(T_{\rm reg}^{(n)} - T_l\right)\, \overline{\varphi} - z_{f,g} + z_\varepsilon|| \\ &\leq\; ||\left(I - T_{\rm reg}^{(n)}\right)^{-1} \left(T_{\rm reg}^{(n)} - T_l\right)\, \overline{\varphi}|| \;+\; \varepsilon\, ||\left(I - T_{\rm reg}^{(n)}\right)^{-1}||. \end{aligned} \tag{18}$$

In next theorem we analyze the estimate (18) for the operators A_n and B_n.

Theorem 2.10 *If we define the family of operators $T_{\rm reg}^{(n)}$ using one of the families in (17) we have*

$$||\left(I - T_{\rm reg}^{(n)}\right)^{-1} \left(T_{\rm reg}^{(n)} - T_l\right)\, \overline{\varphi}|| \to 0 \quad \text{and} \quad ||\left(I - T_{\rm reg}^{(n)}\right)^{-1}|| \to \infty. \tag{19}$$

Proof. i) We analyze the case $T_{\rm reg}^{(n)} = A_n$ first. From the spectral decomposition of T_l follows

$$(I - A_n) \;=\; \int_0^{1-\frac{1}{n}} (1-\lambda)\, dE_\lambda + \int_{1-\frac{1}{n}}^1 dE_\lambda \qquad \text{and} \qquad (A_n - T_l) \;=\; -\int_{1-\frac{1}{n}}^1 \lambda\, dE_\lambda.$$

Now these equalities imply $||(I - A_n)\, \varphi|| \geq 1/n\, ||\varphi||$, and the operator $(I - A_n)$ has an inverse. We also know that $(I - A_n)$ is the identity operator on $Rg(A_n - T_l)$. From this follows

$$(I - A_n)^{-1}(A_n - T_l) \;=\; -\int_{1-\frac{1}{n}}^1 \lambda\, dE_\lambda, \tag{20}$$

[6] We may suppose the real numbers $1 - 1/n$ are not eigenvalues of T_l.

and we can estimate the first term in (19) by

$$||(I-A_n)^{-1}(A_n-T_l)\,\overline{\varphi}||^2 \;\leq\; \int_{1-\frac{1}{n}}^{1} d\langle E_\lambda\,\overline{\varphi},\overline{\varphi}\rangle \;\rightarrow\; 0$$

for $n\rightarrow\infty$. For the second term in (19) we use the identity

$$||(I-A_n)^{-1}|| \;=\; (1-\Lambda(n))^{-1} \;\rightarrow\; \infty \tag{21}$$

for $n\rightarrow\infty$, where $\Lambda(n)$ is the largest eigenvalue of T_l, which is smaller than $(1-1/n)$.

ii) For the case $T^{(n)}_{\text{reg}} = B_n$ we have $(B_n - T_l) \;=-\,T_l^n$ and from the spectral decomposition of T_l follows

$$(I-B_n)^{-1}(B_n-T_l) \;=\; \int_0^1 \frac{\lambda^n}{(1+\lambda^n-\lambda)} dE_\lambda.$$

If we define the functions $\mu(n) := n^{(\frac{1}{1-n})}$ and $\delta(n) := 1-\mu(n)$ we can decompose the operator $(I-B_n)^{-1}(B_n-T_l)$ in $\mathcal{Q}_n+\mathcal{R}_n$ where

$$\mathcal{Q}_n \;=\; \int_0^{1-\delta(n)} \frac{\lambda^n}{(1+\lambda^n-\lambda)}\,dE_\lambda \qquad \text{and} \qquad \mathcal{R}_n \;=\; \int_{1-\delta(n)}^1 \frac{\lambda^n}{(1+\lambda^n-\lambda)}\,dE_\lambda\,.$$

From the convergence $\mu^n(n)\,(1+\mu^n(n)-\mu(n))^{-1} \rightarrow 0$ for $n\rightarrow\infty$ follows $\lim||\mathcal{Q}_n||=0$. The convergence $||\mathcal{R}_n||\rightarrow 0$ follows from inequality $0<\lambda^n\,(1+\lambda^n-\lambda)^{-1}\leq 1,\ \forall\,\lambda\in[1-\delta(n),1],\ \forall n\geq 2$. With this we have proved

$$\lim_{n\rightarrow\infty} ||(I-B_n)^{-1}(B_n-T_l)\,\overline{\varphi}|| \;=\; 0.$$

To obtain the second limit in (19) we deduce from the spectral decomposition of $(I-B_n)^{-1}$ the equality

$$||(I-B_n)^{-1}|| \;=\; (1+\Upsilon^n(n)-\Upsilon(n))^{-1} \;\rightarrow\; \infty$$

for $n\rightarrow\infty$, where $\Upsilon(n)$ is the largest eigenvalue of T_l, which is smaller than $\mu(n)$. $\square$

Our next step it to use *a priori* information about the solution $\overline{\varphi}$ of the fixed point equation $T\,\varphi=\varphi$ in order to find an optimal regularization strategy. Let's suppose there exists a function G with

$$\begin{cases} G:[0,1)\mapsto \mathbb{R}^+ \text{ is continuous and monotone increasing;} \\[2ex] \lim\limits_{\lambda\rightarrow 1^-} G(\lambda) \;=\; \infty; \\[2ex] \displaystyle\int_0^1 G^2(\lambda)\;d\langle E_\lambda\,\overline{\varphi},\overline{\varphi}\rangle \;=M^2 \;<\; \infty. \end{cases} \tag{22}$$

In the next theorem we analyze how the regularity condition in (22) can be used to balance the approximation and regularization errors.

