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On iterative methods for solving ill-posed problems modeled by partial differential equations

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Abstract — We investigate the iterative methods proposed by Maz'ya and Kozlov (see [6, 7]) for solving ill-posed inverse problems modeled by partial differential equations. We consider linear evolutionary problems of elliptic, hyperbolic and parabolic types. Each iteration of the analyzed methods consists in the solution of a well posed problem (boundary value problem or initial value problem respectively). The iterations are described as powers of affine operators, as in [7]. We give alternative convergence proofs for the algorithms by using spectral theory and the fact that the linear parts of these affine operators are non-expansive with additional functional analytical properties (see [9, 10]). Also problems with noisy data are considered and estimates for the convergence rate are obtained under *a priori* regularity assumptions on the problem data.

1. INTRODUCTION

1.1. Main results

We present new convergence proofs for the iterative algorithms proposed in [7] using a functional analytical approach, were each iteration is described using powers of an affine operator T. The key of the proof is to choose a correct topology for the Hilbert space were the iteration takes place and to prove that T_l, the linear component of T, is a *regular asymptotic*, *non-expansive* operator (other properties of T_l such as positiveness, self-adjointness and injectivity are also

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verified). The converse is also proved, i. e. if an iterative procedure converges, the limit point is the solution of the respective problem.

The convergence rate of the iterative method can be estimated when we make appropriate regularity assumptions on the problem data. In the last section some numerical experiments are presented, were we test the algorithm performance for linear elliptic, hyperbolic and parabolic ill-posed problems.

The iterative procedures discussed in this paper were presented in [7] and also treated via semi groups in [1]. The iterative procedure for elliptic Cauchy problems defined in domains of more general type is discussed in [6, 9, 10] and [5]. The iterative procedure concerning parabolic problems is also treated in [12].

1.2. Preliminaries

1.2.1. On non-expansive operators

Let H be a separable Hilbert space endowed with an inner product $\langle \cdot , \cdot \rangle$ and norm $\|\cdot\|$. A linear operator $T : H \to H$ is called *non-expansive* if $\|T\| \le 1$.

An operator $T : H \to H$ is said to be *regular asymptotic* in $x \in H$ if

$$\lim_{k\to\infty} \|T^{k+1}(x) - T^k(x)\| = 0$$

holds true. If the above property holds for every $x \in H$, we say that T is *regular asymptotic* in H.

Next we formulate the results used to prove the convergence of the iterative algorithms analyzed in this paper.

Lemma 1. *Let $T : H \to H$ be a linear non-expansive operator. With Π we denote the orthogonal projector defined on H onto the null space of $(I - T)$. The following assertions are equivalent:*

a) T is regular asymptotic in H;

b) $\lim_{k\to\infty} T^k x = \Pi x$ for all $x \in H$.

A proof of this lemma (even in a more general framework) can be found in [4] (see also [9] and the references cited therein).

Lemma 2. *Let $T : H \to H$ be a linear, non-expansive, regular asymptotic operator such that 1 is not an eigenvalue of T*[1]*. Given $z \in H$ define $S : H \ni x \mapsto Tx + z \in H$. Then for every $x_0 \in H$ the sequence $\{S^k x_0\}$ converges to the uniquely determined solution of the fixed point equation $Sx = x$.*

Proof. Let $\bar{x} \in H$ be the solution of $S\bar{x} = \bar{x}$. Defining $x_k := S^k x_0$ and $\varepsilon_k := \bar{x} - x_k$ one can easily see that $\varepsilon_{k+1} = T\varepsilon_k$, $k \in \mathbb{N}$. Lemma 1 allow us to conclude that $\lim_k \varepsilon_k = \Pi\varepsilon_0$. From the hypothesis we have $\ker(I - T) = \{0\}$, and the lemma follows. □

[1]The set of all eigenvalues of a linear operator T is denoted by $\sigma_p(T)$.

In the next lemma we present a sufficient condition for an operator to be non-expansive and regular asymptotic. For convenience of the reader we include here the proof (see [7]).

Lemma 3. *Let T be a bounded linear operator in H such that for $c > 0$*

$$\|(I-T)x\|^2 \;\le\; c(\|x\|^2 - \|Tx\|^2)\,, \quad \forall x \in H \tag{1}$$

holds true. Then T is non-expansive and regular asymptotic in H.

Proof. The non-expansivity of T follows directly from the inequality

$$0 \;\le\; c^{-1}\|(I-T)x\|^2 \;\le\; \|x\|^2 - \|Tx\|^2, \quad \forall x \in H.$$

Now take $x_0 \in H$. Since $\|T\| \le 1$, the sequence $\|T^k x_0\|^2$ is non-increasing, from what we conclude that $\lim_k(\|T^k x_0\|^2 - \|T^{k+1}x_0\|^2) = 0$. Note that from (1) follows

$$\|T^k x_0 - T^{k+1}x_0\|^2 \;\le\; c(\|T^k x_0\|^2 - \|T^{k+1}x_0\|^2).$$

Putting all together one can see that T is regular asymptotic in H. □

Equivalent to the condition (1) in Lemma 3 is the following one[2]

$$\langle (I-T)x, x\rangle \;\ge\; \frac{c+1}{2c}\,\|(I-T)x\|^2, \quad \forall x \in H, \tag{2}$$

as the following line suggests (see [7])

$$\|x\|^2 \;=\; \|Tx\|^2 - \|(I-T)x\|^2 + 2\langle x, (I-T)x\rangle\,, \quad \forall x \in H.$$

1.2.2. On function spaces

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with smooth boundary and let A be a positive, self-adjoint, unbounded operator densely defined on the Hilbert space $H := L_2(\Omega)$. Let E_λ, $\lambda \in \mathbb{R}$, denote the resolution of the identity associated to A, i.e.