Theorem 2.11 *Let G be a function which satisfy (22) and δ, μ, Λ, Υ be the functions used in the proof of Theorem 2.10. For the two regularization strategies in (17) there exists $n_{\rm opt} \in \mathbb{N}$, such that*

$$||\varphi^{(n_{\rm opt})} - \overline{\varphi}|| \;\le\; ||\varphi^{(n)} - \overline{\varphi}||,$$

for every $n \in \mathbb{N}$. Further $n_{\rm opt}$ is obtained by solving the minimization problem

$$\min_{n\ge 2} \left\{ \frac{M}{G(1-\frac{1}{n})} \;+\; \frac{\varepsilon}{(1-\Lambda(n))} \right\}$$

for the regularization strategy using A_n. For the B_n regularization strategy $n_{\rm opt}$ is obtained as the solution of the minimization problem

$$\min_{n\ge 2} \left\{ \left(\frac{\mu^n(n)\,||\overline{\varphi}||}{1+\mu^n(n)-\mu(n)} \;+\; \frac{M}{G(\mu(n))} \right) \;+\; \frac{\varepsilon}{(1+\Upsilon^n(n)-\Upsilon(n))} \right\}.$$

Proof. We show here only the proof for the regularization strategy A_n, the second case being analog. From (20) and (22) follows

$$\begin{aligned} ||(I-A_n)^{-1}(A_n - T_l)\,\overline{\varphi}||^2 \;&=\; \int_{1-\frac{1}{n}}^{1} \lambda^2 \; d\langle E_\lambda \overline{\varphi}, \overline{\varphi}\rangle \\ &\le\; \frac{1}{G^2(1-\frac{1}{n})} \int_{1-\frac{1}{n}}^{1} G^2(\lambda)\; d\langle E_\lambda \overline{\varphi}, \overline{\varphi}\rangle \\ &\le\; \frac{M^2}{G^2(1-\frac{1}{n})}. \end{aligned}$$

Now inequality (18) and (21) imply

$$||\varphi^{(n)} - \overline{\varphi}|| \;\le\; \frac{M}{G(1-\frac{1}{n})} \;+\; \frac{\varepsilon}{(1-\Lambda(n))},$$

and the assertion follows. $\square$

Remark 2.12 *One can interpret the regularity condition in (22) as follows: With the aid of the function G one can define the unbounded operator*

$$\mathcal{G} \;=\; \int_0^1 G(\lambda)\, dE_\lambda$$

on $H_{00}^{1/2}(\Gamma_2)'$. The existence of the integral in (22) is equivalent to the assumption that $\overline{\varphi}$ belongs to $D(\mathcal{G})$, the domain of $\mathcal{G}$ defined by

$$D(\mathcal{G}) \;:=\; \left\{ \varphi \in H_{00}^{1/2}(\Gamma_2)' \mid \mathcal{G}(\varphi) \in H_{00}^{1/2}(\Gamma_2)' \right\}.$$

3 Numerical experiments

In this section we present some results obtained by the numerical implementation of the iterative algorithm. In the first two examples in sections 3.1 and 3.2 respectively we solve linear consistent problems in a square and in a annular domain. In section 3.3 we exhibit a linear inconsistent problem and in section 3.4 we analyze a non linear consistent problem.

The computation was performed on the IBM–RISC/6000 machines at the Federal University of Santa Catarina. The elliptic mixed boundary value problems that appear in the iteration were solved using the PLTMG package (see [Ban]).

3.1 A linear problem in a square domain

In this example we take $\Omega = (0,1) \times (0,3/4)$ and decompose the boundary $\partial\Omega$ in $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\Gamma_1 := \{(x,0); x \in (0,1)\}\,, \qquad \Gamma_2 := \{(x,3/4); x \in (0,1)\}\,,$$

$$\Gamma_3 := \{(0,y); y \in (0,3/4)\}\,, \quad \Gamma_4 := \{(1,y); y \in (0,3/4)\}\,.$$

Given the Cauchy data $f(x) = \sin(\pi x)$ and $g(x) = 0$ at Γ_1 we reconstruct the (Dirichlet) trace at Γ_2 of the solution of following Cauchy Problem:

$$\left\{ \begin{array}{rcll} \Delta u & = & 0\,, & \text{in } \Omega \\ u & = & f\,, & \text{at } \Gamma_1 \\ u_\nu & = & g\,, & \text{at } \Gamma_1 \\ u & = & 0\,, & \text{at } \Gamma_3 \cup \Gamma_4 \end{array} \right. .$$

The exact solution of this Cauchy Problem is $u^*(x,y) = \cosh(\pi y)\,\sin(\pi x)$.

Each mixed problem is solved using an uniform mesh with 262 913 nodes and linear elements. The trace at Γ_2 of the sequence generated by (IT) is shown (solid line) after 10, 25, 50 and 100 steps at Figure 2. The dotted line represent the trace of the exact solution u^* at Γ_2.

As a stopping criterion we choose $||\psi_{k+1} - \psi_k||_{\infty;\Gamma_2} \leq 10^{-3}$. In this example the iterative sequence converges extremely fast. We observe a slower rate of convergence when the mesh is refined, but the approximation obtained with the same stopping criterion is more accurate.

3.2 A linear problem in an annular domain

In this second example Ω is an annulus centered at the origin with inner and outer radius respectively 1 and 7. Given the Cauchy data $f(\theta) = \sin(\theta)$ and

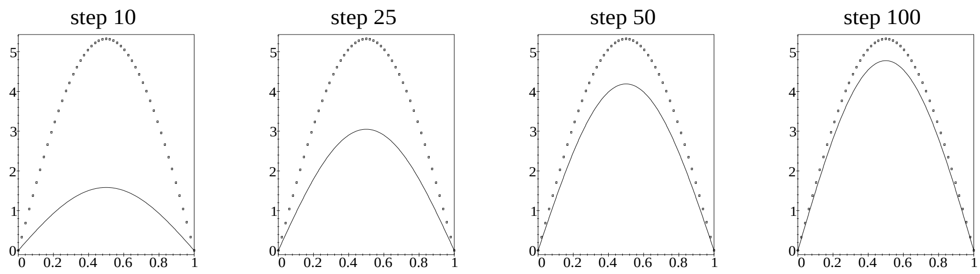


Figure 2: Iterated sequence at the unknown boundary for a linear Cauchy problem at the domain $\Omega = (0,1) \times (0,3/4)$

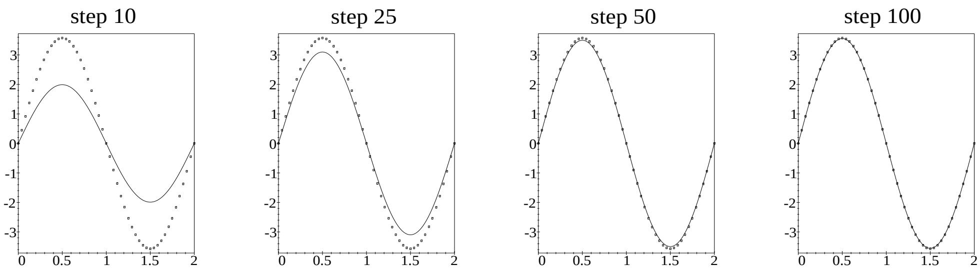


Figure 3: Iterated sequence at the unknown boundary for a linear Cauchy problem at a ring domain with inner and outer radius respectively 1 and 7

$g(\theta) = 0$ at the inner boundary Γ_1, we reconstruct at the outer boundary Γ_2 the trace of the solution of following problem:

$$\left\{ \begin{array}{rcll} \Delta u & = & 0 \ , & \text{in } \Omega \\ u & = & f \ , & \text{at } \Gamma_1 \\ u_\nu & = & g \ , & \text{at } \Gamma_1 \end{array} \right. .$$

The exact solution of this problem is $u^*(x,y) = (r + 1/r)\sin(\theta)/2$. We used a finite element mesh with 61 824 nodes, linear elements and the stopping criterion $||\psi_{k+1} - \psi_k||_{\infty;\Gamma_2} \leq 10^{-4}$. The dotted line in Figure 3 represents the exact solution (note that the x–axis is parameterized from zero to 2π) and the solid line represents the sequence ψ_k generated by (IT).

The iteration gives for this Cauchy problem a better approximation than it does in section 3.1. The reason for this behavior is that the eigenvalues of the operator T_l (defined in (4)) converge to one slower, i.e., they are smaller than we estimated in section 3.1 (see Figure 1).

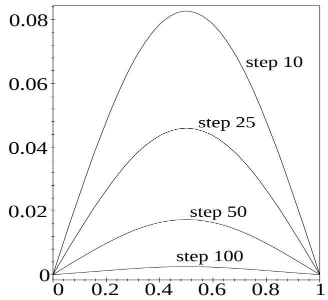


Figure 4: Difference $(\varphi_k - \varphi_{k-1})$ calculated for a linear inconsistent Cauchy problem

3.3 A linear inconsistent problem

In this example we take $\Omega = (0,1)\times(0,1/2)$ and define the boundary segments:

$$\Gamma_1 := \{(x,0); x \in (0,1)\}\,, \qquad \Gamma_2 := \{(x,1/2); x \in (0,1)$$

$$\Gamma_3 := \{(0,y); y \in (0,1/2)\}\,, \quad \Gamma_4 := \{(1,y); y \in (0,1/2)\}.$$