$$\langle A\varphi, \psi\rangle = \int_{\lambda\in\mathbb{R}} \lambda \,\mathrm{d}\langle E_\lambda \varphi, \psi\rangle = \int_0^\infty \lambda\,\mathrm{d}\langle E_\lambda\varphi, \psi\rangle,$$

for $\varphi \in D(A)$, the domain of A, and $\psi \in H$. Note that given $f \in C(\mathbb{R}^+)$ we can define the operator $f(A)$ on H by setting

$$\langle f(A)\varphi, \psi\rangle := \int_0^\infty f(\lambda)\,\mathrm{d}\langle E_\lambda\varphi, \psi\rangle,$$

for every $\varphi \in D(f(A))$ and $\psi \in H$, where the domain of $f(A)$ is defined by

$$D(f(A)) := \Big\{\varphi \in H \;\Big|\; \int_0^\infty f(\lambda)^2\,\mathrm{d}\langle E_\lambda\varphi, \varphi\rangle < \infty\Big\}.$$

[2]Clearly, condition (1) is only sufficient for T being non-expansive and regular asymptotic in H.

Now we are ready to construct a family of Hilbert spaces $\mathcal{H}^s(\Omega)$, $s \geq 0$, as the domain of definition of the powers of A[3]

$$\mathcal{H}^s(\Omega) := \Big\{ \varphi \in H \ | \ \|\varphi\|_s := \Big(\int_0^\infty (1+\lambda^2)^s d\langle E_\lambda \varphi, \varphi \rangle \Big)^{1/2} < \infty \Big\}. \tag{3}$$

The Hilbert spaces $\mathcal{H}^{-s}(\Omega)$ (with $s > 0$) are defined by duality[4]: $\mathcal{H}^{-s} := (\mathcal{H}^s)'$. It follows from the definition that $\mathcal{H}^0(\Omega) = H$. It can also be proved that the embedding $\mathcal{H}^r(\Omega) \hookrightarrow \mathcal{H}^s(\Omega)$ is dense and compact for $r > s$ (see [11], Chapter 1).

An interesting example is $A = (-\Delta)^{1/2}$, where Δ is the Laplace–Beltrami operator on Ω. In this particular case we have the identity $\mathcal{H}^s(\Omega) = H_0^{2s}(\Omega)$, where $H_0^s(\Omega)$ is the Sobolev space of index s according to the definition of Lions and Magenes (see [11], p. 54). One should note that functions in $\mathcal{H}^s(\Omega)$ satisfy null boundary conditions in the sense of the trace operator.

Given $T > 0$ we define the spaces $L_2(0,T;\mathcal{H}^s(\Omega))$ of functions $u : (0,T) \ni t \mapsto u(t) \in \mathcal{H}^s(\Omega)$. These are normed spaces if we consider

$$\|u\|_{2;0,T;s} \ := \ \Big(\int_0^T \|u(t)\|_s^2 \, \mathrm{d}t \Big)^{1/2},$$

as a norm in $L_2(0,T;\mathcal{H}^s(\Omega))$. Finally, we define the spaces $C(0,T;\mathcal{H}^s(\Omega))$ of continuous functions $u : [0,T] \ni t \mapsto u(t) \in \mathcal{H}^s(\Omega)$. The norm on these spaces is given by

$$\|u\|_{\infty;0,T;s} \ := \ \sup_{t\in[0,T]} \|u(t)\|_s .$$

2. THE ILL-POSED PROBLEMS

Let the operator A with discrete spectrum, the set Ω and the Hilbert spaces $\mathcal{H}^s(\Omega)$ be defined as in Section 1.2. In the next three paragraphs we formulate the ill-posed problems that are discussed in this article.

2.1. The elliptic problem:

Given functions $(f,g) \in \mathcal{H}^{1/2}(\Omega) \times \mathcal{H}^{-1/2}(\Omega)$, find $u \in (V_e, \|\cdot\|_{V_e})$, where

$$V_e := L_2(0,T;\mathcal{H}^1(\Omega)),$$
$$\|u\|_{V_e} := \Big(\int_0^T (\|u(t)\|_1^2 + \|\partial_t u(t)\|_0^2) \, \mathrm{d}t \Big)^{1/2},$$

that satisfies

$$\begin{cases} (\partial_t^2 - A^2)u = 0 & \text{in } (0,T)\times\Omega \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), & x \in \Omega. \end{cases} \tag{P_e}$$

[3]For simplicity we may write $\mathcal{H}^s$ instead of $\mathcal{H}^s(\Omega)$.
[4]Alternatively one can define $\mathcal{H}^{-s}(\Omega)$ as the completion of H in the $(-s)$-norm defined in (3).

Note that if $u \in V_e$, then $\partial_t u \in L_2(0,T;H)$ and appropriate trace theorems (see [11]) guarantee that $u(0), u(T) \in \mathcal{H}^{1/2}(\Omega)$ and $\partial_t u(0), \partial_t u(T) \in \mathcal{H}^{-1/2}(\Omega)$.

In this problem we are mostly interested in the value of u for $t = T$, i. e. $u(T,x)$ and $\partial_t u(T,x)$, $x \in \Omega$. This elliptic initial value problem (also called Cauchy problem) is not well posed in the sense of Hadamard (see [2]). This follows from the general representation of the solution of (P_e) given by

$$u(t) = \cosh(At)f + \sinh(At)A^{-1}g. \tag{4}$$

One can construct a sequence of Cauchy data $(f_k, g_k) = (0, g_k)$ using the eigenfunctions of A, such that (f_k, g_k) converge to zero in $\mathcal{H}^{1/2} \times \mathcal{H}^{-1/2}$ while the norm of the solutions $\|u_k\|_{V_e}$ do not.

2.2. The hyperbolic problem:

Given functions $f, g \in \mathcal{H}^1(\Omega)$, find $u \in (V_h, \|\cdot\|_{V_h})$, where

$$\begin{aligned} V_h &:= \{v \in C(0,T;\mathcal{H}^1(\Omega)) \mid \partial_t u \in C(0,T;H)\}, \\ \|u\|_{V_h} &:= \sup_{t\in[0,T]} (\|u(t)\|_1^2 + \|\partial_t u(t)\|_0^2)^{1/2}, \end{aligned}$$

that satisfies

$$\begin{cases} (\partial_t^2 + A^2)u = 0 \qquad \text{in } (0,T)\times\Omega, \\ u(0,x) = f(x), \quad u(T,x) = g(x), \quad x \in \Omega. \end{cases} \tag{P_h}$$

Note that if $u \in V_h$, then $u(0), u(T) \in \mathcal{H}^1(\Omega)$ and $\partial_t u(0), \partial_t u(T) \in H$.

Let's assume that the numbers $k\pi/T$, $k = 1, 2, \ldots$ are not eigenvalues of A[5]. Then this hyperbolic (Dirichlet) boundary value problem is ill-posed if the distance from the set $M := \{k\pi/T; k \in \mathbb{N}\}$ to $\sigma(A)$ (the spectrum of A) is zero. To see this, we take $\lambda_k \in \sigma(A)$ with $\lim_k \operatorname{dist}(\lambda_k, M) = 0$ and g_k the respective (normalized) eigenfunctions. Solving problem (P_h) for the data $(f,g) = (0, g_k)$ one obtains respectively the solutions

$$u_k(t) = \sin(At)\sin(AT)^{-1}g_k = \sin(\lambda_k t)\sin(\lambda_k T)^{-1}g_k, \tag{5}$$

which happens to be unbounded in V_h.

2.3. The parabolic problem:

Given a function $f \in H = L_2(\Omega)$ find $u \in (V_p, \|\cdot\|_{V_p})$, where

$$\begin{aligned} V_p &:= L_2(0,T;\mathcal{H}^1(\Omega)), \\ \|u\|_{V_p} &:= \Big(\int_0^T (\|u(t)\|_1^2 + \|\partial_t u(t)\|_{-1}^2)\,\mathrm{d}t\Big)^{1/2}, \end{aligned}$$

[5]If this condition is not satisfied, one can easily see that problem (P_h) is not uniquely solvable.

that satisfies

$$\begin{cases} (\partial_t + A^2)u = 0 & \text{in } (0,T)\times\Omega, \\ u(T,x) = f(x), & x\in\Omega. \end{cases} \tag{P_p}$$

Note that if $u \in V_p$, then $u(0), u(T) \in H$.

Problem (P_p) corresponds to the well known problem of solving the heat equation backwards in time, which is known to be (severely) ill-posed. This follows from the general representation of the solution of (P_p) given by

$$u(t) = \exp(A^2(T-t))f. \tag{6}$$

Again using the eigenfunctions of A, one can construct a sequence of data f_k converging to zero in H while the norm of the solutions $\|u_k\|_{V_p}$ do not.

3. DESCRIPTION OF THE METHODS

3.1. The iterative procedure for the elliptic problem

Consider problem (P_e) with data $(f,g) \in \mathcal{H}^{1/2}(\Omega)\times\mathcal{H}^{-1/2}(\Omega)$. Given any initial guess $\varphi_0 \in \mathcal{H}^{-1/2}(\Omega)$ for $\partial_t u(T)$ we improve it by solving the following mixed boundary value problems (BVP) of elliptic type:

$$\begin{cases} (\partial_t^2 - A^2)v = 0, & \text{in} \quad (0,T)\times\Omega, \\ v(0) = f, & \partial_t v(T) = \varphi_0, \end{cases}$$

$$\begin{cases} (\partial_t^2 - A^2)w = 0, & \text{in} \quad (0,T)\times\Omega, \\ \partial_t w(0) = g, & w(T) = v(T) \end{cases}$$

and defining $\varphi_1 := \partial_t w(T)$. Each one of the mixed BVP's above has a solution in V_e and consequently $\varphi_1 \in \mathcal{H}^{-1/2}(\Omega)$. Setting $\varphi_0 := \varphi_1$ and repeating this procedure we construct a sequence $\{\varphi_k\}$ in $\mathcal{H}^{-1/2}(\Omega)$.

Our assumptions on the operator A allow the determination of the exact solutions v and w of the above problems, which are given by

$$\begin{aligned} v(t) &= \sinh(At)\cosh(AT)^{-1}A^{-1}\varphi_0 + \cosh(A(t-T))\cosh(AT)^{-1}f, \\ w(t) &= \cosh(At)\cosh(AT)^{-1}v(T) + \sinh(A(t-T))\cosh(AT)^{-1}A^{-1}g. \end{aligned}$$

Finally, we can write

$$\varphi_1 = \partial_t w(T) = \tanh(AT)^2\varphi_0 + \sinh(AT)\cosh(AT)^{-2}Af + \cosh(AT)^{-1}g.$$

Now, defining the affine operator $T_e : \mathcal{H}^{-1/2}(\Omega) \to \mathcal{H}^{-1/2}(\Omega)$ by

$$T_e(\varphi) := \tanh(AT)^2\varphi + z_{f,g}, \tag{7}$$

with $z_{f,g} := \sinh(At)\cosh(AT)^{-2}Af + \cosh(AT)^{-1}g$, the iterative algorithm can be rewritten as

$$\varphi_k = T_e(\varphi_{k-1}) = T_e^k(\varphi_0) = \tanh(AT)^{2k}\varphi_0 + \sum_{j=0}^{k-1} \tanh(AT)^{2j} z_{f,g}. \qquad (8)$$

3.2. The iterative procedure for the hyperbolic problem

Let's now consider problem (P_h) with data $f, g \in \mathcal{H}^1(\Omega)$. Given any initial guess $\varphi_0 \in H$ for $\partial_t u(0)$ we improve it by solving the following initial value problems (IVP) of hyperbolic type[6]:

$$\begin{cases} (\partial_t^2 + A^2)v = 0, & \text{in } (0,T)\times\Omega, \\ \quad v(0) = f, \quad \partial_t v(0) = \varphi_0, \end{cases}$$

$$\begin{cases} (\partial_t^2 + A^2)w = 0, & \text{in } (0,T)\times\Omega, \\ \quad w(T) = g, \quad \partial_t w(T) = \partial_t v(T) \end{cases}$$

and defining $\varphi_1 := \partial_t w(0)$. Each one of the mixed IVP's above has a solution in V_h and consequently $\varphi_1 \in H$. Repeating this procedure we construct a sequence $\{\varphi_k\}$ in H.