For $n \in \mathbb{N}$ we define at Γ_1 the functions:

$$f(x) = 0 \qquad \text{and} \qquad g(x) = \begin{cases} n - n^2|x-\pi/2| & , \ |x-\pi/2| \le 1/n \\ 0 & , \ \text{otherwise} \end{cases} .$$

Using the reflection principle of Schwartz (see [GiTr] or [Le]) one proves that the Cauchy problem

$$\begin{cases} \Delta u &=& 0\,, & \text{in } \Omega \\ u &=& f\,, & \text{at } \Gamma_1 \\ u_\nu &=& g\,, & \text{at } \Gamma_1 \end{cases} ,$$

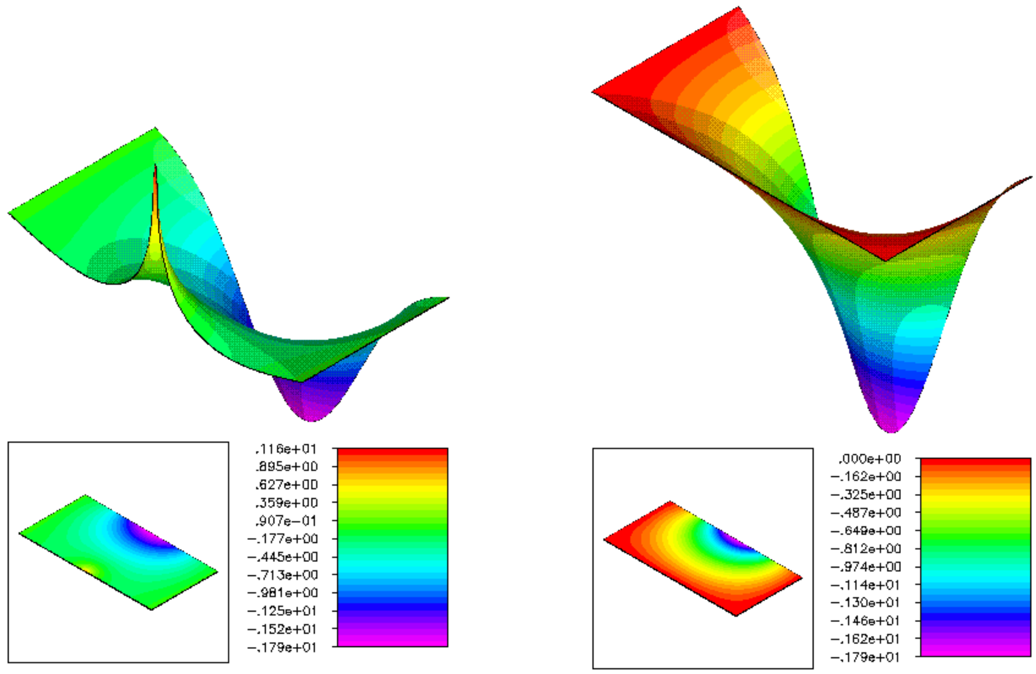
has an analytical solution only if g is itself analytical. So with our choice of data we know a priori that the respective Cauchy problem has no classical solution.

We take $n = 100$ in the definition of g and add to the Cauchy problem above the over determinating condition: $u = 0$ at $\Gamma_3 \cup \Gamma_4$. The iteration is performed as before over a 262 913 node mesh using linear elements and the same stopping criterion as in section 3.1. In Figure 4 the difference $\varphi_k - \varphi_{k-1}$ is plotted for some values of k.

The sequence φ_k converges in the $||\cdot||_\infty$ as fast as it does in section 3.1, but this does not mean that it converges to a solution of the Cauchy problem (see Remark 4.2).

Both sequences w_k and v_k of H^1–functions generated in (IT) converge to the solution of the Cauchy problem, when this problem does have a solution (see Theorem 2.5). In this example, when we analyse the sequences w_k and v_k, we note that they are not converging in $H^1(\Omega)$ to the same limit. In Figures 5 (a) and 5 (b) we show the functions w_{100} and v_{100} respectively.

One observes that on Γ_2 we have $w_{100} \simeq v_{100}$ and $(w_{100})_\nu \simeq (v_{100})_\nu$. The difference $w_k - v_k$ generates also a sequence of H^1–functions with vanishing Cauchy data at Γ_2 but that does not converge to zero at Ω. Such examples are known to exist due to Hadamard (see section 1.2).



(a) (b)

Figure 5: Functions w_k and v_k – at (a) and (b) respectively – after 100 steps for a linear inconsistent Cauchy problem

3.4 A non linear problem

Take Ω the annulus centered at the origin with inner and outer radius respectively 1/2 and 1. This time we decompose the outer component of the boundary in two different ways: $\partial\Omega = \Gamma_1 \cup \Gamma_2 = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$ where

$$\Gamma_1 := \{(x,y) \mid x^2+y^2 = 1,\ x < 0\}, \quad \Gamma_2 := \{(x,y) \mid x^2+y^2 = 1,\ x > 0\}$$

and

$$\tilde{\Gamma}_1 := \{(x,y) \mid x^2+y^2 = 1,\ x < \sqrt{2}/2\}, \quad \tilde{\Gamma}_2 := \{(x,y) \mid x^2+y^2 = 1,\ x > \sqrt{2}/2\}.$$

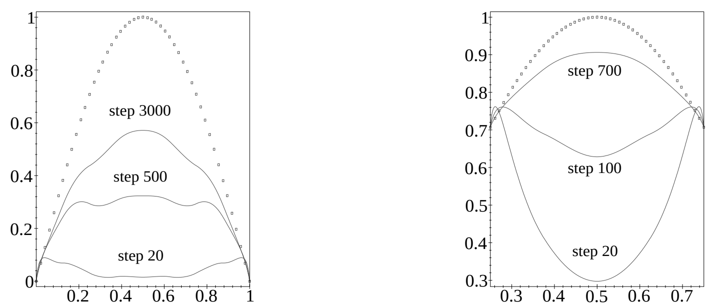
The inner component of $\partial\Omega$ is called $\Gamma_i := \{(x,y);\ x^2+y^2 = 1/4\}$.

Given the Cauchy data $f(\theta) = \sin(\theta)$ and $g(\theta) = 0$ on Γ_1 (respectively $\tilde{\Gamma_1}$) we reconstruct on Γ_2 (respectively $\tilde{\Gamma_2}$) the trace of the solution of the non linear Cauchy problems:

$$\begin{cases} \Delta u + u^3 &= [\,(r + 1/r)\sin(\theta)/2\,]^3\;, \quad \text{in } \Omega \\ u &= f\;, \quad \text{at } \Gamma \\ u_\nu &= g\;, \quad \text{at } \Gamma \\ u &= \frac{5}{4}\sin(\theta)\;, \quad \text{at } \Gamma_i \end{cases},$$

where Γ stands for both Γ_1 and $\tilde{\Gamma_1}$. Both problems have the same solution $u^*(x, y) = (r + 1/r)\sin(\theta)/2$.

A mesh with 82 688 nodes and linear elements is used, the stopping criterion being the same as in section 3.1. In Figure 6 (a) and 6 (b) one can see the exact solution (dotted line) and the iterated sequence ψ_k (solid line) for the Cauchy problems with data given on Γ_1 and $\tilde{\Gamma_1}$ respectively. (note that the x–axis is parameterized from 0 to π in Figure 6 (a) and from $\pi/4$ to $3\pi/4$ in Figure 6 (b))



(a) (b)

Figure 6: Trace of the iteration for a non linear Cauchy problem at the annulus with inner and outer radius respectively 1/2 and 1

Both iterations begin with $\psi_0 \equiv 0$. As it happens in the linear case, the limit of the sequence ψ_k does not depend on the choice of the initial data ψ_0, but the convergence of the iteration is slower for this non linear operator.

Comparing the iteration for both Cauchy problems with data at Γ_1 and $\tilde{\Gamma_1}$ is clear that the amount of known information one has, determines both velocity of convergence and precision of the reconstruction.

4 Concluding Remarks

Remark 4.1 *The classical problem of Hadamard in section 3.1 is numerically treated in [FaMo]. They propose a direct method to solve the Cauchy problem* *on the square* $(0,1) \times (0,1)$ *and obtained a reconstruction with approximately 5% error in the* $L^2(\Gamma_2)$ *norm, while our reconstruction has an error of approximately 4% in this norm (see Figure 2 after 100 iterations).*

The Cauchy problem in section 3.2 is numerically treated in [KuIs], where boundary elements are used on the annulus with inner and outer radius 2 and 6 respectively. The reconstruction error in the $L^\infty(\Gamma_2)$*–norm has the order of* 10^{-2}. *The corresponding error after 100 iteration steps has the same order (see figure 3).*

The difficulty in solving Cauchy problems by using direct methods is that the ill-posedness of the resulting system increases as one tries to refine the numerical model.

Remark 4.2 *A motivation for the convergence of the iteration (in the* L^∞*–norm) even in the inconsistent case can be found on the fact that the numerical discretization* $T_{l,h}$ *of the operator* T_l *we use to generate the sequence* φ_k *is contractive. In fact, the finite element method has the property that only the spaces, and not the differential equation itself, are discretizated (see [Od]). This implies that the eigenvalues of* $T_{l,h}$ *are lower or equal to the respective eigenvalues of* T_l. *As the eigenfunctions of* T_l *(defined at* Γ_2*) cannot be obtained by taking traces of the linear finite elements we are using, we conclude that the eigenvalues*[7] *of* $T_{l,h}$ *are strict smaller than the respective eigenvalues of* T_l.