As in Section 3.1, the assumptions on the operator A allow the determination of the exact solutions v and w of the above problems. In fact we have

$$\begin{aligned} v(t) &= \cos(At)f + \sin(At)A^{-1}\varphi_0, \\ w(t) &= \cos(A(t-T))\,g + \sin(A(t-T))A^{-1}\partial_t v(T). \end{aligned}$$

Finally, we can write

$$\varphi_1 = \partial_t w(0) = \cos(AT)^2\varphi_0 - \cos(AT)\sin(AT)Af \ + \ \sin(AT)g$$

and defining the affine operator $T_h : H \to H$ by

$$T_h(\varphi) := \cos(AT)^2\varphi + z_{f,g}, \qquad (9)$$

with $z_{f,g} := -\cos(AT)\sin(AT)Af + \sin(AT)g$, the iterative method can be rewritten as

$$\varphi_k = T_h(\varphi_{k-1}) = T_h^k(\varphi_0) = \cos(AT)^{2k}\varphi_0 + \sum_{j=0}^{k-1} \cos(AT)^{2j}\, z_{f,g}. \qquad (10)$$

[6]The second problem is considered with reversed time.

3.3. The iterative procedure for the parabolic problem

We consider problem (P_p) with data $f \in H$. Define $\bar{\lambda} := \inf\{\lambda; \lambda \in \sigma(A)\}$ and chose a positive parameter γ such that $\gamma < 2\exp(\bar{\lambda}^2 T)$. Now, given $\varphi_0 \in H$ an initial guess for $u(0)$, the method consists in first solving the IVP of parabolic type:

$$\begin{cases} (\partial_t + A^2)v_0 = 0 & \text{in } (0,T)\times\Omega, \\ v_0(0) = \varphi_0. \end{cases}$$

Then we solve for $k \geq 1$ the sequence of IVP's:

$$\begin{cases} (\partial_t + A^2)v_k = 0 & \text{in } (0,T)\times\Omega, \\ v_k(0) = v_{k-1}(0) - \gamma(v_{k-1}(T) - f). \end{cases}$$

The sequence $\{\varphi_k\}$ is defined by $\varphi_k := v_k(0) \in H$. Note that the analytic solutions of the above problems are given by

$$v_k(t) = \exp(-A^2 t)\varphi_k,$$

and we obtain

$$\varphi_{k+1} = (I - \gamma\exp(-A^2 T))\varphi_k + \gamma f.$$

Now, we define the affine operator $T_p : H \to H$ by

$$T_p(\varphi) := (I - \gamma\exp(-A^2 T))\varphi + z_f, \tag{11}$$

with $z_f := \gamma f$, and we are able to rewrite the iterative algorithm as

$$\begin{aligned} \varphi_k &= T_p(\varphi_{k-1}) = T_p^k(\varphi_0) \\ &= (I - \gamma\exp(-A^2 T))^k\varphi_0 + \sum_{j=0}^{k-1}(I - \gamma\exp(-A^2 T))^j z_f. \end{aligned} \tag{12}$$

4. ANALYSIS OF THE METHODS

4.1. The elliptic case

The linear part of the affine operator T_e defined in (7) is given by $T_{l,e} := \tanh(AT)^2$. We begin the discussion analyzing an important property of problem (P_e).

Lemma 4. *Given $(f,g) \in \mathcal{H}^{1/2} \times \mathcal{H}^{-1/2}$, problem (P_e) has at most one solution in V_e.*

Proof. This result is a generalization of the Cauchy – Kowalewsky theorem. A complete proof can be found in [9]. $\square$

From Lemma 4 follows that if problem (P_e) has a solution $u \in V_e$, then it's Neumann trace $\bar{\varphi} := \partial_t u(T)$ solves the equation $T_e\bar{\varphi} = \bar{\varphi}$. The objective of the iterative method in Section 3.1 is to find a solution of this fixed point equation. The ill-posedness of problem (P_e) can be recognized in the fact that 1 belongs to continuous spectrum of $T_{l,e}$, as one can see in the next lemma.

Lemma 5. *The linear operator $T_{l,e} : \mathcal{H}^{-1/2} \to \mathcal{H}^{-1/2}$ is positive, self-adjoint, injective, non-expansive, regular asymptotic and* 1 *is not an eigenvalue of $T_{l,e}$. Further $T_{l,e}$ satisfies the condition* (1).

Proof. The injectivity follows promptly from Lemma 4. The properties: positiveness, self-adjointness and $1 \notin \sigma_p(T_{l,e})$ follow from the definition of $T_{l,e}$ together with the assumptions on A made in Section 1.2.2 (remember we required in Section 2 that $\sigma(A)$ is discrete).

In order to prove that $T_{l,e}$ is non-expansive and regular asymptotic, it is enough to verify the condition (1) (see Lemma 3). It's easy to see that $T_{l,e}$ satisfies this condition with $c = 1$, if $\sigma(T_{l,e}) \in [0, 1]$. One should note that this last property was already proved above. $\square$

In the next theorem we discuss the convergence of the algorithm described in Section 3.1.

Theorem 1. *Let T_e be the operator defined in* (7) *and $T_{l,e}$ it's linear part. If problem (P_e) in Section 2.1 is consistent*[7] *for the data (f, g), then the sequence $\{\varphi_k\}$ defined in* (8) *converges to $\partial_t u(T)$ in the norm of $\mathcal{H}^{-1/2}(\Omega)$.*

The proof follows from Lemma 5 and Lemma 2 with $z := z_{f,g}$, $T := T_{l,e}$ and $S := T_e$.

The converse of Theorem 1 is also true, i.e. if the sequence $\{\varphi_k\}$ in (8) converges in $\mathcal{H}^{-1/2}(\Omega)$, it converges to the solution of (P_e).

Theorem 2. *If the sequence $\{\varphi_k\}$ defined in* (8) *converges, say to $\bar{\varphi}$, then problem (P_e) is consistent for the Cauchy data (f, g) and it's solution $u \in V_e$ satisfies $\partial_t u(T) = \bar{\varphi}$.*

Proof. If $\lim_k \varphi_k = \bar{\varphi}$, then $T_e\bar{\varphi} = \bar{\varphi}$. Taking $\varphi_0 = \bar{\varphi}$ in the mixed BVP's of Section 3.1 we see that the functions v, w satisfy the same boundary conditions (Dirichlet and Neumann conditions, respectively) at $t = T$. From Lemma 4 we must have $v = w$ and one can see that $u := v = w$ is the solution of (P_e), the identity $\partial_t u(T) = \bar{\varphi}$ being obvious. $\square$

4.2. The hyperbolic case

The linear part of the affine operator T_h defined in (9) is given by $T_{l,h} := \cos(AT)^2$. We start the discussion proving some properties of this operator.

Lemma 6. *The linear operator $T_{l,h} : H \to H$ is positive, self-adjoint, injective, non-expansive, regular asymptotic and* 1 *is not an eigenvalue of $T_{l,h}$. Further $T_{l,h}$ satisfies the condition* (1).

[7]This means that it has a corresponding solution $u \in V_e$.

Proof. The injectivity follows from the assumption $\{k\pi/T; k \in \mathbb{N}\} \cap \sigma(A) = \emptyset$. The properties: positiveness, self-adjointness and $1 \notin \sigma_p(T_{l,h})$ are proved like in Lemma 5.

Again we use Lemma 3 to prove that $T_{l,h}$ is non-expansive and regular asymptotic. Since $\sigma(T_{l,h}) \in [0, 1]$, the condition (1) is obtained analogous as in Lemma 5. □

From Lemma 6 follows that if problem (P_h) has a solution $u \in V_h$, then it's Neumann trace $\bar{\varphi} := \partial_t u(0)$ solves the equation $T_h\bar{\varphi} = \bar{\varphi}$. Just like in the elliptic case (see Section 4.1) the objective of the method in Section 3.2 is to approximate the solution of this fixed point equation. The ill-posedness of problem (P_h) reflects in the fact that 1 belongs to continuous spectrum of $T_{l,h}$ (see Lemma 6). In the next theorem we discuss the convergence of the algorithm described in Section 3.2.

Theorem 3. *Let T_h be the operator defined in* (9) *and $T_{l,h}$ it's linear part. If problem (P_h) in Section 2.1 is consistent for the data (f, g), then the sequence $\{\varphi_k\}$ defined in* (10) *converges to $\partial_t u(0)$ in the norm of H.*

The proof follows from Lemma 6 and Lemma 2 with $z := z_{f,g}$, $T := T_{l,h}$ and $S := T_h$.

The converse of Theorem 3 is also true, i. e. if the sequence $\{\varphi_k\}$ in (10) converges in H, it converges to the solution of (P_h).

Theorem 4. *If the sequence $\{\varphi_k\}$ defined in (10) converges, say to $\bar{\varphi}$, then problem (P_h) is consistent for the Cauchy data (f, g) and it's solution $u \in V_h$ satisfies $\partial_t u(0) = \bar{\varphi}$.*

Proof. If $\lim_k \varphi_k = \bar{\varphi}$, then $T_h\bar{\varphi} = \bar{\varphi}$. Taking $\varphi_0 = \bar{\varphi}$ in the IVP's of section 3.2 we see that the functions v, w satisfy the same Neumann boundary conditions at $t = 0$ and $t = T$. From Lemma 6 we must have $v = w$ and one can see that $u := v = w$ is the solution of (P_h), the identity $\partial_t u(0) = \bar{\varphi}$ being obvious. □

4.3. The parabolic case

The linear part of the affine operator T_p defined in (11) is given by $T_{l,p} := I - \gamma \exp(-A^2T)$. First, we analyze an important property of problem (P_p).

Lemma 7. *Given $f \in H$, problem (P_p) has exactly one solution in V_p.*

This result is suggested by the general representation of the solution given in (6). A complete proof can be found in [11], Chapter 3.

Just like in the other cases, the iterative method in Section 3.3 approximates the solution of the corresponding fixed point equation $T_p\bar{\varphi} = \bar{\varphi}$, which is uniquely solved by the Dirichlet trace $\bar{\varphi} = u(0)$ of the solution $u \in V_p$ of (P_p) (see Lemma 7). Next, we discuss some properties of $T_{l,p}$.

Lemma 8. *The linear operator $T_{l,p} : H \to H$ is self-adjoint, non-expansive, regular asymptotic and 1 is not an eigenvalue of $T_{l,p}$. Further, if $\gamma < 2\exp(\tilde{\lambda}^2 T)$, where $\tilde{\lambda} := (\bar{\lambda}^2 - T^{-1}\ln 2)^{1/2}$, then $T_{l,p}$ is injective and satisfies the condition* (1).