Remark 4.3 *Another discretization of the operator* T_l *was tested. We tried to solve the mixed problems on the square domain using finite differences. This has the advantage of being faster (we must invert 2 stiffness matrices once, but each iteration is than evaluated as a simple matrix vector product) specially if the number of iteration to be computed is large. Our numerical experiments show that the eigenvalues of this second discretization of* T_l *approximate the real eigenvalues better than the discretization by finite elements does. As a direct consequence, the approximation obtained using a finite difference discretization –with the same stopping criterion– is better than the one obtained using finite elements.*

Remark 4.4 *If the differential operator* P *in (1) is non linear, the property*

$$T\varphi \;=\; \varphi,$$

where T *is the operator defined in (4), will only hold if one can guarantees the unicity of solutions for the Cauchy problem governed by* P a priori. *Even if this is the case, one still have to make sure that each mixed problem on the iteration is uniquely solvable.*

[7] Obviously there is only a finite number of them.

We does not prove these properties for the Cauchy problem in section 3.4, but the iteration was performed using different analytical functions as initial data. The velocity of convergence and the limit encountered is the same for all our tests.

In spite of the fact that the convergence of the iteration is much slower in the non linear case, it can be again verified that the fixed point of $T_h\,\varphi = \varphi$ approaches better the fixed point of $T\,\overline{\varphi} = \overline{\varphi}$, if the precision h of the discretization increases.

A Sobolev spaces and trace Theorems

Let $\Omega \in \mathbb{R}^2$ be an open, bounded, regular[8] set with C^∞–boundary $\partial\Omega$, which is splited in $\partial\Omega = \cup_{j=1}^N \overline{\Gamma_j}$, the subsets Γ_j being open, connected and satisfying $\Gamma_i \cap \Gamma_j = \emptyset$ for $1 \le i \ne j \le N$. We denote by γ_j the trace operator with domain $C^\infty(\overline{\Omega})$ and range $C^\infty(\Gamma_j)$. With ν_j we represent the vector field normal to Γ_j.

Given the second order elliptic operator

$$P(u) \;:=\; -\sum_{i,j=1}^{2} D_i(a_{i,j} D_j u) \;+\; \sum_{i=1}^{2} a_i D_i u \;+\; a_0\, u,$$

we represent the co-normal derivative of a function u respective to P by

$$u_{\nu_A} \;=\; \frac{\partial u}{\partial \nu_A} \;:=\; \sum_{i,j=1}^{2} a_{i,j}\, \nu_i\, (D_j u)$$

where A represent the 2×2 matrix $(a_{i,j})$.

We introduce now the Sobolev spaces used in this article. For $s = k + \sigma \in \mathbb{R}^+$ with $k \in \mathbb{N}_0$ and $\sigma \in [0, 1)$, we define

$$H^s(\Omega) := \overline{C^\infty(\overline{\Omega})}^{||\cdot||_{s;\Omega}},\; H^s_0(\Omega) := \overline{C^\infty_0(\Omega)}^{||\cdot||_{s;\Omega}},\; H^s_0(\Omega \cup \Gamma) := \overline{C^\infty_0(\Omega \cup \Gamma)}^{||\cdot||_{s;\Omega}},$$

where the functional $||\cdot||_{s;\Omega}$ is defined by

$$||u||^2_{s;\Omega} \;:=\; \sum_{|\alpha| \le k} ||D^\alpha u||^2_{L^2(\Omega)} \;+\; \sum_{|\alpha| = k} \iint_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}}\, dx\, dy\,.$$

For $s > 0$ we define the space $H^{-s}(\Omega)$ by duality

$$H^{-s}(\Omega) \;=\; \{u \in \mathcal{D}'(\Omega) \;/\; < u, \cdot >_{L^2(\Omega)} \in H^s_0(\Omega)'\}\,.$$

Given the differential operator P as above, we define the space $H^1(\Omega; P)$ as the space of distributions with

$$\{u \in H^1(\Omega) \;/\; P\,u \in L^2(\Omega)\}\,.$$

[8]We mean Ω is locally at one side of $\partial\Omega$.

The last space we need is $H^s_{00}(\Omega)$, $s \in \mathbb{R}^+$. If $s = k + \sigma$ with $k \in \mathbb{N}_0$ and $\sigma \in [0,1)$ we define the functional

$$||u||_{s;00;\Omega} := \left\{ ||u||^2_{s;\Omega} + \sum_{|\alpha|=k} \int_\Omega |D^\alpha u(x)|^2 \, d^{-2\sigma}(x, \partial\Omega) \, dx \right\}^{1/2},$$

where $d(x, \partial\Omega)$ is the Euclidean distance between $x \in \mathbb{R}^n$ and $\partial\Omega$. Now we define $H^s_{00}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to $||\cdot||_{s;00;\Omega}$.

The next theorems can be found in [DaLi] or [Gr1,2] and will be needed in the study of weak solutions of mixed boundary value problems.