Proof. The self-adjointness follows follows from the definition of $T_{l,p}$. Since the inequality $0 < \gamma\exp(-\lambda^2 T) < 2\exp([\bar{\lambda}^2 - \lambda^2]T) < 2$ holds for every $\lambda \in \sigma(A)$, we have $\sigma_p(T_{l,p}) \in (-1,1)$ and the non-expansivity follows. Note that the property $1 \notin \sigma_p(T_{l,p})$ was also proved.

To prove the asymptotic regularity we take $\varphi \in H$ and write $T_{l,p}^{k+1}\varphi - T_{l,p}^k\varphi = T_{l,p}^k\psi$, where $\psi := (T_{l,p} - I)\varphi \in H$. Since $\sigma_p(T_{l,p}) \in (-1,1)$, it follows that $\lim_k T_{l,p}^k\psi = 0$, for all $\psi \in H$.

Now, if γ satisfies the extra assumption, a simple calculation shows that $\sigma_p(T_{l,p}) \in (0,1)$. The injectivity follows immediately and the condition (1) is proved analogous as in Lemma 5. $\square$

In the next theorem we discuss the convergence of the algorithm described in Section 3.3.

Theorem 5. *Let T_p be the operator defined in* (11) *and $T_{l,p}$ it's linear part. Given $f \in H$, let $u \in V_p$ be the uniquely determined solution of problem* (P_p). *Then the sequence $\{\varphi_k\}$ defined in* (12) *converges to $u(0)$ in the norm of H.*

The proof follows from Lemma 8 and Lemma 2 with $z := z_f$, $T := T_{l,p}$ and $S := T_p$.

5. REGULARIZATION

In order to regularize the algorithms proposed in Section 3 we make the following assumptions on the formulation of the respective problems:

(H_e) Given the Cauchy data $(f_\varepsilon, g_\varepsilon) \in \mathcal{H}^{1/2} \times \mathcal{H}^{-1/2}$, there exist consistent Cauchy data $(f,g) \in \mathcal{H}^{1/2} \times \mathcal{H}^{-1/2}$ such that $\|f - f_\varepsilon\|_{1/2} + \|g - g_\varepsilon\|_{-1/2} \le \varepsilon$, where $\varepsilon > 0$.

(H_h) Given the Dirichlet data $(f_\varepsilon, g_\varepsilon) \in \mathcal{H}^1 \times \mathcal{H}^1$, there exist consistent Dirichlet data $(f,g) \in \mathcal{H}^1 \times \mathcal{H}^1$ such that $\|f - f_\varepsilon\|_1 + \|g - g_\varepsilon\|_1 \le \varepsilon$, where $\varepsilon > 0$.

(H_p) The given data $f_\varepsilon \in H$ is such that $\|f - f_\varepsilon\|_H \le \varepsilon$, where $f \in H$ is the Dirichlet trace at $t = T$ of the exact solution of (P_p) and $\varepsilon > 0$.

The assumptions on the data made in (H_e) and (H_h) may look very restrictive. One would prefer $f_\varepsilon \in H = L_2(\Omega)$ in (H_e) and $(f_\varepsilon, g_\varepsilon) \in H \times H$ in (H_h), since these represent measured data. Nevertheless (H_e) and (H_h) are naturally satisfied if we make stronger assumptions on the regularity of the solutions of the corresponding ill-posed problems. This fact is explained in

Lemma 9. *Let $f \in \mathcal{H}^r$, $r > s > 0$, and $f_\varepsilon \in H$ be such that $\|f - f_\varepsilon\|_H^2 \le \varepsilon$, where $\varepsilon > 0$. Then there exists a smoothing operator $S : H \to \mathcal{H}^s$ and a* *positive function* γ *with* $\lim_{x\downarrow 0}\gamma(x) = 0$, *such that* $\tilde{f}_\varepsilon := Sf_\varepsilon \in \mathcal{H}^s$ *satisfies* $\|f - \tilde{f}_\varepsilon\|_s^2 \le \gamma(\varepsilon)$.

Proof. Using the resolution of the identity associated to A, we define for $h > 0$ the operator $S_h : H \to \mathcal{H}^s$ by $S_h := \int_0^{1/h} dE_\lambda$ [8]. Defining $\tilde{f}_\varepsilon := S_h f_\varepsilon \in \mathcal{H}^s$ one can estimate

$$\|f - \tilde{f}_\varepsilon\|_s^2 = \|f - S_h f_\varepsilon \pm S_h f\|_s^2 \le 2\big(\|(I - S_h)f\|_s^2 + \|S_h(f - f_\varepsilon)\|_s^2\big). \tag{13}$$

The first term on the right hand side of (13) can be estimated by

$$\begin{aligned}
\|(I - S_h)f\|_s^2 &= \int_{1/h}^{\infty} (1+\lambda^2)^s \Big[\frac{(1+\lambda^2)^r}{(1+\lambda^2)^r}\Big]\, \mathrm{d}\langle E_\lambda f, f\rangle \\
&\le (1+h^{-2})^{s-r} \int_0^{\infty} (1+\lambda^2)^r\, \mathrm{d}\langle E_\lambda f, f\rangle \\
&= (1+h^{-2})^{s-r} \|f\|_r^2.
\end{aligned}$$

For the second term on the right hand side of (13) we have

$$\begin{aligned}
\|S_h(f - f_\varepsilon)\|_s^2 &= \int_0^{1/h} (1+\lambda^2)^s d\langle E_\lambda(f - f_\varepsilon), (f - f_\varepsilon)\rangle \\
&\le (1+h^{-2})^s \int_0^{1/h} d\langle E_\lambda(f - f_\varepsilon), (f - f_\varepsilon)\rangle \\
&\le (1+h^{-2})^s \|f - f_\varepsilon\|_H^2.
\end{aligned}$$