Theorem A.1 (Neumann trace of a $H^1(\Omega; P)$ distribution) *The operator*

$$\Gamma_{\nu_A} : u \longmapsto \gamma \frac{\partial u}{\partial \nu_A}$$

defined on $C^\infty(\overline{\Omega})$ has only one continuous extension defined at

$$\tilde{\Gamma}_\nu : H^1(\Omega; P) \longrightarrow H^{-1/2}(\partial\Omega).$$

Theorem A.2 (Dirichlet trace on γ_j) *Let $m \in \mathbb{N}$. The operator*

$$u \longmapsto \left\{ \gamma_j u, \ \gamma_j \frac{\partial u}{\partial \nu_j}, \ \ldots, \ \gamma_j \frac{\partial^{m-1} u}{\partial \nu_j^{m-1}} \right\}$$

defined for every $j = 1, \ldots, m$ at $C^\infty(\overline{\Omega})$ has only one continuous extension from

$$H^m(\Omega) \quad \text{onto} \quad \prod_{i=0}^{m-1} H^{m-i-\frac{1}{2}}(\Gamma_j).$$

Remark A.3 *The trace operator in Theorem A.2 has a continuous right inverse. This fact follows from the existence of a prolongation operator $\omega \in \mathcal{L}(H^s(\Gamma_j), H^s(\partial\Omega))$, $s \geq 0$, that is the continuous right inverse of the restriction operator $\rho \in \mathcal{L}(H^s(\partial\Omega), H^s(\Gamma_j))$. For details see [Au] pp. 187-194.*

Theorem A.4 (Neumann trace on γ_j) *The Neumann trace operator defined on $C^\infty(\overline{\Omega})$ with range in $C^\infty(\Gamma_j)$ has only one continuous extension as an operator from*

$$H^1(\Omega; P) \quad \text{into} \quad H^{1/2}_{00}(\Gamma_j)',$$

for $j = 1, \ldots, N$.

B Green's formula

In the analysis of the mixed boundary value problems that appear in the iterative procedure (IT), we make use of a special version of the (first) Green formula. The results presented here are still valid if $\partial\Omega$ only Lipschitz continuous and they can be found in [Au], [DaLi], [Gr1], [Le], [LiMa] or [Tr].

Theorem B.1 *If (u, v) is a pair of functions in $H^2(\Omega) \times H^1(\Omega)$, then we have*

$$\int_\Omega Pu\,v\;dx \;=\; \int_{\partial\Omega} u_{\nu_A}\,v\;d\Gamma \;-\; \int_\Omega (\nabla v)^t A\,\nabla u\;dx\,. \tag{23}$$

Equation (23) is still valid if $(u, v) \in H^1(\Omega; P) \times H^1(\Omega)$.[9]

In next theorem the boundary integral in (23) is replaced by a sum of integrals over each γ_j.

Theorem B.2 *Given functions u in $H^2(\Omega)$ and v in $H^1(\Omega)$ we have*

$$\int_\Omega Pu\,v\;dx \;=\; \sum_{j=1}^{N} \int_{\Gamma_j} u_{\nu_A}\,v\;d\Gamma \;-\; \int_\Omega (\nabla v)^t A\,\nabla u\;dx. \tag{24}$$

This theorem is a consequence of the trace Theorem A.2, that guarantees the boundedness of the inner products in (24) in the sense of $L^2(\Gamma_j)$. Theorem B.2 still holds if $\partial\Omega$ is a $C^{1,1}$ polygon. In the next theorem (see [Le]) we formulate Green's formula in the exact context needed in this paper.

Theorem B.3 *Given $u \in H^1(\Omega; P)$ and $v \in \{v \in H^1(\Omega) \,|\, \gamma_j\, v \in H^{1/2}_{00}(\Gamma_j),$ $j = 1, \ldots, N\}$ we have*

$$\int_\Omega Pu\,v\;dx \;=\; \sum_{j=1}^{N} \int_{\Gamma_j} u_{\nu_A}\,v\;d\Gamma \;-\; \int_\Omega (\nabla v)^t A\,\nabla u\;dx.$$

The next two theorems describe some characteristics of the trace of H^1–functions, that are needed in the formulation of the iterative procedure (IT).

Theorem B.4 *Let $N = 2$, i.e. $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ and $u \in H^1(\Omega)$. If $\gamma_1\, u$ is a $H^{1/2}_{00}(\Gamma_1)$ distribution, then $\gamma_2\, u$ is a $H^{1/2}_{00}(\Gamma_2)$ distribution.*

Theorem B.5 *Let $N = 2$, i.e. $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ and u be a P–harmonic function in $H^1(\Omega)$. If $\gamma_1 \frac{\partial u}{\partial \nu_A}$ belongs to $H^{-1/2}(\Gamma_1)$, than $\gamma_2 \frac{\partial u}{\partial \nu_A}$ belongs to $H^{-1/2}(\Gamma_2)$.* [10]

[9]The last assertion follows from the density of the embedding of $C^\infty(\overline{\Omega})$ into $H^1(\Omega; \Delta)$ together with Theorem A.1.

[10]One should note that $H^{-1/2}(\Gamma_2) = H^{1/2}(\Gamma_2)'$, because of the identity $H^{1/2}_0(\Gamma_2) = H^{1/2}(\Gamma_2)$.