Substituting the last inequalities in (13) we obtain

$$\|f - \tilde{f}_\varepsilon\|_s^2 \le 2\big[(1+h^{-2})^s \varepsilon + (1+h^{-2})^{s-r}\|f\|_r^2\big]. \tag{14}$$

To balance the right hand side of (14) one must choose $h = [(\varepsilon^{-1}\|f\|_r^2)^{1/r} - 1]^{-1/2}$. Now the theorem follows choosing $S := S_h$ and $\gamma(x) := 4x^{(r-s)/r}\|f\|_r^{2s/r}$.
□

Remark 1. Let $f_\varepsilon \in H$ be the given Cauchy data for (P_e). From Lemma 9 follows that when the exact Cauchy data f is better than $\mathcal{H}^{1/2}$, i. e. $f \in \mathcal{H}^r$ for $r > 1/2$, then it is possible to find a $\tilde{f}_\varepsilon$ in $\mathcal{H}^{1/2}$ near to f in the (1/2)–norm. For the hyperbolic case one obtains an analogous result.

Since the affine term $z_{f,g}$ depends continuously on the data (f, g) and z_f depends continuously on f, we conclude from Lemma 9 that under the corresponding assumption it is possible to obtain from the measured data $(f_\varepsilon, g_\varepsilon) \in H^2$ a z_ε satisfying $\|z_{f,g} - z_\varepsilon\| \le \varepsilon'$ (respectively $\|z_f - z_\varepsilon\| \le \varepsilon'$).

Let T be one of the operators defined in (7), (9) or (11) and T_l the corresponding linear part. We want to choose a linear operator R such that given φ_0, the regularized sequence $\tilde{\varphi}_{k+1} := R\tilde{\varphi}_k + z_\varepsilon$ converges faster then the original

[8] Recall that $\int_0^\infty \mathrm{d}E_\lambda$ is the identity operator in H.

one $\varphi_{k+1} := T_l \varphi_k + z_{f,g}$. Simultaneously we have to assure that the difference $\| \lim \tilde{\varphi}_k - \lim \varphi_k \|$ remains small.

In Section 3 we have seen that $T_l = \int_0^\infty F(\lambda) dE_\lambda$, where $F(\lambda)$ is either $\tanh^2(\lambda)$, $\cos^2(\lambda)$ or $(1-\gamma \exp(-\lambda^2 T))$. Given $n \in \mathbb{N}$ we define the regularization operator R_n by

$$R_n := \int_0^n F(\lambda)\, \mathrm{d}E_\lambda.$$

Next we define $\bar{\varphi}$ and φ_n as the fixed points of $\bar{\varphi} = T_l \bar{\varphi} + z_{f,g}$ and $\varphi_n = R\varphi_n + z_\varepsilon$ respectively (note that φ_n exists since R_n is contractive). From the identity

$$\varphi_n - \bar{\varphi} = R_n(\varphi_n - \bar{\varphi}) + (R_n - T_l)\bar{\varphi} + z_\varepsilon - z_{f,g}$$

one obtains the estimate

$$\|\varphi_n - \bar{\varphi}\| \le \|(I - R_n)^{-1}(R_n - T_l)\bar{\varphi}\| + \varepsilon \|(I - R_n)^{-1}\|, \tag{15}$$

which leads us to the following lemma

Lemma 10. *Let T_l represent the linear part of the iterative procedure for one of the problems (P_e), (P_h) or (P_p). Given the corresponding family of operators R_n defined as above, we have*

$$\lim_{n\to\infty} \|(I - R_n)^{-1}(R_n - T_l)\bar{\varphi}\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|(I - R_n)^{-1}\| = \infty.$$

Proof. Since $(I - R_n)$ is the identity operator on $\mathrm{Rg}(R_n - T_l)$, the first assertion follows from the innequality[9]

$$\|(I - R_n)^{-1}(R_n - T_l)\bar{\varphi}\|_s^2 \le \int_n^\infty (1 + \lambda^2)^s \, \mathrm{d}\langle E_\lambda \bar{\varphi}, \bar{\varphi}\rangle.$$

The second assertion follows from the identity $\|(I - R_n)^{-1}\| = (1 - \Lambda(n))^{-1}$, where $\Lambda(n) := \max\{\lambda \in \sigma(A); \lambda < n\}$, and the fact that $A : H \to H$ is unbounded. □

Now making *a priori* assumptions on the regularity of $\bar{\varphi}$, we obtain from Lemma 10 and the estimate (15) the desired regularization result.

Lemma 11. *If there exists a positive monotone increasing function $G \in C(\mathbb{R}^+)$ with*

$$\lim_{\lambda\to\infty} G(\lambda) = \infty \qquad \text{and} \qquad \int_0^\infty (1 + \lambda^2)^s G^2(\lambda)\, \mathrm{d}\langle E_\lambda \bar{\varphi}, \bar{\varphi}\rangle = M^2 < \infty,$$

then exists an optimal choice of $n^* \in \mathbb{N}$ such that $\|\varphi_{n^*} - \bar{\varphi}\| \le \|\varphi_n - \bar{\varphi}\|$ for all $n \in \mathbb{N}$. Further n^* solves the minimization problem

$$\min_{n\in\mathbb{N}} \left\{ M G^{-1}(n) + \varepsilon(1 + \Lambda(n))^{-1} \right\}.$$

[9]Here $s \in \mathbb{R}$ must be chosen according to the space where the iteration takes place.