C Mixed boundary value problems

For the analysis of mixed problems we need a type of *Poincaré inequality* on the space $H_0^1(\Omega \cup \Gamma_j)$. This is obtained with theorem

Theorem C.1 *Given a function $u \in H_0^1(\Omega \cup \Gamma_j)$ we have*

$$||u||_{L^2(\Omega)} \;\leq\; c\,||\nabla u||_{L^2(\Omega)},$$

where the constant c depends only on Ω.[11]

We analyze the following mixed problem problem at Ω. Given the functions $f \in H^{1/2}(\Gamma_1)$ and $g \in H_{00}^{1/2}(\Gamma_2)'$, find a H^1–solution of

$$(GP) \qquad \begin{cases} \Delta u = 0 & ,\ \text{in } \Omega \\ u = f & ,\ \text{at } \Gamma_1 \\ u_\nu = g & ,\ \text{at } \Gamma_2 \end{cases} .$$

Existence, unicity and continuous dependency of the data for (GP) are given by the following theorem of Lax–Milgramm type.

Theorem C.2 *For every pair of data $(f,g) \in H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_2)'$ the problem (GP) has a unique solution $u \in H^1(\Omega)$. Further it holds*

$$||u||_{H^1(\Omega)} \;\leq\; c\left(||f||_{H^{1/2}(\Gamma_1)} \;+\; ||g||_{H_{00}^{1/2}(\Gamma_2)'}\right). \tag{25}$$

The next theorem investigates the regularity of the H^1–solution of (GP). For a detailed proof see [Gr1,2] or [Wn].

Theorem C.3 *For boundary data $f \in H^{3/2}(\Gamma_1)$ and $g \in H^{1/2}(\Gamma_2)$, the H^1–solution u of (GP) can be written as*

$$u \;=\; h \;+\; \sum_{i=1}^{2} \alpha_i\, u_i\,, \tag{26}$$

where $h \in H^2(\Omega)$, $\alpha_i \in \mathbb{R}$ and u_i are the singular H^1–functions

$$u_i(r,\theta) \;=\; r_i^{1/2} \sin\frac{\theta_i}{2}.$$

Here r_1 (respectively r_2) is the distance from $z = (r,\theta) \in \Omega$ to the contact point p_a between Γ_1 and Γ_2 (respectively p_b between Γ_2 and Γ_1); θ_1 (respectively θ_2) is the angle between $z - p_a$ (respectively $z - p_b$) and the line tangent to $\partial\Omega$ at p_a (respectively at p_b) in the direction of Γ_1.

[11] For a detailed proof see [Tr] pp. 69.

D Unicity results for Cauchy problems

What we present now is a generalisation of some classical results concerning the theory of diferential operators with analytical coefficients. We use the Cauchy–Kowalewsky and Holmgren theorems[12] together with a regularity theorem for weak solutions of elliptic equations, to guarantee the uniqueness of H^1–solutions of Cauchy problems.[13]

Theorem D.1 *Let L be a linear differential operator of order 2 with $C^\infty(\Omega)$–coefficients, where $\Omega \subset \mathbb{R}^n$ is an open regular set. Define $a(\cdot,\cdot)$ the respective bilinear form, which is supposed to be strong coercive. If the distribution u is a solution of $Lu = \psi$ with $\psi \in H^k_{\text{loc}}(\Omega)$, $k \in \mathbb{N}$, then $u \in H^{k+2}_{\text{loc}}(\Omega)$.*[14]

Remark D.2 *Under the same assumptions as in Theorem D.1 it follows from the assumption $\psi \in C^\infty(\Omega)$ that u belongs to $C^\infty(\Omega)$.*

We can now state the uniqueness result for Cauchy problems.

Theorem D.3 *Let Ω be an open, bounded and simply connected set of $\mathbb{R}^2$ with analytical boundary $\partial\Omega$. Let Γ be an open simply connected subset of $\partial\Omega$ and the differential operator L defined as in Theorem D.1. Then the Cauchy–Problem*

$$\begin{cases} Lu = \psi & \text{, in } \Omega \\ u = f & \text{, at } \Gamma \\ u_\nu = g & \text{, at } \Gamma \end{cases}$$

has for $\psi \in L^2(\Omega)$, $f \in H^{1/2}(\Gamma)$ and $g \in H^{1/2}_{00}(\Gamma)'$ at most one solution in $H^1(\Omega)$.

For the special case $L = \Delta$, the Laplace operator, the assumptions relative to Ω can be weakened. For this operator Theorem D.3 still holds even if Ω is not supposed to be simply connected.

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[12] The Cauchy–Kowalewsky theorem can be found in [Fo] pp. 69, [DaLi] or [Jo]; for details on the Holmgren theorem one may see [Jo] pp. 65 or [DaLi].

[13] See also [Is].

[14] The theorem still holds for differential operators of order $2m$. In this case we conclude $u \in H^{k+2m}_{\text{loc}}(\Omega)$.

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