Proof. From Lemma 10 follows

$$\begin{aligned}\|(I-R_n)^{-1}(R_n-T_l)\bar{\varphi}\|_s^2 &\leq \int_n^\infty (1+\lambda^2)^s \frac{G^2(\lambda)}{G^2(\lambda)}\,\mathrm{d}\langle E_\lambda\bar{\varphi},\bar{\varphi}\rangle \\ &\leq G^{-2}(n)\int_n^\infty (1+\lambda^2)^s G^2(\lambda)\,\mathrm{d}\langle E_\lambda\bar{\varphi},\bar{\varphi}\rangle \\ &\leq M^2G^{-2}(n).\end{aligned}$$

From (15) we obtain

$$\|\varphi_n-\bar{\varphi}\| \leq MG^{-1}(n)+\varepsilon(1+\Lambda(n))^{-1}$$

and the theorem follows. □

6. NUMERICAL RESULTS

6.1. A parabolic reconstruction problem

We consider the heat equation $a^2\partial_t u = \Delta u$ at $(0,T)\times\Omega$, where $\Omega=(0,1)\times(0,1)$. The solution $u(0)$ of the reconstruction problem is shown in Figure 1. It consists of a $L^2(\Omega)$ function added to a polynomial of fourth degree.

In the first example we choose $a^2=8$ and take as problem data $f:=u(T)$ evaluated at $T=0.0625$. The iterative procedure is started with $\varphi_0\equiv 0$ and we chose the parameter $\gamma=2$, which is in agreement with Lemma 8. In Figure 2 one can see the data f of the reconstruction problem and the iteration error after 10^6 steps.

In the second example we choose $a^2=2$ and set $f:=u(T)$ for $T=0.0625$, where $u(0)$ is the same as before. The iterative procedure is started with $\varphi_0\equiv 0$. In Figure 3 one can see the problem data f and the iteration error after 10^6 steps.

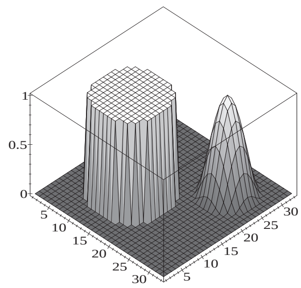

Figure 1. Solution of the reconstruction problem ($u(0)$)

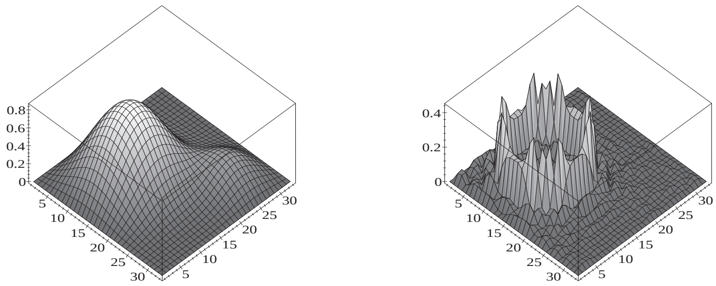

Figure 2. Temperature profile $u(T)$ for $a^2 = 8$ and iteration error after 10^6 steps

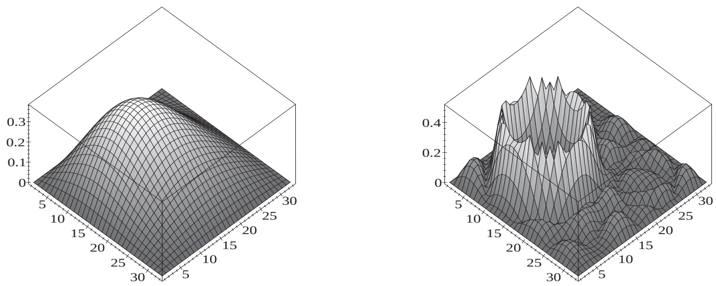

Figure 3. Temperature profile $u(T)$ for $a^2 = 2$ and iteration error after 10^6 steps

Table 1. Evolution of the relative error in the L^2-norm (parabolic problem)

	10 steps	10^3 steps	10^4 steps	10^5 steps	10^6 steps
$a^2 = 8$	32.5%	25.7%	23.5%	22.5%	20.9%
$a^2 = 2$	49.8%	42.2%	40.1%	36.2%	31.4%

Table 2. Evolution of the relative error in the L^2-norm (elliptic problem)

	10^2 steps	10^3 steps	10^5 steps	10^6 steps	10^8 steps	10^9 steps
$k=1$	48.33%	5.34%	5.30%	5.30%	5.30%	5.30%
$k=2$	99.86%	98.61%	29.72%	11.28%	11.28%	11.28%
$k=3$	99.99%	99.98%	99.74%	97.44%	21.79%	18.42%

One should note that $\bar{\varphi} := u(0)$ is a fixed point of the numerical iteration. This follows from the fact that $f = u(T)$ was obtained by solving a direct problem.

In both examples the reconstruction is much better at the part of the domain where the initial condition is smooth. In Table 1 we present the evolution of the iteration error $\varphi_k - u(0)$ for the two examples above.

Note also that the convergence speed decays exponentially as we iterate. This is a consequence of the exponential behaviour of the eigenvalues of $T_{l,p}$ (see Paragraph 4.3).

6.2. An elliptic reconstruction problem

We consider next the Laplace equation $\partial_t^2 u + \partial_x^2 u = 0$ at $(0,T) \times \Omega$, where $\Omega = (0,1)$. Given $k \in \mathbb{N}$ we choose the Cauchy data $f \equiv 0$, $g_k = \sin(k\pi x)$ and try to reconstruct the corresponding traces $\partial_t u(T)$ at the final time $T = 1$[10].

In Table 2 the evolution of the relative reconstruction error for three distinct values of k is presented. From this data one can see that if g can be expanded in a Fourier series, it's first coefficient will be accurately reconstructed after 10^4 steps, while the second one only after 10^6 steps, etc.

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[10] Note that $g_k = \sin(k\pi x)$ are eigenfunctions of $T_{l,e}$ with corresponding eigenvalues $\lambda_k = \tanh(k\pi)^2$. The solutions of the reconstruction problems are given by $\cosh(k\pi)\sin(k\pi x)$.

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